Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVIII(2002), Nr. 1, 39 - 52

Dedicated to Costică MUSTĂŢA on his 60^{th} anniversary

STRONGLY δ-CONTINUOUS FUNCTIONS AND TOPOLOGIES ON FUNCTION SPACES

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Abstract. In this paper we study the function space of strongly δ-continuous functions and have generalised some basic results of R. Arens and J. Dugundji.

MSC: 54C35, 54C10

Keywords : Strongly δ-splitting , Strongly δ-conjoining.

Introduction

The concept of 'proper' and 'admissible' topology in function space were first introduced by Arens and Dugundji in [1]. These terminologies subsequently changed to 'splitting' and 'conjoining' respectively. In [4] we introduced the notion of strongly δ-continuous function and the concepts of δ-splitting and δ-conjoining topology . Here we introduce strong δ-splitting toplogy and strong δ-conjoining topology and obtain some of its properties. We also introduce the notion of 'strong δ^*' by slightly changing the definition of 'strong δ' and try to find its behaviour in relation to the notion of 'strong δ'. We also construct some examples of strongly δ-splitting topology and try to find the behaviour of $N-R$ topology [3] on the set of δ-continuous functions and also on the set of strong δ-continuous functions. Lastly, we have defined the δ-upper limit of a net and have investigated the relations between different types of convergence through it.

§1. Prerequisites, Definitions & Theorems

Definition 1.1 [5] : Let X be a topological space . A set S in X is said to be regular open (respectively regular closed) if $Int.(cl.S) = S$ (respectively $Cl.(int.S) = S$). A point $x \in S$ is said to be a δ-cluster point of S if $S \cap U \neq \emptyset$, for every regular open set U containing x. The set of all δ-cluster points of S is called the δ-closure of S and is denoted by $[S]_\delta$. If $[S]_\delta = S$, then S is said to be δ-closed . The complement of a δ-closed set is called a δ-open set .

For every topological space (X, τ), the collection of all δ-open sets forms a topology for X, which is weaker than τ. This topology τ^* has a base consisting of all regular open sets in (X, τ).

Definition 1.2 [5] : A function $f : X \to Y$ is called a δ-continuous function iff for every regular open set V of Y, $f^{-1}(V)$ is δ-open in X. This can be alternatively defined as follows : a function $f : X \to Y$ is δ-continuous at a point $x \in X$ iff for every regular open nbd. V of $f(x)$ in Y, $\exists$ a δ-open nbd. U of x such that $f(U) \subseteq V$.

Definition 1.3 [4]: A function $f : X \to Y$ is strongly δ-continuous at a point $x \in X$ iff for any open nbd. V of $f(x)$ in Y , $\exists$ a δ-open nbd. U of x in X such that $f(U) \subseteq V$; instead of taking an arbitrary nbd. of $f(x)$ we could take a sub-basic open set containing $f(x)$ as well.

The set of all strongly δ-continuous functions would be denoted by $SD(X, Y)$, whereas $D(X, Y)$ would denote the set of all δ-continuous functions from X to Y.

Obviously every strongly δ-continuous function is always continuous and the converse does also hold if X is regular.

Definition 1.4 [2] :A set $A \subset (X, \tau)$ is said to be N-closed in X or simply N-closed, if for any cover of A by τ-open sets , there exists a finite sub-collection the interiors of the closures of which cover A; interiors and closures are of course w.r.t τ. A space (X, τ) is said to be nearly compact iff X is N-closed in X.

Definition 1.5 [3] : The $N - R$ topology on $D(X, Y)$ (or $SD(X, Y)$) is generated by the sets of the form

$$\{T(C, U) \ : \ C \text{ is } N\text{-closed in } X \text{and } U \text{ regular open in } Y\},$$

where $T(C, U) = \{f \in D(X, Y) \ : \ f(C) \subseteq U\}$.

Definition 1.6 [5] : A net $\{x_\lambda : \lambda \in \Lambda\}$ in (X, τ) is said to δ-converge to a point $x \in X$ iff every regular open nbd. of x contains the net eventually ; we write $x_\lambda \xrightarrow{\delta} x$.

Theorem 1.7 [5]. *A function $f : X \to Y$ is δ-continuous iff $\{f(x_\lambda)\}_{\lambda \in \Lambda}$ δ-converges to $f(x)$ for each $x \in X$ and for each net $\{x_\lambda\}_{\lambda \in \Lambda}$ δ-converging to x.*

Theorem 1.8 [4]. *A net $\{f_\mu \ : \ \mu \in M\}$ in $D(X, Y)$ is said to be δ-continuously convergent to $f \in D(X, Y)$, if for any net $\{x_\lambda \ : \ \lambda \in \Lambda\}$ in X such that $x_\lambda \xrightarrow{\delta} x$, the net $f_\mu(x_\lambda) \xrightarrow{\delta} f(x)$.*

Notation 1.9 : By C_δ we denote the class of all pairs $(\{f_\lambda : \lambda \in \Lambda\}, f)$ where $\{f_\lambda : \lambda \in \Lambda\}$ is a net in $D(X, Y)$ which δ-continuously converges to $f \in D(X, Y)$. If τ is a topology on $D(X, Y)$ then by $(C(\tau))_\delta$, we denote the class of all pairs $(\{f_\lambda : \lambda \in \Lambda\}, f)$, where $\{f_\lambda : \lambda \in \Lambda\}$ is a net in $D(X, Y)$ which δ-converges to f in the τ-topology.

Definition 1.10 [4] ; Given three spaces X, Y, Z a function $\alpha(x, y) = z$ can be regarded as a map from $X \times Y$ to Z or as a family of maps $Y \to Z$ with X a parametric space .

For notation let $\alpha : X \times Y \to Z$ be δ-continuous at $y \in Y$ for each fixed $x \in X$, the formula $[\tilde{\alpha}(x)](y) = \alpha(x, y) \cdots (1)$ defines $\tilde{\alpha}(x) : Y \to Z$ which is δ-continuous i.e., $\tilde{\alpha}(x) \in D(Y, Z)$. So $\tilde{\alpha} : X \to D(Y, Z)$ is generated from the original mapping $\alpha : X \times Y \to Z$ as given .

Conversely given an $\tilde{\alpha} : X \to D(Y, Z)$, the formula (1) defines an $\alpha : X \times Y \to Z$ which is δ-continuous at $y \in Y$ for each fixed $x \in X$. Two maps $\alpha : X \times Y \to Z$ and $\tilde{\alpha} : x \to D(Y, Z)$ related by the formula (1) are called associates.

Definition 1.11 : A topology τ on $D(Y, Z)$ is called δ-splitting iff for every space X the δ-continuity of a map $\alpha : X \times Y \to Z$ implies the δ-continuity of the map $\tilde{\alpha} : X \to D(Y, Z)$.

Definition 1.12 : A topology τ on $D(Y, Z)$ is called δ-conjoining iff for every space X the δ-continuity of a map $\tilde{\alpha} : X \to D(Y, Z)$ implies the δ-continuity of the map $\alpha : X \times Y \to Z$.

Theorem 1.13. *A topology τ on $D(Y, Z)$ is δ-conjoining iff the evaluation map $P : D(Y, Z) \times Y \to Z$ defined by $P(f, y) = f(y)$ is δ-continuous.*

Proof : It suffices to observe that $\alpha = P \circ (\tilde{\alpha} \times 1)$, where 1 is the identity function in Y. Moreover the evaluation function i.e. is the associate of the identity mapping on $D(Y, Z)$.

Theorem 1.14 [4]: *A topology τ on $D(Y, Z)$ is δ-splitting iff $C_\delta \subseteq (C(\tau))_\delta$*

Theorem 1.15 [4] : A topology τ on $D(Y, Z)$ is δ-conjoining iff $(C(\tau))_\delta \subseteq C_\delta$.

§2. Strong δ-notions

We have already defined strong δ-continuity of a function in 1.3.

Definition 2.1 : A net $\{f_\mu : \mu \in M\}$ in $SD(X, Y)$ is said to be strongly δ-continuously convergent to $f \in SD(X, Y)$, iff for any net $\{x_\lambda : \lambda \in \Lambda\}$ in X which δ-converges to $x \in X$, we have the net $\{f_\mu(x_\lambda) : (\lambda, \mu) \in \Lambda \times M\}$ converging to $f(x)$ in Y.

Theorem 2.2 [4]. *A function $f : X \to Y$ is strongly δ-continuous at a point $x \in X$ iff for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X for which $x_\lambda \xrightarrow{\delta} x$, we have $f(x_\lambda) \to f(x)$ in Y.*

Notation 2.3 : By C^*_δ we denote the class of all pairs $(\{f_\lambda : \lambda \in \Lambda\}, f)$ where $\{f_\lambda : \lambda \in \Lambda\}$ is a net in $SD(Y, Z)$ which strongly δ-continuously converges to $f \in SD(Y, Z)$. If τ is a topology on $SD(Y, Z)$ then by $(C^*(\tau))_\delta$, we denote the class of all pairs $(\{f_\lambda : \lambda \in \Lambda\}, f)$, where $\{f_\lambda : \lambda \in \Lambda\}$ is a net in $SD(Y, Z)$ which converges to $f \in SD(Y, Z)$ in the τ-topology.

Definition 2.4 : A topology τ on $SD(Y,Z)$ is called strongly δ-splitting iff for every space X the strong δ-continuity of a map $\alpha : X \times Y \to Z$ implies the strong δ-continuity of the map $\tilde{\alpha} : X \to SD(Y,Z)$.

Definition 2.5 : A topology τ on $SD(Y,Z)$ is called strongly δ-conjoining iff for every space X the strong δ-continuity of a map $\tilde{\alpha} : X \to SD(Y,Z)$ implies the strong δ-continuity of the map $\alpha : X \times Y \to Z$.

Theorem 2.6. *A topology τ on $SD(Y,Z)$ is strongly δ-conjoining iff the evaluation map $P : SD(Y,Z) \times Y \to Z$ defined by $P(f,y) = f(y)$ is strongly δ-continuous. The proof is straight forward in view of theorem 1.13 and with the fact that the composition of two strongly δ-continuous function is strongly δ-continuous.*

Lemma 2.7 [4] : If X and Y are topological spaces & A, B are N-closed sets in X and Y respectuvely . If W is a δ-open set containing $A \times B$ in the product space $X \times Y$, then there are δ-open sets U & V respectively such that $A \subset U$, $B \subset V$, $U \times V \subset W$.

Notation 2.8 : Let $\mathcal{A}$ be a family of spaces . A topology τ on $SD(Y,Z)$ is called strongly $\delta_{\mathcal{A}}$-splitting (respectively strongly $\delta_{\mathcal{A}}$-conjoining) iff for an element X of $\mathcal{A}$, the strong δ-continuity of a map $\alpha : X \times Y \to Z$ (respectively a map $\tilde{\beta} : X \to SD(Y,Z)$) implies the strong δ-continuity of the map $\tilde{\alpha} : X \to SD(Y,Z)$ (respectively $\beta : X \times Y \to Z$)

Note 2.9 : If $\mathcal{A}$ is the family of all spaces the notions of strongly $\delta_{\mathcal{A}}$-splitting and strongly $\delta_{\mathcal{A}}$-conjoining coincides with the notions of strongly δ-splitting and strongly δ-conjoining,

Theorem 2.10. *A topology τ on $SD(Y,Z)$ is strongly δ-splitting iff $C^*_\delta \subseteq (C^*(\tau))_\delta$.*

Proof : Let τ be a strongly δ-splitting topology on $SD(Y,Z)$ and let $(\{f_\lambda : \lambda \in \Lambda\}, f) \in C^*_\delta$. We prove that $\{f_\lambda : \lambda \in \Lambda\}$ converges to f in the τ topology. Λ is a directed set and let us add a point ∞ to Λ such that $\infty \not\in \Lambda$; to ascertain the natural order relations between ∞ and members of Λ, let us take $\infty \geq \lambda \ \forall \lambda \in \Lambda$. We then topologize $X = \Lambda \cup \{\infty\}$ defining any singleton $\{\lambda\}$,$\lambda \in \Lambda$ to be open and nbds. of ∞ the sets $\{\lambda \in X : \lambda \geq \lambda_0 \text{ for some } \lambda_0 \in \Lambda\}$. Let $\alpha : X \times Y \to Z$ be a map for which $\alpha(\lambda, y) = f_\lambda(y), \lambda \neq \infty$ and $\alpha(\infty, y) = f(y)$ for every $y \in Y$. The map α is strongly δ-continuous. Obviously $\tilde{\alpha}(\lambda) = f_\lambda$ and $\tilde{\alpha}(\infty) = f$. Since the topology τ is strongly δ-splitting , the map $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous.

By strong δ-continuity of $\tilde{\alpha}$, we have that for every open nbd. U of f in $SD(Y,Z)$, there exists a δ-open nbd. V of ∞ in X such that $\tilde{\alpha}(V) \subseteq U$.

By definition of the topology of X $\exists$ an element $\lambda_0 \in \Lambda$ such that $\lambda \in V \ \forall \lambda \in \Lambda$ with $\lambda \geq \lambda_0$. Hence $f_\lambda \in U \ \forall \lambda \in \Lambda$ with $\lambda \geq \lambda_0$ i.e., the net $\{f_\lambda : \lambda \in \Lambda\}$ converges to f in the τ topology. Thus $C^*_\delta \subseteq (C^*(\tau))_\delta$.

Conversely let τ be a topology on $SD(Y,Z)$ such that $C^*_\delta \subseteq (C^*(\tau))_\delta$. We have to prove that τ is strongly δ-splitting. Let X be an aritrary space and let $\alpha : X \times Y \to Z$ be a strongly δ-continuous map. Consider the map $\tilde{\alpha} : X \to SD(Y,Z)$ ($SD(Y,Z)$ is endowed with the τ topology). We have to prove that $\tilde{\alpha}$ is strongly δ-continuous. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X which δ-converges to x. We prove that the net $\tilde{\alpha}(x_\lambda)$ converges to $\tilde{\alpha}(x)$.

Let $\{y_\mu : \mu \in M\}$ be a net in Y which δ-converges to y in Y. Since the map α is strongly δ-continuous and the net $\{(x_\lambda, y_\mu) : (\lambda,\mu) \in \Lambda \times M\}$ of $X \times Y$ δ-converges to (x,y) in $X \times Y$, we have $\alpha(x_\lambda, y_\mu) \longrightarrow \alpha(x,y)$. This means that $\alpha_{x_\lambda}(y_\mu) \longrightarrow \alpha_x(y)$. Thus the net $\{\tilde{\alpha}(x_\lambda) : \lambda \in \Lambda\}$ converges to $\tilde{\alpha}(x)$. Thus the map $\tilde{\alpha}$ is strongly δ-continuous and hence τ is strongly δ-splitting.

Theorem 2.11. *A topology τ on $SD(Y,Z)$ is strongly δ-conjoining if and only if $(C^*(\tau))_\delta \subseteq C^*_\delta$*

Proof : Let τ be a strongly δ-conjoining topology . Let X be the space as in the theorem of 2.10. Let $(\{f_\lambda : \lambda \in \Lambda\}, f) \in (C^*(\tau))_\delta$. Clearly the map $\alpha : X \to SD(Y,Z)$ is strongly δ-continuous where $\alpha(\lambda) = f_\lambda$ and $\alpha(\infty) = f$. Then the map $\tilde{\alpha} : X \times Y \to Z$ is strongly δ-continuous. We have to prove that $(\{f_\lambda : \lambda \in \Lambda\}, f) \in C^*_\delta$. Then it is sufficient to prove that if $\{y_\mu : \mu \in M\}$ is a net in Y which δ-converge to y in Y, then the net $\{f_\lambda(y_\mu) : (\lambda,\mu) \in \Lambda \times M\}$ converges to $f(y)$. But the net $\{\lambda : \lambda \in \Lambda\}$ in X δ-converges to ∞ in X. Hence the net $\{(\lambda, y_\mu) : (\lambda,\mu) \in \Lambda \times M\}$ in $X \times Y$ δ-converges to (∞, y) in $X \times Y$. Since the map $\tilde{\alpha}$ is strongly δ-continuous , the net $\{\tilde{\alpha}(\lambda, y_\mu) \equiv \alpha(\lambda)(y_\mu) \equiv f_\lambda(y_\mu), (\lambda,\mu) \in \Lambda \times M\}$ converges to $\tilde{\alpha}(\infty, y) \equiv f(y)$.

Conversely , let τ be a topology on $SD(Y,Z)$ such that $(C^*(\tau))_\delta \subseteq C^*_\delta$.

We prove that the topology τ is strongly δ-conjoining . Let X be an arbitrary space and let $\alpha : X \to SD(Y,Z)$ ($SD(Y,Z)$ be endowed with the τ topology) be a strongly δ-continuous map . We prove that the map $\tilde{\alpha} : X \times Y \to Z$ is strongly δ-continuous . Let $\{(x_\lambda, y_\mu) : (\lambda,\mu) \in \Lambda \times M\}$ be a net in $X \times Y$ which δ-converge to (x,y). We prove that the net $\{\tilde{\alpha}(x_\lambda, y_\mu) : (\lambda,\mu) \in \Lambda \times M\}$ in Z converges to $\tilde{\alpha}(x,y)$.

Since the net $\{x_\lambda : \lambda \in \Lambda\}$ δ-converges to x in X and the map α is strongly δ-continuous , the net $\{\alpha(x_\lambda) : \lambda \in \Lambda\}$ converges to $\alpha(x)$. Thus by assumption the net $\{\alpha(x_\lambda) : \lambda \in \Lambda\}$ strongly δ-continuously converges to $\alpha(x)$. Now since the net $\{y_\mu : \mu \in M\}$ δ-converges to y , the net $\{\alpha(x_\lambda)(y_\mu) \equiv \tilde{\alpha}(x_\lambda, y_\mu) : (\lambda,\mu) \in \Lambda \times M\}$ converges to $\alpha(x)(y) = \tilde{\alpha}(x,y)$. Hence the topology τ is strongly δ-conjoining.

Theorem 2.12. *A topology τ on $SD(Y,Z)$ is simultaneously strongly δ-splitting and strongly δ-conjoining iff $C^*_\delta = (C^*(\tau))_\delta$.* The proof of this theorem follows from theorems 2.10 & 2.11.

Theorem 2.13. *A topology τ on $SD(Y,Z)$ is strongly δ-splitting iff is strongly $\delta_{\mathcal{A}}$-splitting , where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.*

Proof : It is enough to prove that if τ is strongly $\delta_{\mathcal{A}}$-splitting , where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point, then the topology τ is strongly δ-splitting.

Let $(\{f_\lambda : \lambda \in \Lambda\}, f) \in C^*_\delta$. We have to prove that $(\{f_\lambda : \lambda \in \Lambda\}$ converges to f in the τ topology.

Let $X = \Lambda \cup \{\infty\}$, where ∞ is a symbol such that $\infty \geq \lambda$ for every $\lambda \in \Lambda$. Then we topologize $X = \Lambda \cup \{\infty\}$ by defining any singleton $\{\lambda\}, \lambda \in \Lambda$ to be open and nbds. of ∞ the sets $\{\lambda \in X : \lambda \geq \lambda_0$ for some $\lambda_0 \in \Lambda\}$. Clearly the element ∞ is the unique non-isolated point of the space X and thus $X \in \mathcal{A}$.

We consider the map $\alpha : X \times Y \to Z$ by setting $\alpha(\lambda, y) = f_\lambda(y)$ & $\alpha(\infty, y) = f(y)$. Obviously the map α is strongly δ-continuous . Now we prove that $\{f_\lambda : \lambda \in \Lambda\}$ converges to f in the τ topology.

Let $U \in \tau$ be an open nbd. of f. Now the topology τ is strongly $\delta_{\mathcal{A}}$-splitting . Hence the map $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous . Also $\tilde{\alpha}(\infty) = f$ & $\tilde{\alpha}(\lambda) = f_\lambda, \lambda \neq \infty$. Thus $\exists$ a δ-open nbd. V of ∞ such that $\tilde{\alpha}(V) \subseteq U$.

Since the set V is an δ-open nbd. of ∞ in X $\exists$ an element $\lambda_0 \in \Lambda$ such that $\lambda \in V, \forall \lambda \geq \lambda_0$.Hence $\tilde{\alpha}(\lambda) = f_\lambda \in U \ \forall \ \lambda \in \Lambda$ with $\lambda \geq \lambda_0$. Thus the net $\tilde{\alpha}(\lambda) = \{f_\lambda : \lambda \in \Lambda\}$ converges to f in the τ toplogy and hence τ is strongly δ-splitting.

Theorem 2.14. *A topology τ on $SD(Y,Z)$ is strongly δ-conjoining iff is strongly $\delta_{\mathcal{A}}$-conjoining , where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.*

The proof is similar to theorem 2.13.

§3. Strong δ^* notions on function space

Definition 3.1 : A topology τ on $SD(Y,Z)$ is called strongly δ^*-splitting iff for every space X,the strong δ-continuity of a map $\alpha : X \times Y \to Z$ implies the δ-continuity of the map $\tilde{\alpha} : X \to SD(Y,Z)$.

Definition 3.2 : A topology τ on $SD(Y,Z)$ is called strongly δ^*-conjoining iff for every space X,the δ-continuity of a map $\tilde{\alpha} : X \to SD(Y,Z)$ implies the strong δ-continuity of the map $\alpha : X \times Y \to Z$.

Theorem 3.3. *The following propositions are true :*

(1) Let τ be a strongly δ-splitting topology on $SD(Y,Z)$. Then the topology τ is strongly δ^*-splitting.

(2) Let τ be a strongly δ-conjoining topology on $SD(Y,Z)$. Then the topology τ is strongly δ^*-conjoining.

The proof of the theorem is clear.

Theorem 3.4. *A toplogy τ on $SD(Y,Z)$ is strongly δ^*-conjoining iff the evaluation map $P : SD(Y,Z) \rightarrow Z$ defined by $P(f,y) = f(y)$ is strongly δ-continuous.*

Proof : Clearly , the identity map $\tilde{\alpha} \equiv 1 : SD(Y,Z) \rightarrow SD(Y,Z)$, where $SD(Y,Z)$ is endowed with τ topology is δ-continuous , since τ is strongly δ^*-conjoining the map $\alpha \equiv P : SD(Y,Z) \times Z$ is strongly δ-continuous.

Conversely , let X be aa space ,$\tilde{\alpha} : X \rightarrow SD(Y,Z)$ be a δ-continuous map and $1 : Y \rightarrow Y$ be the identity map . Cleaarly the map $\tilde{\alpha} \times 1 : X \times Y \rightarrow SD(Y,Z) \times Y$ is also δ-continuous in the product space. Also it is given that the evaluation map $P : SD(Y,Z) \times Y \rightarrow Z$ is strongly δ-continuous . Then the composition map $P \circ (\tilde{\alpha} \times 1) : X \times Y \rightarrow Z$ is strongly δ-continuous and $\alpha = P \circ (\tilde{\alpha} \times 1)$. Thus the topology τ is strongly δ^*-conjoining.

§4. Examples of strongly δ-splitting topology

Example 4.1: The trivial topology on $SD(Y,Z)$ is clearly strongly δ-splitting.

Example 4.2 : The pointwise topology τ_p on $SD(Y,Z)$ is strongly δ-splitting.

Indeed, let X be any arbitrary space and let $\alpha : X \times Y \rightarrow Z$ be a strongly δ-continuous map. We have to show that $\tilde{\alpha} : X \rightarrow SD(Y,Z)$ is strongly δ-continuous. Let $x \in X$ and let $\tilde{\alpha}(x) \in T(\{y\},U)$, where $y \in Y$ and U be an open set of Z. Then we have $\tilde{\alpha}(x)(y) = \alpha(x,y) \in U$. Since α is strongly δ-continuous so $\exists$ δ-open nbds. W_1 & W_2 of x & y respectively such that $\alpha(W_1 \times W_2) \subseteq U$. Which implies that $\tilde{\alpha}(W_1) \in T(\{y\},U)$ and thus the map $\tilde{\alpha}$ is strongly δ-continuous.

Lemma 4.3 : Let $\alpha : X \times Y \rightarrow Z$ be a strongly δ-continuous map, O be an open set of Z, K be a compact subset of Y and $x \in X$ be such that $\{x\} \times K \subseteq \alpha^{-1}(O)$. Then $\exists$ a δ-open nbd. V_x of x such that $V_x \times K \subseteq \alpha^{-1}(O)$.

Proof : Let $y \in K$. Then $(x,y) \in \alpha^{-1}(O)$ which implies $\alpha(x,y) \in O$. Since α is strongly δ-continuous so $\exists$ a δ-open nbd. V_x^y of x and an open nbd. V_y of y such that $V_x^y \times Int.cl.(V_y) \subseteq \alpha^{-1}(O)$. Also we have $K \subseteq \cup\{V_y : y \in K\}$. Since K is compact so $\exists$ open sets $V_{y_1}, ..., V_{y_k}$ such that $K \subseteq V_{y_1} \cup .... \cup V_{y_k}$.

Let $V_x = V_x^{y_1} \cap V_x^{y_2} \cap .... \cap V_x^{y_k}$ & $V'_y = V_{y_1} \cup ...., \cup V_{y_k}$. Then V_x is a δ-open nbd. of x (since intersection of finite number of δ-open set is δ-open). We prove that $V_x \times K \subseteq \alpha^{-1}(O)$.

let $(x_1,y_1) \in V_x \times K \subseteq V_x \times Int.cl.(V'_y)$. Then $x_1 \in V_x^{y_i}$ for all $i = 1,2,...,k$ and $y_1 \in V_{y_j}$ for some $j = 1,2,...,k$. Thus $(x_1,y_1) \in V_x^{y_p} \times V_{y_p}$ for

$1 \leq p \leq k$. Which is a subset of $\alpha^{-1}(O)$. Hence $V_x \times K \subseteq \alpha^{-1}(O)$.

Example 4.4 : The compact open topology τ_c on $SD(Y,Z)$ is strongly δ-splitting.

Let X be an arbitrary topological space and let $\alpha : X \times Y \to Z$ be a strongly δ-continuous map. We have to show that $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous. Let $x \in X$ and let $\tilde{\alpha}(x) \in T(K,U)$ where K is a compact subset of Y and U be an open set in Z. We prove that $\exists$ δ-open set W containing x in X such that $\tilde{\alpha}(W) \subseteq T(K,U)$. We have $\{x\} \times K \subseteq \alpha^{-1}(U)$. By the above lemma 4.3 $\exists$ δ-open nbd. W of x such that $W \times K \subseteq \alpha^{-1}(U)$. Thus $\alpha(W \times K) \subseteq U$ and hence $\tilde{\alpha}(W) \subseteq T(K,U)$.Hence $\tilde{\alpha}$ is strongly δ-continuous.

Example 4.5 : A topology τ on $SD(Y,Z)$ is generated by the sets of the form $\{P(C,U) : C$ is a N-closed subset of Y & U be an open set in $Z\}$ where $P(C,U) = \{f \in SD(Y,Z) : f(C) \subseteq U\}$. This topology τ on $SD(Y,Z)$ is strongly δ-splitting.To this end we first show that for a strongly δ-continuous map $\alpha : X \times Y \to Z$, if U be an open set in Z and C be a N-closed subset of Y and $x \in X$ be such that $\{x\} \times C \subseteq \alpha^{-1}(U)$, then $\exists$ a δ-open nbd. V_x of x such that $V_x \times C \subseteq \alpha^{-1}(U)$.

Indeed for every $y \in C$, we have $(x,y) \in \alpha^{-1}(U)$ and therefore $\alpha(x,y) \in U$. Since α is strongly δ-continuous $\exists$ δ-open nbds. V_x^y & V_y of x & y respectively such that $V_x^y \times V_y \subseteq \alpha^{-1}(U)$. Also we have $C \subseteq \cup\{V_y : y \in C\}$. Since C is N-closed , $\exists$ δ-open sets $V_{y_1}, ..., V_{y_n}$ such that $C \subseteq V_{y_1} \cup, ..., \cup V_{y_n}$.

Let $V_x = V_x^{y_1} \cap .... \cap V_x^{y_n}$ & $V_y' = V_{y_1} \cup .... \cup V_{y_n}$. We prove that $V_x \times C \subseteq \alpha^{-1}(U)$. Let $(x_1, y_1) \in V_x \times C \subseteq V_x \times V_y'$. Then $x_1 \in V_x^{y_i}$ for all $i = 1, ..., n$. & $y_1 \in V_{y_j}$ for some $j = 1, ..., n$. Thus $(x_1, y_1) \in V_x^{y_p} \times V_{y_p}$ for some p , $1 \leq p \leq n$. Which is a subset of $\alpha^{-1}(U)$. Thus $V_x \times C \subseteq \alpha^{-1}(U)$.

Now we prove that τ is strongly δ-splitting. Let X be any arbitrary space and let $\alpha : X \times Y \to Z$ be a strongly δ-continuous map . We have to show that the map $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous . Let $x \in X$ and let $\tilde{\alpha}(x) \in P(C,U)$, where C is N-closed set in Y and U an open set in Z. We have $\{x\} \times C \subseteq \alpha^{-1}(U)$. Then what we have just proved above ,$\exists$ a δ-open nbd. W of x such that $W \times C \subseteq \alpha^{-1}(U)$. Thus $\alpha(W \times C) \subseteq U$ and so $\tilde{\alpha}(W) \subseteq P(C,U)$. Thus the map $\tilde{\alpha}$ is strongly δ-continuous.

Remarks 4.6 : All examples that we have discussed above remain valid for the case of strongly δ^*-notions.

§5. Splittingness & conjoiningness of N-R-Topology

One natural question may come up, is there exists any topology on $SD(Y,Z)$ which is strongly δ-splitting as well as strongly δ-conjoining.

Theorem 5.1. *The N-R topology on $SD(Y,Z)$ is strongly δ-splitting.*

Proof : The N-R topology τ on $SD(Y,Z)$ is generated by the sets of the form $T(C,U) = \{f \in SD(Y,Z) : f(C) \subseteq U\}$, where C is a N-closed subset in Y and U be regular open in Z. Since regular open sets are δ-open so in the subbasic open set of the N-R topology we can take U to be a δ-open subset of Z.

For this we first prove that for a strongly δ-continuous map $\alpha : X \times Y \to Z$, if U be an δ-open set in Z and C be a N-closed subset of Y and $x \in X$ be such that $\{x\} \times C \subseteq \alpha^{-1}(U)$, then $\exists$ a δ-open nbd. V_x of x such that $V_x \times C \subseteq \alpha^{-1}(U)$.

Now for every $y \in C$, we have $(x,y) \in \alpha^{-1}(U)$ and hence $\alpha(x,y) \in U$. Since α is strongly δ-continuous $\exists$ δ-open nbds. V_x^y & V_y of x & y respectively such that $V_x^y \times V_y \subseteq \alpha^{-1}(U)$, since δ-open sets are open sets. Also we have $C \subseteq \cup\{V_y : y \in C\}$. Since C is N-closed , $\exists$ δ-open sets $V_{y_1}, ..., V_{y_n}$ such that $C \subseteq V_{y_1} \cup, ..., \cup V_{y_n}$.

Let $V_x = V_x^{y_1} \cap .... \cap V_x^{y_n}$ & $V_y' = V_{y_1} \cup .... \cup V_{y_n}$. We prove that $V_x \times C \subseteq \alpha^{-1}(U)$. Let $(x_1, y_1) \in V_x \times C \subseteq V_x \times V_y'$. Then $x_1 \in V_x^{y_i}$ for all $i = 1, ..., n$. & $y_1 \in V_{y_j}$ for some $j = 1, ..., n$. Thus $(x_1, y_1) \in V_x^{y_p} \times V_{y_p}$ for some p , $1 \leq p \leq n$, which is a subset of $\alpha^{-1}(U)$. Thus $V_x \times C \subseteq \alpha^{-1}(U)$.

Next we show that the N-R topology is strongly δ-splitting. Let X be any arbitrary space and let $\alpha : X \times Y \to Z$ be a strongly δ-continuous map . We have to show that the map $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous . Let $x \in X$ and let $\tilde{\alpha}(x) \in T(C,U)$, where C is N-closed set in Y and U a δ-open set in Z. We have $\{x\} \times C \subseteq \alpha^{-1}(U)$. Then by above ,$\exists$ a δ-open nbd. W of x such that $W \times C \subseteq \alpha^{-1}(U)$. Thus $\alpha(W \times C) \subseteq U$ and so $\tilde{\alpha}(W) \subseteq T(C,U)$. Thus the map $\tilde{\alpha}$ is strongly δ-continuous.

Theorem 5.2. *On the set $SD(Y,Z)$ there exists the greatest strongly δ-splitting topology.*

Proof : Let $\{T_\alpha\}$ be the set of all strongly δ-splitting topologies on the set $SD(Y,Z)$. Let τ be the topology having the members of $\cup_\alpha T_\alpha$ as subbasis . We prove that τ is the greatest strongly δ-splitting topology . Then it is enough to prove that τ is strongly δ-splitting topology. Let X be any arbitrary space and let $\alpha : X \times Y \to Z$ be a strongly δ-continuous map. We have to show that the map $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous. ($SD(Y,Z)$ is endowed with the τ topology). Since any subbasic open set $U \in \tau$ belongs to some strongly δ-splitting topology T_α, we must have $\tilde{\alpha}^{-1}(U)$ is δ-open in X and hence $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous.

Theorem 5.3. a) *A topology larger than a strongly δ-conjoining topology is also strongly δ-conjoining.*

b) *A topology smaller than a strongly δ-splitting topology is also strongly*

δ-splitting.

Proof : Let τ be a strongly δ-conjoining topology and $\tau \subset \sigma$. Since the identity map $1 : SD_\sigma(Y,Z) \to SD_\tau(Y,Z)$ is strongly δ-continuous and the strongly δ-conjoining property of τ gives that the map $P : SD_\tau(Y,Z) \times Y \to Z$ is strongly δ-continuous. So we get $SD_\sigma(Y,Z) \times Y \to Z$ is also strongly δ-continuous. Thus σ is strongly δ-conjoining.The proof of (b) is similar.

Theorem 5.4. *Any strongly δ-conjoining topology is larger than any strongly δ-splitting topology.*

Proof : Let τ be a strongly δ-conjoining and σ be a strongly δ-splitting topology on $SD(Y,Z)$. Then for any arbitrary space X , the strong δ-cotinuity of the map $\tilde{\alpha} : X \to SD_\tau(Y,Z)$ implies $\alpha : X \times Y \to Z$ is strongly δ-continuous (as τ is strongly δ-conjoining) which implies $\tilde{\alpha} : X \to SD_\sigma(Y,Z)$ is strongly δ-continuous (as σ is strongly δ-splitting). Thus we find that $1 : SD_\tau(Y,Z) \to SD_\sigma(Y,Z)$ is strongly δ-continuous. Which shows that $\sigma \subset \tau$.

Theorem 5.5. *On the set $SD(Y,Z)$, the N-R topology is the smallest strongly δ-conjoining topology if Y is locally nearly compact T_2 & Z is semiregular.*

Proof : First we show that the N-R topology is strongly δ-conjoining. Any sub-basic open set of the N-R topology on $SD(Y,Z)$ is

$$T(C,U) = \{f \in SD(Y,Z) \ : \ f(C) \subseteq U\}$$

where C is a N-closed set in Y & U regular open in Z.

Let X be an arbitrary topological space and it is given that the map $\tilde{\alpha} : X \to SD(Y,Z)$ is strongly δ-continuous . We have to show that $\alpha : X \times Y \to Z$ is strongly δ-continuous.

Let V be an sub-basic open set in Z. Let $y \in Y$ and P' be a regular open nbd. of y in Y. Since Y is locally nearly compact T_2 so $\exists$ an open set M containing y such that $\bar{M} \subset P'$ with $\bar{M}$ N-closed. Then $T(\bar{M},V)$ is a sub-basic open set in N-R topology on $SD(Y,Z)$. Since $\tilde{\alpha}$ is strongly δ-continuous so there exists a regular open set W in X such that $\tilde{\alpha}(W) \subset T(\bar{M},V)$. Then for any $x \in W$, $\tilde{\alpha}(x) \in T(\bar{M},V) \Rightarrow \tilde{\alpha}(x)(y) \in V($ as $y \in \bar{M}) \Rightarrow \alpha(x,y) \in V$. Thus $\alpha(W \times \bar{M}) \subset V$. So for any sub-basic open set V of Z , $\exists$ a regular open nbd. $W \times \bar{M}$ of (x,y) in the product space $X \times Y$ such that $\alpha(W \times \bar{M}) \subset V$. Hence α is strongly δ-continuous.

Now we show that it is the smallest among all the strongly δ-conjoining topology that can be given on $SD(Y,Z)$.
Let σ be a topology on $SD(Y,Z)$ which is strongly δ-conjoining. We show that $T(C,U)$ is σ-open in order to show that the N-R topology is the smallest one . Now in view of theorem 2.6 the map $P : SD(Y,Z) \times Y \to Z$ defined by $P(f,y) = f(y)$ is strongly δ-continuous. Then the set $V' =$

$(SD(Y,Z) \times Y) \cap P^{-1}(U)$ is δ-open in $SD(Y,Z) \times Y$. If $f \in T(C,U)$ then $f(C) \subset U$ i.e., $\{f\} \times C \subset P^{-1}(U)$ i.e., $\{f\} \times C \subset V'$. Now $\{f\}$ is N-closed in $SD(Y,Z)$ & C is so in Y so by lemma 2.7, $\exists$ δ-open sets N of f in σ-topology such that $N \times C \subset P^{-1}(U)$. So for each $f \in N$, $f(C) \subset U \Rightarrow N \subset T(C,U)$ and so $f \in N \subset T(C,U)$. Thus $T(C,U)$ is σ-open. As a partial converse of Theorem 5.5 we can now state and prove the following theorem.

Theorem 5.6. *Let X be a non-regular T_2 topological space in which for every δ-open set U and a point $p \in U$, $\exists$ a strongly δ-continuous function $f : X \to [0,1]$ such that $f(p) = \{1\} \,\&\, f(X \setminus U) = \{0\}$; if $SD(X,[0,1])$ be endowed with N-R topology $\Im$ then X must be locally nearly compact if $P : SD(X,[0,1]) \times X \to [0,1]$ is strongly δ-continuous.*

Proof : Let $F : X \to [0,1]$ be defined by $F(x) = 0 \; \forall x \in X$. Then obviously $F \in SD(X,[0,1])$. Let W_0 be a nbd. of 0 in $[0,1]$ which does not contain 1. By the strong δ-continuity of F , $\exists$ a $\Im$ nbd. U of F and a nbd. V of x in X such that $y \in Int.(cl.V)$ & $g \in Int.(cl.U)$ imply $g(y) \in W_0 \cdots (1)$. We show that $\bar{V}$ is N-closed.

Suppose $\mathcal{U}$ is a δ-open covering of $\bar{V}$; since $\bar{V}$ is the closure of an open set it is regularly closed and hence δ-closed; thus $X \setminus \bar{V}$ is δ-open and $\mathcal{U} \cup \{X \setminus \bar{V}\}$ is a δ-open cover of X.

Since U is a $\Im$-nbd. of F ,$\exists$ $A_1, A_2, ..., A_n$ N-closed in X & $U_1, U_2, ..., U_n$ regular open in $[0,1]$ such that $F \in T(A_1,U_1) \cap .... \cap T(A_n,U_n) \subset U$.

Let $G = Int.\bar{V} \setminus (A_1 \cup .... \cup A_n)$. Obviously $A_1 \cup .... \cup A_n$ is N-closed in X and hence δ-closed and let if possible $p \in G$ then $\exists$ a strongly δ-continuous function $r : X \to [0,1]$ such that $r(p) = \{1\}$ & $r(X \setminus G) = \{0\}$. Now $r \in SD(X,[0,1])$; also $A_1 \cup .... \cup A_n \subset X \setminus G$ and thus $r(A_1) = \cdots = r(A_n) = \{0\}$. Since $F(A_1) = \cdots = F(A_n) = \{0\}$, $0 \in U$; for $i = 1, ..., n$ and as such $r \in T(A_1,U_1) \cap .... \cap T(A_n,U_n) \subset U$. But $r(p) = \{1\}$ & $1 \not\in W_0$, where as $r \in U \subset Int.(cl.U)$ & $p \in Int.(cl.V)$ should imply $r(p) \in W_0$ (from (1)).

Thus we arrive at a contradiction ; this contradiction shows that $G = \emptyset$. For $i = 1, ..., n$, now $Int.\bar{V} \subseteq A_1 \cup ... \cup A_n$. But $A_1 \cup ... \cup A_n$ is a closed set and thus $\bar{V} \subseteq A_1 \cup ... \cup A_n$. Now $\mathcal{U} \cup \{X \setminus \bar{V}\}$ is a δ-open cover of $A_1 \cup ... \cup A_n$; since each A_i is N-closed , $A_i \subseteq W_{i_1} \cup .... \cup W i_{m_i}$ where each $W_{i_{m_i}}$ is chosen from $\mathcal{U} \cup \{X \setminus \bar{V}\}$.

Thus $\bar{V}$ has a finite subcovering from $\mathcal{U}$ (in fact $X \setminus \bar{V}$ adjoined to $\mathcal{U}$ need not occur among the members of the finite subcovering). Thus $\bar{V}$ is N-closed.

Note 5.7 : X with the properties stated in the theorem does exist ; infact $[0,1]$ with the countable complement extension topology [6] satisfies this condition.

Conclusion 5.8 : Now we are in a position to give answear to our question that we bring at begining of this article. In viewing the above results we can conclude that the N-R topology on $SD(Y,Z)$ is the smallest strongly δ-conjoining and the largest strongly δ-splitting topology provided Y a locally nearly compact T_2 space and Z a semiregular space.

§6. δ-upper limit of a net

Definition 6.1 : With $P(X)$– the power set of a topological space X and $\mathcal{A}' = \{A_\lambda : \lambda \in \Lambda\} \subset P(X)$, where Λ is a directed set , we define the δ-upper limit for $\mathcal{A}'$ as the set of all points $x \in X$ such that for every $\lambda_0 \in \Lambda$ and every δ-open nbd. U of x in X $\exists$ an element $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0 \,\&\, A_\lambda \cap U \neq \emptyset$. We denote the δ-upper limit for $\mathcal{A}'$ by $\delta - \overline{\lim_{\Lambda}}(A_\lambda)$.

Theorem 6.2[4]. *A net $\{f_\lambda : \lambda \in \Lambda\}$ on $D(X,Y)$ δ–continuously converges to $f \in D(X,Y)$ iff $\delta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subseteq f^{-1}(K)$, for every δ-closed subset K of Y.*

Definition 6.3 : Let $\mathcal{O}(Y)$ be the family of all δ-open sets of the space Y and let $\mathcal{A} \subseteq \mathcal{O}(Y)$.

We define $C^*_\delta(\mathcal{A})$ on the set $D(X,Y)$ as follows : a pair $(\{f_\lambda : \lambda \in \Lambda\}, f) \in C^*_\delta(\mathcal{A})$, where $\{f_\lambda : \lambda \in \Lambda\}$ is a net in $D(X,Y)$ & $f \in D(X,Y)$ if

$$f^{-1}(U) \subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(U))$$

or equivalently

$$\delta - \overline{\lim_{\Lambda}} f_\lambda^{-1}(K)) \subseteq f^{-1}(K)$$

where $K = Y \setminus U$, for every $U \in \mathcal{A}$. Obviously if $\mathcal{A} = \mathcal{O}(Y)$, then $C^*_\delta(\mathcal{A}) = C_\delta$

Lemma 6.4 : Let $\mathcal{K}(Y)$ be the family of all δ-closed subset of the space Y and let $\{f_\lambda : \lambda \in \Lambda\}$ be a net in $D(X,Y)$. Then the following are true.

(1)$\delta - \overline{\lim_{\Lambda}}(\cup\{f_\lambda^{-1}(K_i) : i = 1,...,n\}) = \cup\{\delta - \overline{\lim_{\Lambda}} f_\lambda^{-1}(K_i) : i = 1,...,n\}$, where $K_i \in \mathcal{K}(Y)$ for all $i = 1,...,n$.

(2)$\delta - \overline{\lim_{\Lambda}}(\cap\{f_\lambda^{-1}(K_i) : i \in I\}) \subseteq \cap\{\delta - \overline{\lim_{\Lambda}} f_\lambda^{-1}(K_i) : i \in I\}$, where $K_i \in \mathcal{K}(Y)$ for every $i \in I$.

Proof. (1) It is easy to see that

$$\cup\{\delta - \overline{\lim_{\Lambda}} f_\lambda^{-1}(K_i) : i = 1,...,n\} \subseteq \delta - \overline{\lim_{\Lambda}}(\cup\{f_\lambda^{-1}(K_i) : i = 1,...,n\})$$

We prove the reverse inclusion .

Let $x \in \delta - \overline{\lim_{\Lambda}}(\cup\{f_\lambda^{-1}(K_i) : i = 1,...,n\})$. Then for every $\lambda_0 \in \Lambda$ and for every δ-open nbd. U_x of x $\exists\, \lambda \in \Lambda, \lambda \geq \lambda_0$ such that

$$U_x \cap (\cup\{f_\lambda^{-1}(K_i) : i = 1,...,n\}) \neq \emptyset \;\Rightarrow\; \cup\{U_x \cap f_\lambda^{-1}(K_i) : i = 1,...,n\} \neq \emptyset$$

Let $x \not\in \cup\{\delta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K_i)) \ : \ i = 1, ..., n\}$. Then $x \not\in \delta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K_i))$ for every $i = 1, ..., n$. This means that for every $i = 1, ..., n$ $\exists \lambda_0^i \in \Lambda$ and a δ-open nbd. U_x^i of x such that $U_x^i \cap f_\lambda^{-1}(K_i) = \emptyset$ for every $\lambda \in \Lambda$ with $\lambda \geq \lambda_0^i$. Let $\lambda_0 \in \Lambda$ be such that $\lambda_0 \geq \lambda_0^i$ for every $i = 1, ..., n$, and let $U_x = \cap_{i=1}^n U_x^i$. Then for every $\lambda \in \Lambda, \lambda \geq \lambda_0$, we have

$$\cup\{U_x \cap f_\lambda^{-1}(K_i) \ , \ i = 1, ..., n\} = \emptyset.$$

which is a contradiction and thus $x \in \{\delta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K_i)) \ : \ i = 1, ..., n\}$.

(2) The proof is immediate.

Theorem 6.5. *The following propositions are true :*

(1) $C_\delta \subseteq C_\delta^*(\mathcal{A})$

(2) Let $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{O}(Y)$. Then $C_\delta^*(\mathcal{A}') \subseteq C_\delta^*(\mathcal{A})$.

(3) Let $\mathcal{A}_i \subset \mathcal{O}(Y)\,, i \in I$. Then $\cap\{C_\delta^*(\mathcal{A}_i) \ : \ i \in I\} = C_\delta^*(\cup\{\mathcal{A}_i \ : \ i \in I\})$.

(4) Let $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{O}(Y)$. Let every element of $\mathcal{A}'$ is the intersection of finitely many elements of $\mathcal{A}$. Then $C_\delta^*(\mathcal{A}) \subseteq C_\delta^*(\mathcal{A}')$

Proof. The proof of (1),(2)& (3) are clear from the definition .To prove (4), let $(\{f_\lambda \ , \ \lambda \in \Lambda\}, f) \in C_\delta^*(\mathcal{A})$ and let $U \in \mathcal{A}'$. We have to prove that $\delta - \overline{\lim_{\Lambda}} f_\lambda^{-1}(K)) \subseteq f^{-1}(K)$, where $K = Y \setminus U$

$$\text{i.e., } f^{-1}(U) \subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(U))$$

Now every element of $\mathcal{A}'$ is the intersection of finitely many elements of $\mathcal{A}$, so $\exists U_1, ..., U_n \in \mathcal{A}$ such that $U = \cap\{U_i \ : \ i = 1, ..., n\}$ and $f^{-1}(U_i) \subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(U_i))$, for every $i = 1, ..., n$. Hence we have $f^{-1}(U) = f^{-1}(\cap\{U_i \ : \ i = 1, ..., n\}) = \cap\{f^{-1}(U_i) \ : \ i = 1, ..., n\} \subseteq \cap\{X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(U_i)) \ : \ i = 1, ..., n\} = X \setminus \cup\{\delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(U_i)) \ : \ i = 1, ..., n\} = X \setminus \delta - \overline{\lim_{\Lambda}}(\cup\{X \setminus f_\lambda^{-1}(U_i) \ : \ i = 1, ..., n\})$ (by lamma 6.4) $= X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus \cap\{f_\lambda^{-1}(U_i) \ : \ i = 1, ..., n\}) = X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(\cap\{U_i \ : \ i = 1, ..., n\})) = X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(U))$

Theorem 6.6. *Let $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{O}(Y)$ and let every elements of $\mathcal{A}'$ is the union of elements of $\mathcal{A}$. Then $C_\delta^*(\mathcal{A}) \subseteq C_\delta^*(\mathcal{A}')$.*

Proof. Let $(\{f_\lambda \ , \ \lambda \in \Lambda\}, f) \in C_\delta^*(\mathcal{A})$ and let $V \in \mathcal{A}'$. We have to prove that

$$f^{-1}(V) \subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(V))$$

Now from the given condition $\exists$ $V_i \in \mathcal{A}, i \in I$ such that $V = \cup\{V_i \ : \ i = 1, ..., n\}$ and $f^{-1}(V_i) \subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(V_i))$ for every $i \in I$. Hence we have

$$\begin{aligned}
f^{-1}(V) &= f^{-1}(\cup\{V_i \ : \ i \in I\}) \\
&= \cup\{f^{-1}(V_i) \ : \ i \in I\} && (1)\\
&\subseteq \cup\{X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(V_i)) \ : \ i \in I\} && (2)\\
&= X \setminus \cap\{\delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(V_i)) \ : \ i \in I\} && (3)\\
&\subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(\cap\{X \setminus f_{\lambda}^{-1}(V_i) \ : \ i \in I\}) \text{ (by lamma 6.4)} && (4)\\
&= X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus \cup\{f_{\lambda}^{-1}(V_i) \ : \ i \in I\}) && (5)\\
&= X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(\cup\{V_i \ : \ i \in I\})) && (6)\\
&= X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(V)) && (7)
\end{aligned}$$

Theorem 6.7 : *Let $\mathcal{A} \subseteq \mathcal{O}(Y)$ and let $\mathcal{A}'$ be the family of all δ-open sets for which every element is the union of elements $\mathcal{A}_i$; $i \in I$ such that every $\mathcal{A}_i$, $i \in I$ is the intersection of finitely many elements of $\mathcal{A}$. Then $C^*_\delta(\mathcal{A}) \subseteq C^*_\delta(\mathcal{A}')$*

Proof : Let $(\{f_\lambda \ , \ \lambda \in \Lambda\}, f) \in C^*_\delta(\mathcal{A})$ and let $V \in \mathcal{A}'$. We have to prove that
$f^{-1}(V) \subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(V))$.

By assumption $\exists\, V_1^i, ..., V_{m(i)}^i \ \in \mathcal{A}$, $i \in I$ such that $\mathcal{A}_i = \cap\{V_k^i : k = 1, ..., m(i)\}$, $V = \cup\{\cap\{V_k^i : k = 1, ..., m(i)\} : i \in I\}$, $f^{-1}(V_k^i) \subseteq X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(V_k^i))$ for every $i \in I$ & $k = 1, ..., m(i)$. Hence we have

$$\begin{aligned}
f^{-1}(V) &= f^{-1}(\cup\{\cap\{V_k^i \ : \ k = 1, ..., m(i)\} : i \in I\}) \\
&= \cup\{f^{-1}(\cap\{V_k^i \ : \ k = 1, ..., m(i)\}) : i \in I\} \\
&= \cup\{\cap\{f^{-1}(V_k^i) \ : \ k = 1, ..., m(i)\} : i \in I\} \\
&\subseteq \cup\{\cap\{X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(V_k^i)) \ : \ k = 1, ..., m(i)\} : i \in I\} \\
&= \cup\{X \setminus \cup\{\delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(V_k^i)) \ : \ k = 1, ..., m(i)\} : i \in I\} \\
&= \cup\{X \setminus \delta - \overline{\lim_{\Lambda}}(\cup\{X \setminus f_{\lambda}^{-1}(V_k^i) \ : \ k = 1, ..., m(i)\}) : i \in I\} \\
&= \cup\{X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus \cap\{f_{\lambda}^{-1}(V_k^i) \ : \ k = 1, ..., m(i)\}) : i \in I\} \\
&= \cup\{X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}\{\cap(V_k^i) \ : \ k = 1, ..., m(i)\}) : i \in I\} \\
&= X \setminus \cap\{\delta - \overline{\lim_{\Lambda}}(X \setminus f_{\lambda}^{-1}(\cap\{V_k^i \ : \ k = 1, ..., m(i)\})) : i \in I\} \\
&\subseteq X \setminus \delta - \overline{\lim_{\Lambda}}\{\cap(X \setminus \{f_{\lambda}^{-1}(\cap\{V_k^i \ : \ k = 1, ..., m(i)\})) : i \in I\}
\end{aligned}$$

$$
\begin{aligned}
&= X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus \cup\{f_\lambda^{-1}(\cap\{V_k^i \ : \ k = 1, ..., m(i)\})\} : i \in I) \\
&= X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}\{\cup(\cap\{V_k^i \ : \ k = 1, ..., m(i)\}) : i \in I\}) \\
&= X \setminus \delta - \overline{\lim_{\Lambda}}(X \setminus f_\lambda^{-1}(V)).
\end{aligned}
$$

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Received: 11. 05. 2002

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