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# GEOMETRIC PROPERTIES OF EXTENDED HOLOMORPHIC FUNCTIONS

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**Abstract.** In this article I have determined some geometrical properties of the real surface $(S) \quad \bar{r} = \bar{i}y + \bar{j}u(x,y) + \bar{k}v(x,y)$ attached to a monogenous quaternion.

The notion of monogenous quaternion was introduced in [1] and represents a prolongement of the ollomorphic functions in the four-dimension space. The difference between these ollomorphic functions and the monogenous functions analyzed in [4] is the fact that the functions introduced in [1] can be particularized as ollomorphic functions from the usual complex analysis. The analyzed geometrical properties: The Gauss Curvature, the curves' torsion on coordinates, the area of the ollomorphic surface element. The notion of monogenous quaternion given under relation (1) as well as its properties constitute the author's original contribution

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We take a quaternion type function of two real variables
$\mathcal{K}(x,y) = \mathcal{P}(x,y) + iQ(x,y) + jR(x,y) + \bar{k}\,S(x,y) \doteq D \subset \mathbb{R}^2 \to \mathcal{C}$

$\mathcal{C}$ = the quaternion's set ($(x,y)$ = pair of real variables), which we choose wishing to extend the complex numbers and the Cauchy- Riemann monogenity conditions of form [1]

$$\mathcal{K}(x,y) = x + iy + ju(x,y) + kv(x,y) \tag{1}$$

where

$$\begin{aligned} i^2 &= -j^2 = k^2 = +1 \\ ij &= k; \quad ki = j; \; j \cdot k = i \\ ji &= -k; \; ik = -j; \; kj = -i \end{aligned}$$

We showed in [1] that this function verifies the following conditions:

$$\begin{aligned} \frac{\partial \mathcal{K}}{\partial x} &= (1, 0, u_x, v_x) \\ \frac{\partial \mathcal{K}}{\partial y} &= (0, 1, u_y, v_y) \end{aligned} \tag{2}$$

If $\mathcal{K}(x,y) \in C^2(\mathbb{R}^2)$ and we use the Cauchy-Riemann's conditions for functions $w(x,y) = (u(x,y), v(x,y))$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

with $\triangle u = 0$ and $\triangle v = 0$ we obtain for the costernion $\mathcal{K}(x,y)$ given by relation (1) the following monogenity conditions

$$\begin{cases} \frac{\partial \mathcal{K}}{\partial x} \cdot \frac{\partial \mathcal{K}}{\partial y} = i\left[1 + (u_x)^2 + (u_y)^2\right] = i\left[1 + (v_x)^2 + (v_y)^2\right] \\ \frac{1}{2}\left(\frac{\partial \mathcal{K}}{\partial x} + i\frac{\partial \mathcal{K}}{\partial y}\right) = (1, 0, u_x - v_y, v_x + u_y) = (1,0,0,0) = scalar \\ \frac{1}{2}\left(\frac{\partial \mathcal{K}}{\partial x} - i\frac{\partial \mathcal{K}}{\partial y}\right) = (0, 0, \frac{1}{2}(u_x + v_y), \frac{1}{2}(v_x - u_y)) = \\ \qquad = (0, 0, u_x, v_x) = (0, 0, v_y, -u_y) \end{cases} \tag{3}$$

**Observations:** The first condition in (3) will be considered a monogenity condition for the cuaternion $\mathcal{K}(x,y)$. The conditions two and three in relation (3) can be regarded as extentions of the derivatives $\frac{\partial w}{\partial \overline{z}}$ respectively $\frac{\partial w}{\partial z}$ of the function $w(x,y)$ from the complex analysis.

We take the surface

$$(S) \quad \overline{r} = \overline{i}y + \overline{j}u(x,y) + \overline{k}v(x,y) \tag{4}$$

which is a vector (quaternion reduced to vector)

$$\mathcal{K} = x + \overline{r} \; (scalar + vectorial \;\; function) \tag{4*}$$

Let's further analyse some geometric properties of this surface (S) from relation (4)

$$\begin{matrix} \overline{r}_x = (0, u_x^{'}, v_x^{'}) \\ \overline{r}_y = (1, u_y^{'}, v_y^{'}) \end{matrix} \quad \overline{r}_x \times \overline{r}_y = \overline{i}\frac{D(u,v)}{D(x,y)} + \overline{j} \cdot v_x^{'} - \overline{k} \cdot u_x^{'} \tag{5}$$

In Cauchy-Riemann's monogenity conditions then

$$\overline{r}_x \times \overline{r}_y = \overline{i}\frac{D(u,v)}{D(x,y)} - \overline{j} \cdot u_y - \overline{k}v_y \tag{6}$$

We calculate $EG - F^2$ for surface $(S)$ given by (4)

$$\begin{matrix} E = (u_x^{'})^2 + (v_x^{'})^2; \\ F = 0; \\ G = 1 + u_x^{'2} + v_y^{'2} \\ EG - F^2 = (u_x^{'})^2 + (v_x^{'})^2 + (u_y^{'} v_x^{'} + u_x^{'} v_y^{'})^2 \end{matrix} \tag{7}$$

If $\mathcal{K}$ is a monogenous function, then

$$EG - F^2 = |\bar{r}_x \times \bar{r}_y|^2 \tag{8}$$

**Co-ordinate curves on (S)**

The tangents to the co-ordinate curves will be respectively

$$\begin{aligned} \bar{r}_{x_0} &= \bar{i}\, y + \bar{j}\, u(x_0, y) + \bar{k} v(x_0, y) \\ \bar{r}_{y_0} &= \bar{i}\, y_0 + \bar{j} u(x, y_0) + \bar{k} v(x, y_0) \end{aligned} \tag{9}$$

$$\begin{cases} \overrightarrow{T}_{x_0} = (0, u_y', v_y') \\ \overrightarrow{T}_{y_0} = (1, u_x', v_x') \end{cases} \tag{10}$$

We find that $\overrightarrow{T}_{x_0} \cdot \overrightarrow{T}_{y_0} = 0 + u_x' u_y' + v_x' v_y' = 0$ in the monogenity case, thus $\overrightarrow{T}_{x_0} \perp \overrightarrow{T}_{y_0}$; the co-ordinate curves are orthogonal for $F \equiv 0$, and the first fundamental form of the surface is $ds^2 = EG - F^2 = EG$.

The co-ordinate curves' network constitutes an isotherm network (isotherm system), and about metrics $ds^2 = \lambda(x, y)[(dx)^2 + (dy)^2]$ where $\lambda(x, y) = \dfrac{1}{(du)^2 + (dv)^2}$ we say that it has an isotherm form in the monogenity case [4].

**The area of a surface portion**

$$\sigma = \iint_D \sqrt{EG - F^2}\, dxdy = \iint_D \begin{vmatrix} u_x' & u_y' \\ v_x & v_y \end{vmatrix} dxdy = Area(S).$$

The differential equation of the geodesic lines of surface $(S)$ is $(\overline{N}, d\bar{r}, d^2\bar{r}) = 0$ where $\overline{N} = \bar{r}_x \times \bar{r}_y$ thus $\overline{N}(u_x' v_y' - u_y' v_x', v_x', -u_x')$.

If

$$\begin{cases} r_x = (1, 0, u_x', v_x') \\ r_y = (0, 1, u_y', v_y') \end{cases}$$

$$\begin{aligned} r''_{x^2} &= (0, 0, u''_{x^2}, v''_{x^2}) \\ r''_{y^2} &= (0, 0, u''_{y^2}, v''_{y^2}) \\ r''_{xy} &= (0, 0, u''_{xy}, v''_{xy}) = r''_{yx} \end{aligned}$$

**The Gauss Curvature**

In the case when $F = 0$ for surface $(S)$, the Gauss curvature can be calculated [4] with the following formula

$$K = -\frac{1}{2E}\left[\left(\frac{E_u}{E}\right)_u + \left(\frac{E_v}{E}\right)_v\right].$$

For the surface considered in (4) the Gauss curvature is (see [3]) for surfaces with a izoterm metrics:

$$ds^2 = E(du^2 + dv^2)$$

$$K = -\frac{1}{2E}\left[\left(\frac{E_x}{E}\right)_x + \left(\frac{E_y}{E}\right)_y\right].$$

Making the calculations:

$$E_x^2 + E_y^2 = 2\left[(u''_{x^2})^2 + (u''_{y^2})^2 + 2(y''_{xy})^2\right].$$

We obtain

$$(u''_{x^2})^2 - (u''_{y^2})^2 = (u''_{x^2} - u''_{y^2})(u''_{x^2} + u''_{y^2}) = 0$$

and

$$\left(\frac{E_x}{E}\right)_x + \left(\frac{E_y}{E}\right)_x = \frac{1}{E^2}\left\{E\left[E''_{x^2} + E''_{y^2}\right] - (E_x)^2 - (E_y)^2\right] =$$

$$\frac{1}{E^2}\left\{-2(u'_x)^2(u''_{x^2})^2 - 2(u'_y)^2(u''_{y^2})^2 + 2(u''_{x^2})^2 \cdot (u'_y)^2 + 2(u''_{y^2})^2 \cdot (u'_x)^2\right] =$$

$$= \frac{2}{E^2}\left[(u'_y)^2\left[(u''_{x^2})^2 - (u''_{y^2})^2\right] + (u'_x)^2\left[(u''_{y^2})^2 - (u''_{x^2})^2\right]\right] =$$

$$= 2\frac{\left[(u''_{x^2})^2 - (u''_{y^2})^2\right]}{E^2} \cdot \left[(u'_y)^2 - (u'_x)^2\right] = 0.$$

On this surface $(S)$ $\overline{r} = iy + ju(x,y) + kv(x,y)$ in conditions $[C - R]$ of monogenity the curvature Gauss $= 0$.(for surfaces with a izoterm metrics).

**The calculation of curves' torsion on surface**

The surface $(S)$

$$\overline{r} = iy + ju(x,y) + kv(x,y) \tag{4}$$

attached to the quaternion $K = x + \overline{r}$ allows the curves on surface

$$\begin{aligned} r_x &= (0, u'_x, v'_x) \\ r_y &= (1, u'_y, v'_y) \end{aligned}$$

and

$$\left(\frac{1}{T}\right)_{y_0} = \frac{1}{A^2+B^2+C^2}\begin{vmatrix} 0 & u'_x & v'_x \\ 0 & u''_{x^2} & v''_{x^2} \\ 0 & u'''_{x^3} & v'''_{x^3} \end{vmatrix} = 0$$

The curve $y = y_0$ =constant is of null torsion, thus the curve is plane.

The curve $x = x_0$ =constant has the torsion

$$\left(\frac{1}{T}\right)_{x_0} = \frac{1}{A^2+B^2+C^2}\begin{vmatrix} 0 & u'_y & v'_y \\ 0 & u''_{y^2} & v''_{y^2} \\ 0 & u'''_{y^3} & v'''_{y^3} \end{vmatrix} = \frac{1}{|\overline{N}|^2}\left[u''_{y^2} \cdot v'''_{y_3} - u''_{y^3} \cdot v''_{y_2}\right] \neq 0.$$

usually

**The curvature of co-ordinate curves**

$$\frac{1}{R} = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}.$$

Let there be $\boxed{y = y_0}$

$$\begin{matrix} r_x = (0, u'_x, v'_x) \\ r_y = (1, u'_y, v'_y) \end{matrix} ; r'_x \times r''_{x^2} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & u'_x & v'_x \\ 0 & u''_{x^2} & v''_{x^2} \end{vmatrix} = \overline{i}(u'_x v''_{x^2} - v'_x u''_{x^2}).$$

$$K_{y_0} = \frac{|u'_x v''_{x^2} - v'_x u''_{x^2}|}{[(u'_x)^2 + (v'_x)^2]^{3/2}}$$

The curve $\boxed{x = x_0}$ has

$$\begin{matrix} \overline{r}'_y = (1, u'_y, v'_y) \\ \overline{r}''_{y^2} = (0, u''_{y^2}, v''_{y^2}) \end{matrix} ; \overline{r}'_y \times \overline{r}''_{y^2} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & u'_y & v'_y \\ 0 & u''_{y^2} & v''_{y^2} \end{vmatrix} = \overline{i}(u'_y v''_{y^2} - v'_y u''_{y^2}) - \overline{j} \cdot v''_{y^2} + \overline{k} u''_{y^2}.$$

$$K_{x_0} = \frac{\sqrt{(u'_y v''_{y^2} - v'_y u''_{y^2}) + (v''_{y^2})^2 + (u''_{y^2})^2}}{\left[\sqrt{1 + (u'_y)^2 + (v'_y)^2}\right]^3} = \frac{\sqrt{\left[\left(\frac{v_y}{u_y}\right)'_y\right]^2 + (v''_{y^2})^2 + (u''_{y^2})^2}}{G^{3/2}}$$

We find that the curvatures of co-ordinate curves are not constant, although the Gauss curvature defined [3] as product of two curvatures of co-ordinate curves is variable $(K_{max})(K_{min})$.

Generally, curves $x = x_0$ and $y = y_0$ are not straight ($K_{x_0} \neq 0$, $K_{y_0} \neq 0$). In the monogenity case

$$(K_{max})(K_{min}) = 0 \equiv \textit{The Gauss Curvature.}$$

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