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A note on partially ordered topological spaces and a special type of lower semicontinuous function

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ABSTRACT. θ-closed partial order in a topological space has been studied in details. $\theta^\star$-lower semicontinuity of a function to a hyperspace has been introduced and such functions are compared to the multifunctions. Lastly the $\theta^\star$-lower semicontinuity of some special types of functions is studied.

1. INTRODUCTION

In [1] Ganguly and Bandyopadhyay introduced the concept of θ-closed partial order in a topological space. In the first section of the paper we have tried to examine this special type of order in details. In the next section the concept of $\theta^\star$-lower semicontinuous function has been introduced from a topological space X to the hyperspace of a topological space Y along with Vietoris topology and its usual order relation; such functions have been compared to their analogues in the collection of multifunctions. In the last section we use θ-closed partial order of a topological space X to consider the $\theta^\star$-lower semicontinuity of some special type of functions on X.

2. PARTIALLY ORDERED TOPOLOGICAL SPACE

Definition 2.1. [2] Let X be a topological space and '$\leq$' be a partial order in it. For each subset $A \subseteq X$ let,

$$\uparrow A = \{x \in X : a \leq x, \text{ for some } a \in A\} \quad \text{and}$$
$$\downarrow A = \{x \in X : x \leq a, \text{ for some } a \in A\}.$$

The sets $\uparrow A$ and $\downarrow A$ are called the increasing hull of A and decreasing hull of A respectively.

It is easy to verify that, for any $A, B \subseteq X$,
(i) $A \subseteq \uparrow A, A \subseteq \downarrow A$;
(ii) $A \subseteq B \Rightarrow \uparrow A \subseteq \uparrow B$ and $\downarrow A \subseteq \downarrow B$;
(iii) $\uparrow (A \cup B) = \uparrow A \cup \uparrow B, \downarrow (A \cup B) = \downarrow A \cup \downarrow B$;
(iv) $\uparrow (A \cap B) \subseteq \uparrow A \cap \uparrow B, \downarrow (A \cap B) \subseteq \downarrow A \cap \downarrow B$.

Definition 2.2. [1] A partial order '$\leq$' on a topological space X is a θ-closed order if its graph $\{(x, y) \in X \times X : x \leq y\}$ is a θ-closed subset of $X \times X$.

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Definition 2.3. A partial order '$\leq$' on a topological space X is an almost regular order iff for every regularly closed set $A \subseteq X$ and $x \in X$ with $a \not\leq x, \forall a \in A$, $\exists$ neighbourhoods (nbds. in short)V and W of A and x respectively in X such that $\uparrow V \cap \downarrow W = \Phi$.

Theorem 2.4. *The partial order '$\leq$' on a topological space X is a θ-closed order iff for every $x, y \in X$ with $x \not\leq y$, there exists nbds. U, V of x, y respectively in X such that $\uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi$.*

Proof. Let the partial order '$\leq$' on X be θ-closed and $x, y \in X$ with $x \not\leq y$. Then (x, y) does not belong to the graph G (say) of '$\leq$'. Since G is θ-closed, $\exists$ nbds. U of x and V of y in X such that $\overline{U \times V} \cap G = \Phi$ i.e. $\overline{U} \times \overline{V} \cap G = \Phi$,which means that if $u \in \overline{U}$ and $v \in \overline{V}$ then $u \not\leq v$. We claim that $\uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi$. If not, $\exists\ z \in \uparrow (\overline{U}) \bigcap \downarrow (\overline{V})$. So, $\exists\ a \in \overline{U}, b \in \overline{V}$ such that $a \leq z$ and $z \leq b$. Then by transitivity of '$\leq$', $a \leq b$ which implies $(a, b) \in G$ – a contradiction.
Conversely, let the condition holds. Let $(x, y) \in X \times X \setminus G$. Then $x \not\leq y$. So by hypothesis, $\exists$ nbds. U of x and V of y in X such that $\uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi$. We claim that $\overline{U \times V} \cap G = \Phi$. If not, $\exists\ (a, b) \in \overline{U \times V} \cap G \Rightarrow a \in \overline{U}, b \in \overline{V}$ and $a \leq b$. Thus $b \in \uparrow (\overline{U})$. Also $b \in \downarrow (\overline{V})$ [since $\overline{V} \subseteq \downarrow (\overline{V})$] – contradicts that $\uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi$. This proves that (x, y) is not a θ-contact point [6] of G.
Consequently, G is θ-closed. □

Corollary 2.5. *Let '$\leq$' be a θ-closed order in a topological space X. Then $\uparrow (a)$ and $\downarrow (a)$ are θ-closed for each $a \in X$.*

Proof. Let $a \in X$ and $b \in X \setminus \uparrow (a)$. Then $a \not\leq b$. Since '$\leq$' is a θ-closed order, $\exists$ nbds. U, V of a, b respectively in X such that $\uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi$, [by theorem 2.4]. Now $\overline{V} \bigcap \uparrow (a) \subseteq \uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi$. Consequently, b cannot be a θ-contact point of $\uparrow (a)$. So $\uparrow (a)$ is θ-closed.
Similarly $\downarrow (a)$ is θ-closed. □

Corollary 2.6. *Every topological space X, equipped with a θ-closed order '$\leq$' is a Urysohn space.*

Proof. Let $a, b \in X$ with $a \neq b$. Then either $a \not\leq b$ or $b \not\leq a$. Let us assume that $a \not\leq b$.
Since '$\leq$' is a θ-closed order, $\exists$ nbds. U, V of a, b respectively in X such that $\uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi$, [by theorem 2.4]. Now, $\overline{U} \cap \overline{V} \subseteq \uparrow (\overline{U}) \bigcap \downarrow (\overline{V}) = \Phi \Rightarrow X$ is a Urysohn space. □

Corollary 2.7. *Let X be a topological space equipped with a θ-closed order '$\leq$'. Let $H \subseteq X$ be an H-set* [6] *in X. Then both $\uparrow H$ and $\downarrow H$ are θ-closed.*

Proof. Let $a \in X \setminus \uparrow H$. Then $h \not\leq a, \forall h \in H$. Since '$\leq$' is θ-closed, for each $h \in H$, $\exists$ open nbds. U_h, V_h of h and a respectively in X such that $\uparrow (\overline{U_h}) \bigcap \downarrow (\overline{V_h}) = \Phi$. [by theorem 2.4]. Now, $\{U_h : h \in H\}$ is an open cover of H. Since H is an H-set in X, $\exists$ a finite subset $H_0 \subseteq H$ such that $\bigcup_{h \in H_0} \overline{U_h} \supseteq H$. Let $V = \bigcap_{h \in H_0} V_h$. Then V is an open nbd. of a in X. Now $\overline{V} \bigcap \uparrow H \subseteq \downarrow \overline{V} \bigcap \uparrow (\bigcup_{h \in H_0} \overline{U_h}) \subseteq (\bigcap_{h \in H_0} \downarrow \overline{V_h}) \bigcap (\bigcup_{h \in H_0} \uparrow \overline{U_h}) = \Phi$. [since $\uparrow (\overline{U_h}) \bigcap \downarrow (\overline{V_h}) = \Phi, \forall h \in H_0$] Thus, a is not a

θ-contact point of $\uparrow H$. Consequently $\uparrow H$ is θ-closed.
Similarly, $\downarrow H$ is θ-closed. $\square$

Corollary 2.8. *If '$\leq$' is a θ-closed order on a topological space X and X is H-closed, then '$\leq$' is an almost regular order.*

Proof. Let A be a regular closed set and $x \in X$ be such that $y \not\leq x, \forall y \in A$. Then for each $y \in A$, $\exists$ open nbds. U_y and V_y of y and x respectively in X such that, $\uparrow (\overline{U_y}) \bigcap \downarrow (\overline{V_y}) = \Phi$. [by theorem 2.4]. A being a regular closed set in an H-closed space X, it is an H-closed subspace [7] and hence an H-set. Now $\{U_y : y \in A\}$ is an open cover of A and A is an H-set. So $\exists$ a finite subset $A_0 \subseteq A$ such that $\bigcup_{y\in A_0} \overline{U_y} \supseteq A$. Let $V = \bigcap_{y\in A_0} V_y$. Then V is an open nbd. of x in X. Now, $\downarrow (\overline{V}) \cap A \subseteq (\bigcap_{y\in A_0} \downarrow \overline{V_y}) \bigcap (\bigcup_{y\in A_0} \uparrow \overline{U_y}) = \Phi$. [since $\uparrow (\overline{U_y}) \bigcap \downarrow (\overline{V_y}) = \Phi, \forall y \in A$] $\Rightarrow A \subseteq X \setminus \downarrow (\overline{V})$. Again $\downarrow \overline{V}$ is θ-closed [by corollary 2.7] since, $\overline{V}$ is an H-set [7]. So $X \setminus \downarrow \overline{V}$ is an open nbd. of A. We claim that, $\uparrow (X \setminus \downarrow \overline{V}) \bigcap \downarrow \overline{V} = \Phi$. If not, $\exists\ z \in \downarrow \overline{V} \bigcap \uparrow (X \setminus \downarrow \overline{V})$. So $\exists\ w \in X \setminus \downarrow \overline{V}$ such that $w \leq z \Rightarrow w \in \downarrow \overline{V}$ – a contradiction. Therefore $\uparrow (X \setminus \downarrow \overline{V}) \bigcap \downarrow V = \Phi$. This completes the proof. $\square$

3. Functions into hyperspaces

In this article we shall discuss about a hyperspace [2] and the functions into a hyperspace.

Let X be a topological space and 2^X be the collection of all nonempty closed subsets of X. There have been various endeavors to topologize 2^X. The most commonly used topology is the Vietoris topology [3]. This topology is constructed as follows:

For each subset $S \subseteq X$ we denote, $S^+ = \{A \in 2^X : A \subseteq S\}$ and $S^- = \{A \in 2^X : A \cap S \neq \Phi\}$. The Vietoris topology on 2^X is one generated by the subbase $\{W^+ : W \text{ is open in } X\} \bigcup \{W^- : W \text{ is open in } X\}$. Now, the usual inclusion relation '$\subseteq$' induces a partial order on 2^X.

Since $V_1^+ \cap V_2^+ \cap \cdots \cap V_n^+ = (V_1 \cap V_2 \cap \cdots \cap V_n)^+$, a basic open set of the Vietoris topology is of the form, $V_1^- \cap \cdots \cap V_n^- \cap V_0^+$, where V_i is open in X for $i = 0, 1, \ldots, n$. The space 2^X with the Vietoris topology is usually known as a 'hyperspace'.

Proposition 3.1. $\uparrow (V_1^- \cap \cdots \cap V_n^-) = V_1^- \cap \cdots \cap V_n^-$.

Proof. $A \in \uparrow (V_1^- \cap \cdots \cap V_n^-) \Rightarrow \exists\ B \in V_1^- \cap \cdots \cap V_n^-$ such that $B \subseteq A$.
$\Rightarrow A \cap V_i \neq \Phi, \forall i = 1, \ldots, n$ [since $B \cap V_i \neq \Phi, \forall i = 1, \ldots, n$] $\Rightarrow A \in V_1^- \cap \cdots \cap V_n^-$. Thus, $\uparrow (V_1^- \cap \cdots \cap V_n^-) \subseteq V_1^- \cap \cdots \cap V_n^-$. The reverse inclusion follows from definition 2.1. $\square$

Proposition 3.2. $\downarrow (V_1^- \cap \cdots \cap V_n^-) = 2^X$

Proof. Let $A \in 2^X$. Since $A \subseteq X$ and $X \in (V_1^- \cap \cdots \cap V_n^-)$ so it follows that $A \in \downarrow (V_1^- \cap \cdots \cap V_n^-)$. Thus $2^X \subseteq \downarrow (V_1^- \cap \cdots \cap V_n^-)$. Reverse inclusion is obvious. $\square$

Proposition 3.3. *If X be a T_1-space and $V_i \subseteq V_0$, for $i = 1, \ldots, n$ then* $\uparrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+) = V_1^- \cap \cdots \cap V_n^-$

Proof. Let, $A \in V_1^- \cap \cdots \cap V_n^-$. Let $x_i \in A \cap V_i, i = 1, \ldots, n$ [since $A \cap V_i \neq \Phi, i = 1, \ldots, n$]. Now $\{x_1, \ldots, x_n\} \subseteq A \cap V_0$ [since $V_i \subseteq V_0, i = 1, \ldots, n$] and $\{x_1, \ldots, x_n\}$ is closed in X, since X is T_1. Therefore, $\{x_1, \ldots, x_n\} \in V_1^- \cap \cdots \cap V_n^- \cap V_0^+$.Consequently $A \in \uparrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+)$.Thus $V_1^- \cap \cdots \cap V_n^- \subseteq \uparrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+)$.
Conversely let $A \in \uparrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+)$. Then $\exists\ B \in V_1^- \cap \cdots \cap V_n^- \cap V_0^+$ such that $B \subseteq A$. Therefore $B \cap V_i \neq \Phi, i = 1, \ldots, n$. So $A \cap V_i \neq \Phi, i = 1, \ldots, n$. Consequently $A \in V_1^- \cap \cdots \cap V_n^-$. Therefore $\uparrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq V_1^- \cap \cdots \cap V_n^-$. □

Proposition 3.4. *If X be a T_1-space and $V_i \subseteq V_0, i = 1, \ldots, n$ then $\downarrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+) = V_0^+$.*

Proof. Let $A \in V_0^+$. We choose $x_i \in V_i, i = 1, \ldots, n$. Then $B = \{x_1, \ldots, x_n\} \subseteq V_0$ [since $V_i \subseteq V_0, i = 1, \ldots, n$] and B is closed in X [since X is T_1]. Therefore $A \cup B$ is a closed subset of X and $A \subseteq A \cup B$ and $A \cup B \in V_1^- \cap \cdots \cap V_n^- \cap V_0^+$. Consequently $A \in \downarrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+)$. Therefore $V_0^+ \subseteq \downarrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+)$.
Conversely, let $A \in \downarrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+)$. So $\exists\ B \in V_1^- \cap \cdots \cap V_n^- \cap V_0^+$ such that $A \subseteq B$. Since $B \subseteq V_0$ so $A \subseteq V_0$. Consequently, $A \in V_0^+$. Therefore $\downarrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq V_0^+$. □

Definition 3.5. A topological space X equipped with a θ-closed partial order '$\leq$' is said to be a θ-partially ordered space(θ-PO space in short) if $\downarrow V$ is θ-open for every θ-open set V of X.

Theorem 3.6. *If X is a T_3-space then the space 2^X equipped with the Vietoris topology and the usual set-inclusion as the partial order, is a θ-PO space.*

Proof. First we shall show that '$\subseteq$' is a θ-closed order in 2^X.
Let $K, L \in 2^X$ be such that $K \nsubseteq L$. Then $\exists\ p \in K$ such that $p \notin L$. Since L is closed in X and X is regular, $\exists$ two disjoint open sets U, V in X such that $p \in U$ and $L \subseteq V$. Now $U \cap V = \Phi \Rightarrow U \cap \overline{V} = \Phi$. Since X is regular, $\exists$ an open nbd. W of p in X such that $p \in W \subseteq \overline{W} \subseteq U$. Therefore $\overline{W} \cap \overline{V} = \Phi$. Now $K \cap W \neq \Phi$ [since $p \in K \cap W] \Rightarrow k \in W^-$. And $L \subseteq V \Rightarrow L \in V^+$. Now $\uparrow (\overline{W^-}) \bigcap \downarrow (\overline{V^+}) = (\uparrow (\bar{W})^-) \bigcap (\downarrow (\bar{V})^+) = (\bar{W})^- \cap (\bar{V})^+ = \Phi$ [since $\overline{W} \cap \overline{V} = \Phi$]. Then by theorem 2.4, '$\subseteq$' is a θ-closed order in 2^X.
Now let, G be any θ-open set in 2^X and $F_0 \in \downarrow G$. Then $\exists\ K_0 \in G$ such that $F_0 \subseteq K_0$. Since G is θ-open in 2^X, $\exists$ open sets $V_0, V_1, \ldots, V_n$ in X such that $K_0 \in (V_1^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq \overline{(V_1^- \cap \cdots \cap V_n^- \cap V_0^+)} \subseteq G$ and $V_i \subseteq V_0$, for, $i = 1, 2, \ldots, n$.
$\Rightarrow K_0 \in \downarrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq \downarrow (\bar{V}_1^- \cap \cdots \cap \bar{V}_n^- \cap \bar{V}_0^+) \subseteq \downarrow G$.
$\Rightarrow K_0 \in V_0^+ \subseteq \bar{V}_0^+ \subseteq \downarrow G$
$\Rightarrow F_0 \in V_0^+ \subseteq \overline{V_0^+} \subseteq \downarrow G$ [since $F_0 \subseteq K_0$ and $\bar{V}_0^+ = \overline{V_0^+}$]. This shows that $\downarrow G$ is θ-open in 2^X. This completes the proof. □

Definition 3.7. A function $f : X \to Y$, Y being equipped with a partial order '$\leq$', is called $\theta^\star$-lower semicontinuous with respect to '$\leq$' at $x \in X$ iff for every open nbd. V of $f(x)$ in Y, $\exists$ an open nbd. U of x in X such that $f(\overline{U}) \subseteq \uparrow V$.

f is $\theta^\star$-lower semicontinuous with respect to '$\leq$' iff it is $\theta^\star$-lower semicontinuous at each point of X.

Theorem 3.8. *Let Y be a T_1-space and 2^Y have the Vietoris topology. Then a function $\Phi : X \to 2^Y$ is $\theta^\star$-lower semicontinuous with respect to '$\subseteq$' iff $\Phi^{-1}(V^-)$ is θ-open in X whenever V is an open subset of Y.*

Proof. Let Φ be $\theta^\star$-lower semicontinuous with respect to '$\subseteq$' and V be any open subset of Y.
Let $a \in \Phi^{-1}(V^-)$. Then $\Phi(a) \in V^-$. Since Φ is $\theta^\star$-lower semicontinuous so $\exists$ an open nbd. U of a in X such that $\Phi(\overline{U}) \subseteq \uparrow (V^-) = V^-$ [by proposition 3.1]
$\Rightarrow a \in U \subseteq \overline{U} \subseteq \Phi^{-1}(V^-)$. This shows that $\Phi^{-1}(V^-)$ is $\theta-$open.
Conversely, let the condition holds. Let $a \in X$ and G be any open nbd. of $\Phi(a)$ in 2^Y. Then $\exists$ open sets $V_0, V_1, \ldots, V_n$ in Y such that $\Phi(a) \in V_1^- \cap \cdots \cap V_n^- \cap V_0^+ \subseteq G$.
We define, $U = \Phi^{-1}(V_1^-) \cap \cdots \cap \Phi^{-1}(V_n^-)$.
By hypothesis U is θ-open [since finite intersection of θ-open sets is again θ-open] and $a \in U$. So $\exists$ an open nbd. W of a in X such that $a \in W \subseteq \overline{W} \subseteq U \Rightarrow \Phi(a) \in \Phi(W) \subseteq \Phi(\overline{W}) \subseteq \Phi(U) \subseteq V_1^- \cap \cdots \cap V_n^- =\uparrow (V_1^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq\uparrow G$ [by proposition 3.3]. This shows that, Φ is $\theta^\star$-lower semicontinuous. □

4. MULTIFUNCTIONS

In the previous article, we have studied about functions into a hyperspace. These functions are nothing but set-valued functions or multifunctions. In this article we shall treat them as the ordinary multifunction and compare the two different aspects.

Mukherjee ,Raychaudhuri and Sinha introduced lower-$\theta^\star$-continuous multifunctions in [4]; in the same way the concept of lower-$\theta^\star$-semicontinuous multifunction can also be introduced.

Definition 4.1. A multifunction $F : X \to Y$, where X, Y are topological spaces, is called lower-$\theta^\star$-semicontinuous function iff for each $x_0 \in X$ and each open set V in Y with $F(x_0) \cap V \neq \Phi$, there is an open nbd. U of x_0 such that $F(x) \cap V \neq \Phi$ for each $x \in \overline{U}$.

Definition 4.2. [4] A multifunction $F : X \to Y$ is called $\theta^\star$-closed if whenever $x \in X, y \in Y$ and $y \notin F(x)$, there exists open nbds. U, V of x, y in X and Y respectively such that $p \in \overline{U} \Rightarrow F(p) \cap V \neq \Phi$.

Theorem 4.3. [4] *If $F : X \to Y$ be a multifunction which is $\theta^\star$-closed, then $F(x)$ is closed in Y, for each $x \in X$.*

Theorem 4.4. *Let $F : X \to Y$ be a multifunction, where X, Y are topological spaces and Y is a T_1-space. If F be lower-$\theta^\star$-semicontinuous and $\theta^\star$-closed then*

$$\left.\begin{array}{rcl} f : X & \to & 2^Y \\ x & \mapsto & F(x) \end{array}\right\}$$

is $\theta^\star$-lower semicontinuous, when 2^Y is endowed with Vietoris topology.

Proof. The function f is well-defined by theorem 4.3.
Let V be any open set in Y and $a \in f^{-1}(V^-)$. Then $f(a) \in V^-$ i.e. $F(a) \cap V \neq$

Φ. Since F is lower-$\theta^\star$-semicontinuous, $\exists$ an open nbd. U of a in X such that $F(x) \cap V \neq \Phi, \forall\, x \in \overline{U}$
$\Rightarrow f(x) \in V^-, \forall\, x \in \overline{U}$
$\Rightarrow \overline{U} \subseteq f^{-1}(V^-)$.
Therefore $a \in U \subseteq \overline{U} \subseteq f^{-1}(V^-)$.
Thus $f^{-1}(V^-)$ is θ-open for each open set V in Y.
Consequently, f is $\theta^\star$-lower semicontinuous [by theorem 3.8]. □

Theorem 4.5. *Let X be a topological space and Y be a T_1-space. Let $f : X \to 2^Y$ be a $\theta^\star$-lower semicontinuous function, where 2^Y is endowed with Vietoris topology. Then the multifunction,*

$$\left.\begin{array}{rcl} F : X & \to & Y \\ x & \mapsto & f(x) \end{array}\right\}$$

is lower-$\theta^\star$-semicontinuous.

Proof. Let $x_0 \in X$ and V be open in Y such that $F(x_0) \cap V \neq \Phi$ i.e. $f(x_0) \in V^-$ i.e. $x_0 \in f^{-1}(V^-)$. Since f is $\theta^\star$-lower semicontinuous function, $f^{-1}(V^-)$ is θ-open in X [by theorem 3.8]. So $\exists$ an open nbd. U of x_0 in X such that, $x_0 \in U \subseteq \overline{U} \subseteq f^{-1}(V^-)$
$\Rightarrow f(\overline{U}) \subseteq V^-$ i.e. $f(x) \in V^-, \forall x \in \overline{U}$ i.e. $F(x) \cap V \neq \Phi, \forall x \in \overline{U}$. Thus F is lower-$\theta^\star$-semicontinuous. □

5. Some special multifunctions

In this article, we discuss the $\theta^\star$-lower semicontinuity of a very special type of multifunction. Since the consideration of either a hyperspace or an ordinary space as the codomain of a multifunction is immaterial, as seen from the previous article, we discuss the $\theta^\star$-lower semicontinuity of the multifunction in the hyperspace-setting.

Definition 5.1. We define a pair of functions $i, d : X \to 2^X$, where X is a topological space equipped with a partial order '$\leq$' which is assumed to be a θ-closed order, as follows:-

$$i(x) = \uparrow (x) \text{ and } d(x) = \downarrow (x)$$

Since '$\leq$' is a θ-closed order, $\uparrow (x)$ & $\downarrow (x)$ are θ-closed [by corollary 2.5]. So the functions 'i' and 'd' are well-defined.

Theorem 5.2. (i) *The function $i : X \to 2^X$ is $\theta^\star$-lower semicontinuous with respect to '$\subseteq$' iff $\downarrow V$ is θ-open in X for every open set V of X.*

(ii) *The function $d : X \to 2^X$ is $\theta^\star$-lower semicontinuous with respect to '$\subseteq$' iff $\uparrow V$ is θ-open in X for every open set V of X.*

Proof. (i) Let V be any open set in X. Now, $i^{-1}(V^-) = \{x \in X : i(x) \in V^-\} = \{x \in X : \uparrow (x) \cap V \neq \Phi\} = \{x \in X : x \leq y, \text{ for some } y \in V\} = \downarrow V$
It now clearly follows from theorem 3.8 that, 'i' is $\theta^\star$-lower semicontinuous with respect to '$\subseteq$' iff $\downarrow V$ is θ-open in X.

(ii) The result follows from the following fact.
Let V be any open set in X. Now, $d^{-1}(V^-) = \{x \in X : d(x) \in V^-\} = \{x \in X : \downarrow (x) \cap V \neq \Phi\} = \{x \in X : y \leq x, \text{ for some } y \in V\} = \uparrow V$. □

Theorem 5.3. *If $F : X \to Y,\ Y$ being equipped with a θ-closed order '$\leq$' be a set-valued mapping such that $F(x)$ is an H-set in Y and if F is lower-$\theta^\star$- semicontinuous and $\downarrow V$ is open for each open V of Y, then*

$$\left.\begin{array}{rcl} f : X & \to & 2^Y \\ x & \mapsto & \uparrow F(x) \end{array}\right\}$$

is $\theta^\star$-lower semicontinuous.

Proof. Since $F(x)$ is an H-set in Y and '$\leq$' is a θ-closed order, $\uparrow F(x)$ is θ-closed [by corollary 2.7]. So f is well-defined.
Let V be open in Y. Now, $f^{-1}(V^-) = \{x \in X :\uparrow F(x) \in V^-\} = \{x \in X :\uparrow F(x) \cap V \neq \Phi\} = \{x \in X : F(x) \cap \downarrow V \neq \Phi\}$. Let, $x_0 \in f^{-1}(V^-)$. Then $F(X_0) \cap \downarrow V \neq \Phi$. Since $\downarrow V$ is open [by hypothesis] and F is lower-$\theta^\star$-semicontinuous $\exists$ an open nbd. U of x_0 in X such that $F(x) \cap \downarrow V \neq \Phi, \forall x \in \overline{U} \Rightarrow \overline{U} \subseteq f^{-1}(V^-)$ i.e. $x_0 \in U \subseteq \overline{U} \subseteq f^{-1}(V^-)$. Thus $f^{-1}(V^-)$ is θ-open in X. Consequently, f is $\theta^\star$-lower semicontinuous [by theorem 3.8].

We can get a similar result if we take $\downarrow F(x)$ instead of $\uparrow F(x)$ in the above theorem with only changing $\uparrow V$ instead of $\downarrow V$ in the hypothesis. □

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