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On convex feasibility problems

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ABSTRACT. In this paper we consider a projection method for convex feasibility problem that is known to converge only weakly. Exploiting a property concerning the intersection of a family of convex closed sets, we present a condition that makes them strongly convergent, without additional assumptions.

1. INTRODUCTION

The convex feasibility problem consists in finding a point in the intersection of a convex sets. Initially, this problem arose in the constrain optimization problems for trying to guess an initial "feasible" point, that is a point which satisfies the constrains. Often, these constrains are defined by linear inequalities and so the feasible set is the intersection of a number of halfspaces. Later, it was proved that the convex feasibility problem have great utility and board applicability in many areas, spreading on modern mathematical and physical science to economics and even medical practices, like: statistics (linear prediction theory), image reconstruction with applications in computerized tomography, radiation therapy treatment planning, electron microscopy, signal processing, and the like. A complete and exhaustive study on algorithms for solving convex feasibility problem, including comments about their applications and an excellent bibliography, was given by H.H. Bausche and J.M. Borwein [3].

The projection algorithms it seems to be the common way for solving this problem The idea is to use the projection of the current iterate onto certain set from the intersection family (the strategy of selecting this set leads to a particular algorithm) and so to yielding a sequence of points that is supposed to converge to a solution. This idea was used (it seems for the first time) in [1, 11] for solving a system of linear inequalities (the authors named their method as "relaxation algorithm"). Generalizations for convex sets in real n-dimensional spaces were given in [8, 10]. Bergman [5] considered the classical projection method for the case of m intersecting closed convex sets (M_i) in a real Hilbert space. He showed that, given an arbitrary starting point x_0, the sequence generated by the projection algorithm converges weakly to a point in $M = \bigcap_{i=1}^{m} M_i$. In [9] certain regularity conditions on the sets were described that guaranteed strong convergence of the iterations. In recent papers, other conditions for strong convergence have been given, for example in [3, 2, 6, 4].

In this paper we present a simple condition that insure the strong convergence of the sequence generated by the projection algorithm.

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2. PRELIMINARIES

Let $\mathcal{H}$ be a real Hilbert space with scalar product $\langle\cdot|\cdot\rangle$, norm $\|\cdot\|$, and distance d. Let $T : D \subset \mathcal{H} \to \mathcal{H}$ be a nonlinear mapping, and let $F(T)$ denotes the set of fixed points of T in D. In the following we will assume that $F(T) \neq \emptyset$. According to [12], the mapping T is said to be quasi-nonexpansive if $\|Tx - x^*\| \leq \|x - x^*\|$, $\forall\, x \in D, x^* \in F(T)$.

Remark 2.1. The notion of quasi-nonexpansivity has been introduced by Tricomi [13] for real-valued fuctions and subsequently studied in [13, 7] for mapping in Hilbert or Banach spaces.

Let $d(x, E)$ denotes the distance between a point $x \in \mathcal{H}$ and a set $E \subset \mathcal{H}$, that is $d(x, E) = \inf_{y\in E} \|x - y\|$.

We shall use the following general theorem concerning the convergence of the simple iterates for quasi-nonexpansive mappings.

Theorem 2.1. *Suppose that $T : D \subset \mathcal{H} \to \mathcal{H}$ is a quasi-nonexpansive mapping and that $F(T)$ is nonempty and closed. Let $x_0 \in D$ such that $x_k = T^k_{x_0} \in D$, $k = 1, 2, \cdots$. Then the sequence $\{x_k\}$ converges to a fixed point of T if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $d(x_{k_j}, F(T)) \to 0$ as $j \to \infty$.*

Here, as usual, T^k denotes the k iterate of T.

Remark 2.2. Theorem 2.1 is a slight generalization of the first result of [12] and its proof is similar. Essentially, Theorem 2.1 replaced the condition of continuity of T, from the original result, by the condition of closedness of $F(T)$. It is easy to see that the latter condition is weaker, and, as it will result, is essential for our development.

3. THE MAIN RESULT

We first prove the following lemma.

Lemma 3.1. *Let $M_i \subset \mathcal{H}(i = 1, \cdots, m)$ be a family of convex sets such that Int $\bigcap M_i$ is nonempty and bounded and let $\{x_k\}$ be a sequence of $\mathcal{H}$ such that $d(x_k, M_i) \to 0$ as $k \to \infty$ for each i. Then $d(x_k, \bigcap M_i) \to 0$, as $k \to \infty$.*

Proof. We assume that $o \in Int \bigcap M_i$. Then there exists a closed ball $D(o, r) = \{x \in \mathcal{H} : \|x\| \leq r\} \subset \bigcap M_i$. Let ϵ be a given real number, $0 < \epsilon < 1$, and let C be a constant such that $\|x\| \leq C - 1$ for all $x \in \bigcap M_i$, which is possible, because $\bigcap M_i$ is bounded.

Since $d(x_k, M_i) \to 0$ as $k \to \infty$, for each index i, there exists a sequence $\{y^{(i)}_k\}_{k\in N} \subset M_i$ such that $\|y^{(i)}_k - x_k\| \to 0$ as $k \to \infty$. Let

$$z_k = \left(1 - \frac{C}{\varepsilon}\right)(y^{(i)}_k - x_k), k = 0, 1, \cdots . \tag{3.1}$$

There exists a number $k_i(\varepsilon)$ such that if $k \geq k_i(\varepsilon)$ then $\|y^{(i)}_k\| \leq \dfrac{r}{\left|1 - \frac{C}{\varepsilon}\right|}$ and so $\|z_k\| \leq r$, that is $z_k \in \bigcap M_i$.

On the other hand, from equation 3.1 we obtain

$$\left(1-\frac{\varepsilon}{C}\right)x_k=\frac{\varepsilon}{C}z_k+\left(1-\frac{\varepsilon}{C}\right)y_k^{(i)},$$

and for $k \geq k_i(\varepsilon)$ we have $\left(1-\dfrac{\varepsilon}{C}\right)x_k \in M_i$, because $y_k^{(i)}$, $z_k \in M_i$ and M_i are convex.

Now, let $k_0(\varepsilon) = \max_i k_i(\varepsilon)$. Then, for $k \ \geq k_0(\varepsilon)$ it follows that $\left(1-\dfrac{\varepsilon}{C}\right)x_k \in \bigcap M_i$ and

$$d(x_k,\bigcap M_i) \leq \|x_k-\left(1-\frac{\varepsilon}{C}\right)x_k\| = \frac{\varepsilon}{C-\varepsilon}\Big\|\left(1-\frac{\varepsilon}{C}\right)x_k\Big\| < \varepsilon,$$

which end the proof. □

Apparently, the condition that $Int \bigcap M_i$ is nonempty and bounded is very strong. The following example shows that this condition cannot be replaced by the weaker condition $\bigcap M_i \neq \emptyset$, which seems to be more natural.

Example. Suppose that $\mathcal{H}$ is the real three-dimensional space, that the set M_i is a cone (A) and the set M_2 is a tangent plane $(ABCD)$. The situation is depicted in Figure 1.

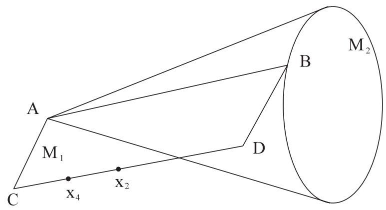

Fig.1. Example

The plane $(ABCD)$ is tangent to the cone along the generatrix (AB) and hence $M_1 \bigcap M_2 = (AB)$. Now, let us consider a sequence $\{x_k\}$ in the plane $(ABCD)$ such that $d(x_k,(AB)) = \delta = const.$ and $\|x_k\| \to \infty$ as $k \to \infty$. It is clear that $d(x_k, M_2) \to 0$ as $k \to \infty$ and $d(x_k, M_1 = 0$ for all k; but $d(x_k, M_1 \bigcap M_2) = \delta > 0$. Therefore, the conclusion of Lemma 3.1 is not true.

In the following, we shall suppose that $M_i(i = 1, \cdots, m)$ are closed and convex sets of $\mathcal{H}$. Let $P(x, i)$ be the projection of an $x \in \mathcal{H}$ onto M_i and let i_x be the smallest index such that $\|x - P(x, i_x)\| = \max_i \|x - P(x, i)\|$. We define the mapping $T : \mathcal{H} \to \mathcal{H}$ by $Tx = P(x, i_x)$. It is easy to see that $x \in \bigcap M_i$ if and only if $Tx = x$; hence if and only if x is a fixed point of T. In other words, $F(T) = \bigcap M_i$.

Let $\lambda \in (0, 2)$ and let $T_\lambda = I - \lambda(I - T)$, where I is the identity mapping of $\mathcal{H}$ into $\mathcal{H}$. Obviously, $F(T_\lambda) = F(T)$.

Theorem 3.2. *Let $M_i(i = 1,\ldots,m)$ be a family of closed and convex sets of $\mathcal{H}$ such that $Int \bigcap M_i$ is nonempty and bounded. Then the sequence x_k given by $X_k = T_\lambda^k x_0$ converges (strongly) to a point of $\bigcap M_i$ for all $x_0 \in \mathcal{H}$.*

Proof. Since $F(T_\lambda) = \bigcap M_i$ is a closed set, it suffices to show that T_λ is quasi-nonexpansive on $\mathcal{H}$ and that $d(x_k, \bigcap M_i) \to 0$ as $k \to 0$. Then Theorem 3.2 follows from Theorem 2.1.

Let $x \in \mathcal{H}$ and $y \in \bigcap M_I$. Since $P(x, i_x)$ is the projection of x onto M_{i_x} and $y \in M_{i_x}$, we have

$$\langle Tx - y, x - Tx\rangle = \langle P(x, i_x) - y, x - P(x, i_x)\rangle \geq 0,$$

and

$$(3.2)\qquad \begin{aligned} \|T_\lambda x - y\|^2 &= \|x - y\|^2 - 2\lambda\langle x - y, x - Tx\rangle + \lambda^2\|x - Tx\|^2 \\ &= \|x - y\|^2 - \lambda(2 - \lambda)\|x - Tx\|^2 - 2\lambda\langle Tx - y, x - Tx\rangle \\ &\leq \|x - y\|^2 - \lambda(2 - \lambda)\|x - Tx\|^2. \end{aligned}$$

Therefore, we have

$$(3.3)\qquad \|T_\lambda x - y\| \leq \|x - y\|, \forall x \in \mathcal{H}, y \in \bigcap M_i,$$

and T_λ is quasi-nonexpansive on $\mathcal{H}$.

Now, since $x_{k+1} = T_\lambda x_k$, from (3.3) it follows that the sequence $\{\|x_k - y\|\}$ is monotone decreasing and bounded, therefore $\|x_k - y\| \to \delta_y$ as $k \to \infty$, for each $y \in \bigcap M_i$. From Equation 2 we obtain

$$\|x_k - Tx_k\|^2 \leq \frac{1}{\lambda(2 - \lambda)}\left(\|x_k - y\|^2 - \|x_{k+1} - y\|^2\right)$$

and hence $\|x_k - Tx_k\| \to 0$ as $k \to \infty$. But $\|x - P(x, i)\| \leq \|x - Tx\|$ for each i. Therefore $d(x_k, M_i) = \|x_k - P(x_k, i)\| \to 0$ as $k \to \infty$ and Theorem 3.2 is proved. □

Remark 3.3. It is easy to see that the mapping $T : \mathcal{H} \to \mathcal{H}$ defined above ($Tx = P(x, i_x)$ is not continuous. Indeed, let $m = 2$ and let x be a point of $\mathcal{H}$ such that $d(x, M_1) = d(x, M_2)$. Now, let $\{x_k\}$ be a sequence such that $x_k \to x$ as $k \to \infty$ and $d(x_k, M_1) < d(x_k, M_2)$ for all k. Then $\lim Tx_k = \lim P(x_k, M_2) = P(x, M_2)$; but $Tx = P(x, M_1)$, that is T is not continuous at x.

Theorem 3.2 extends to real Hilbert spaces a result of Eremin [8], which is in turn a generalization of the Motzkin-Agmon-Schoenberg relaxation algorithm for inequalities. Note that the conditions of Eremin's theorem, for the finite dimensional case are somewhat weaker; more precisely, it is required only that $\bigcap M_i \neq \emptyset$, while our theorem requires that $Int \bigcap M_i \neq \emptyset$ are bounded.

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