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About a class of linear and positive operators

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ABSTRACT. In this article we shall give a general property to a class of linear and positive operators and then, through particular cases we shall obtain statements verified by the Bernstein, Schurer and Stancu operators.

1. INTRODUCTION

Let m be a non zero natural number and $B_m : C([0,1]) \to C([0,1)]$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(1.1)\qquad (B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right),$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$(1.2)\qquad p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k},$$

for any $x \in [0,1]$ and any $k \in \{0,1,\ldots,m\}$.

In 1932, E. Voronovskaja proved the result contained in the following theorem.

Theorem 1.1. ([10]). *Let $f \in C([0,1])$ be a two times derivable function at the point $x \in [0,1]$. Then the equality*

$$(1.3)\qquad \lim_{m\to\infty} m\left[(B_m f)(x) - f(x)\right] = \frac{x(1-x)}{2} f''(x)$$

holds.

For the positive integers m and p, m non zero, F. Schurer (see [6] or [7]) introduced and studied, in 1962, the operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, defined for any function $f \in C([0,1+p])$ by

$$(1.4)\qquad \left(\widetilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right),$$

where $\widetilde{p}_{m,k}(x)$ denotes the fundamental Bernstein-Schurer polynomials, defined as follows

$$(1.5)\qquad \widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} = p_{m+p,k}(x)$$

for any $x \in [0,1]$ and any $k \in \{0,1,\ldots,m+p\}$.

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In 2002, D. Bărbosu proved the result contained in the following theorem.

Theorem 1.2. ([2]). *Let* $f \in C([0,1+p])$ *be a two times derivable function in the point* $x \in [0,1+p]$. *Then the equality*

$$\lim_{m\to\infty}(m+p)\left[\left(\widetilde{B}_{m,p}f\right)(x)-f(x)\right]=px\,f'(x)+\frac{x(1-x)}{2}\,f''(x) \tag{1.6}$$

holds, where $p\in\mathbb{N}$.

For $0\le\alpha\le\beta$ and m a non zero natural number, let $P_m^{(\alpha,\beta)}: C([0,1])\to C([0,1])$ defined for any function $f\in C([0,1])$ by

$$\left(P_m^{(\alpha,\beta)}f\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.7}$$

for any $x\in[0,1]$.

The operators $P_m^{(\alpha,\beta)}$, m a non zero natural number, are named Bernstein-Stancu operators, introduced and studied in 1969 by D.D. Stancu (see [9]).

In [9] is given the result contained in the following theorem.

Theorem 1.3. *Let* $f\in C\left([0,1]\right)$ *be a two times differentiable function at the point* $x\in[0,1]$. *Then the equality*

$$\lim_{m\to\infty}(m+\beta)\left[\left(P_m^{(\alpha,\beta)}f\right)(x)-f\left(x\right)\right]=\left(\alpha-\beta x\right)f'\left(x\right)+\frac{x\left(1-x\right)}{2}f''\left(x\right) \tag{1.8}$$

holds.

Let m and i be positive integers, $m\neq 0$, and $T_{m,i}(x)$ be the polynomials

$$T_{m,i}(x)=\sum_{k=0}^{m}(k-mx)^i p_{m,k}(x), \tag{1.9}$$

for any $x\in[0,1]$ (see [7]), and $\widetilde{T}_{m,i}$ the polynomials

$$\widetilde{T}_{m,i}(x)=\sum_{k=0}^{m+p}\left[k-(m+p)x\right]^i\,\widetilde{p}_{m,k}(x)=T_{m+p,i}(x) \tag{1.10}$$

for any $x\in[0,1]$, where $p\in\mathbb{N}$ (see [7] and [3]).

In [4] are given the results contained in the following theorems.

Theorem 1.4. *Let* $f:[0,1]\to\mathbb{R}$ *be a* s *times differentiable function at the point* $x\in[0,1]$, $s\in\mathbb{N}$, s *even. Then*

$$\lim_{m\to\infty}m^{\frac{s}{2}}\left[(B_mf)(x)-\sum_{i=0}^{s}\frac{1}{m^i i!}\,T_{m,i}(x)f^{(i)}(x)\right]=0. \tag{1.11}$$

Theorem 1.5. *If* $i\in\mathbb{N}$, *then*

$$\lim_{m\to\infty}\frac{T_{m,i}(x)}{m^{\left[\frac{i}{2}\right]}}=\left[x(1-x)\right]^{\left[\frac{i}{2}\right]}\left(a_ix+b_i\right), \tag{1.12}$$

for any $x \in [0,1]$*, where*

$$(1.13)\quad a_i = \begin{cases} 0, \text{ if } i \text{ is even or } i = 1 \\ -(i-1)!! \sum_{k=1}^{\left[\frac{i}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, \text{ if } i \text{ is odd}, \ i \geq 3 \end{cases}$$

and

$$(1.14)\quad b_i = \begin{cases} 1, \quad \text{if} \quad i = 0 \\ 0, \quad \text{if} \quad i = 1 \\ (i-1)!!, \quad \text{if } i \text{ is even}, \ i \geq 2 \\ \frac{1}{2}(i-1)!! \sum_{k=1}^{\left[\frac{i}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, \text{ if } i \text{ is odd}, \ i \geq 3. \end{cases}$$

Theorem 1.6. *Let* $f : [0,1] \to \mathbb{R}$ *be a* s *times derivable function in the point* $x \in [0,1]$*,* $s \in \mathbb{N}$*,* s *even.*
If $s = 0$*, then*

$$(1.15)\quad \lim_{m\to\infty} (B_m f)(x) = f(x),$$

and if $s \geq 2$*, then*

$$(1.16)\quad \lim_{m\to\infty} m^{\frac{s}{2}} \left[(B_m f)(x) - \sum_{i=0}^{s-1} \frac{1}{m^i i!} T_{m,i}(x) f^{(i)}(x) \right] = = \frac{(s-1)!!}{s!} \left[x(1-x)\right]^{\frac{s}{2}} f^{(s)}(x).$$

2. Preliminaries

Let a, b, a', b' be real numbers, $I \subset \mathbb{R}$ interval, $a < b$, $a' < b'$, $[a,b] \subset I$, $[a',b'] \subset I$ and $[a,b] \cap [a',b'] \neq \phi$. For m non zero natural number, consider a system of nodes $(x_{m,k})_{k=0}^{m}$ in the interval $[a,b]$, so that

$$a = x_{m,0} < x_{m,1} < \cdots < x_{m,m} = b,$$

and the functions $p^*_{m,k} : I \to \mathbb{R}$ with the property that $p^*_{m,k}(x) \geq 0$, for any $x \in [a',b']$ and for any $k \in \{0,1,\ldots,m\}$.
For m non zero natural number and $i \in [0,\infty)$ define

$$(2.17)\quad T^*_{m,i}(x) = m^i \sum_{k=0}^{m} (x_{m,k} - x)^i p^*_{m,k}(x),$$

for any $x \in I$.

Also define the operators L^*_m, m non zero natural number, by

$$L^*_m : E([a,b]) \to F(I)$$

$$(2.18)\quad (L^*_m f)(x) = \sum_{k=0}^{m} p^*_{m,k}(x) f(x_{m,k})$$

for any $x \in I$ and any $f \in E([a,b])$, where $E([a,b])$, $F(I)$ are the subsets of the set of real functions defined on the interval $[a,b]$, respectively the interval I.

In the following, let s be a fixed natural number, s even and we suppose that the operators $(L_m^*)_{m\geq 1}$ verify the conditions: there exists the smallest $\alpha_i \in [0,\infty)$ so that

$$\lim_{m\to\infty} \frac{T^*_{m,i}(x)}{m^{\alpha_i}} = B_i(x) \in \mathbb{R} \tag{2.19}$$

for any $x \in [a',b']$, $i \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{2.20}$$

Proposition 2.1. *For m non zero natural number, the L_m^* operators are linear and positive.*

Proof. The proof follows immediately. □

3. Main results

Theorem 3.7. *Let $\gamma \in \mathbb{R}$ that verifies the inequality $\gamma < s+2-\alpha_{s+2}$ (where α_{s+2} has been defined in $(2.1)-(2.4)$).*
If $\delta > 0$ and $x \in [a',b']$ then

$$\lim_{m\to\infty} m^\gamma \sum_{|x_{m,k}-x|\geq\delta} (x_{m,k}-x)^s p^*_{m,k}(x) = 0. \tag{3.21}$$

Proof. We have

$$\sum_{|x_{m,k}-x|\geq\delta} (x_{m,k}-x)^s p^*_{m,k}(x) \leq \frac{1}{\delta^2} \sum_{|x_{m,k}-x|\geq\delta} (x_{m,k}-x)^{s+2} p^*_{m,k}(x) \leq$$
$$\leq \frac{1}{\delta^2} \sum_{k=0}^{m} (x_{m,k}-x)^{s+2} p^*_{m,k}(x) = \frac{1}{\delta^2 m^{s+2}} T^*_{m,s+2}(x),$$

so

$$m^\gamma \sum_{|x_{m,k}-x|\geq\delta} (x_{m,k}-x)^s p^*_{m,k}(x) \leq \frac{1}{\delta^2 m^{s+2-\gamma}} T^*_{m,s+2}(x). \tag{3.22}$$

But $\frac{1}{\delta^2 m^{s+2-\gamma}} T^*_{m,s+2}(x) = \frac{1}{\delta^2 m^{s+2-\gamma-\alpha_{s+2}}} \frac{T^*_{m,s+2}(x)}{m^{\alpha_{s+2}}}$ and because $\gamma < s+2-\alpha_{s+2}$, we get $s+2-\gamma-\alpha_{s+2} > 0$. Taking (2.3) into account, which means $\lim_{m\to\infty} \frac{T^*_{m,s+2}(x)}{m^{\alpha_{s+2}}} = B_{s+2}(x) \in \mathbb{R}$, it results that

$$\lim_{m\to\infty} \frac{1}{\delta^2 m^{s+2-\gamma-\alpha_{s+2}}} \frac{T^*_{m,s+2}(x)}{m^{\alpha s+2}} = 0.$$

Considering the limit computed above, the fact that s is even and (3.2), we obtain (3.1). □

Theorem 3.8. *Let* $f : [a,b] \to \mathbb{R}$*, be a* s *times differentiable function at the point* $x \in [a,b] \cap [a',b']$*. If there exists* $M \in (0,\infty)$ *so that*

$$\sum_{k=0}^{m} p^*_{m,k}(x) \le M, \tag{3.23}$$

for any non zero natural number m*, then*

$$\lim_{m\to\infty} m^{s-\alpha_s} \left[(L^*_m f)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!} T^*_{m,i}(x) f^{(i)}(x) \right] = 0. \tag{3.24}$$

Proof. According to Taylor's theorem for the function f around x, we have

$$f(t) = \sum_{i=0}^{s} \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x) \tag{3.25}$$

where μ is a bounded function and $\lim_{t\to x} \mu(t-x) = 0$, or

$$\forall\, \varepsilon > 0,\ \exists\, \delta_\varepsilon > 0,\ \forall\, h \in (a-b, b-a),\ |h| < \delta_\varepsilon, \quad \text{we have} \quad |\mu(h)| < \varepsilon. \tag{3.26}$$

If we replace t with $x_{m,k}$ in (3.5), multiply by $p^*_{m,k}(x)$ and sum after k, when $k \in \{0,1,\ldots,m\}$, we obtain

$$\begin{aligned} (L^*_m f)(x) &= \sum_{k=0}^{m} \sum_{i=0}^{s} \frac{(x_{m,k}-x)^i}{i!} p^*_{m,k}(x) f^{(i)}(x) + \\ &+ \sum_{k=0}^{m} (x_{m,k}-x)^s p^*_{m,k}(x) \mu(x_{m,k}-x) = \\ &= \sum_{i=0}^{s} \frac{1}{m^i i!} \left[m^i \sum_{k=0}^{m} (x_{m,k}-x)^i p^*_{m,k}(x) \right] f^{(i)}(x) + \\ &+ \sum_{k=0}^{m} (x_{m,k}-x)^s p^*_{m,k}(x) \mu(x_{m,k}-x)\,, \end{aligned}$$

or

$$(L^*_m f)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!} T^*_{m,i}(x) f^{(i)}(x) = \sum_{k=0}^{m} (x_{m,k}-x)^s p^*_{m,k}(x) \mu(x_{m,k}-x),$$

and thus

$$m^{s-\alpha_s} \left[(L^*_m f)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!} T^*_{m,i}(x) f^{(i)}(x) \right] = (R_m f)(x), \tag{3.27}$$

where

$$(R_m f)(x) = m^{s-\alpha_s} \sum_{k=0}^{m} (x_{m,k}-x)^s p^*_{m,k}(x) \mu(x_{m,k}-x). \tag{3.28}$$

Consider δ_ε from (3.6), $I_m = \{0, 1, \ldots, m\}$, $I_{m,1}\,\{k \in I_m : |x_{m,k} - x| < \delta_\varepsilon\}$ and $I_{m,2} = \{k \in I_m : |x_{m,k} - x| \geq \delta_\varepsilon\}$. Then

$$\begin{aligned} |(R_m f)(x)| &\leq m^{s-\alpha_s} \sum_{k=0}^{m} (x_{m,k} - x)^s p^*_{m,k}(x)\,|\mu(x_{m,k} - x)| = \\ &= m^{s-\alpha_s} \sum_{k \in I_{m,1}} (x_{m,k} - x)^s p^*_{m,k}(x)\,|\mu(x_{m,k} - x)| + \\ &+ m^{s-\alpha_s} \sum_{k \in I_{m,2}} (x_{m,k} - x)^s p^*_{m,k}(x)\,|\mu(x_{m,k} - x)| \end{aligned}$$

taking (3.6) into account, and considering the fact that μ is bounded, $\sup\limits_{t \in (a-b, b-a)} |\mu(t)| = \eta$, we have

$$\begin{aligned} |(R_m f)(x)| \leq m^{s-\alpha_s} \varepsilon \sum_{k \in I_{m,1}} (x_{m,k} - x)^s p^*_{m,k}(x) + \\ + m^{s-\alpha_s} \mu \sum_{k \in I_{m,2}} (x_{m,k} - x)^s p^*_{m,k}(x). \end{aligned} \tag{3.29}$$

But $(x_{m,k} - x)^s \leq (b-a)^s$ and so

$$\sum_{k \in I_{m,1}} (x_{m,k} - x)^s p^*_{m,k}(x) \leq (b-a)^s \sum_{k \in I_{m,1}} p^*_{m,k}(x) \leq (b-a)^s \sum_{k=0}^{m} p^*_{m,k}(x). \tag{3.30}$$

Taking (3.3) and (3.10) into account, we have that

$$m^{s-\alpha_s} \varepsilon \sum_{k \in I_{m,1}} (x_{m,k} - x)^s p^*_{m,k}(x) \leq m^{s-\alpha_s} \varepsilon (b-a)^s M. \tag{3.31}$$

Considering (2.4) we have that $s - \alpha_s < s + 2 - \alpha_{s+2}$ and then from Theorem 3.1 we obtain $\lim\limits_{m\to\infty} m^{s-\alpha_s} \sum\limits_{k \in I_{m,2}} (x_{m,k} - x)^s p^*_{m,k}(x) = 0$, thus for ε from (3.6), there exists $m(\varepsilon) \in \mathbb{N}$, $\forall\, m \in \mathbb{N}$, $m \geq m(\varepsilon)$, so that

$$m^{s-\alpha_s} \eta \sum_{k \in I_{m,2}} (x_{m,k} - x)^s p^*_{m,k}(x) < \varepsilon. \tag{3.32}$$

Choose $\varepsilon = \dfrac{1}{m[m^{s-\alpha_s}(b-a)^s M + 1]}$ and there exists $m(\varepsilon) \in \mathbb{N}$, for any $m \in \mathbb{N}$, $m \geq m(\varepsilon)$, from (3.9) - (3.12) it results that $|(R_m f)(x)| < \dfrac{1}{m}$, and so

$$\lim_{m\to\infty} (R_m f)(x) = 0. \tag{3.33}$$

From (3.7) and (3.13), (3.4) follows. □

Remark 3.1. The number M in (3.3) depends on the fixed number x in Theorem 3.2.

Corollary 3.1. *Under the assumptions of Theorem 3.2, if $s \neq 0$, then*

$$\lim_{m\to\infty} m^{s-\alpha_s} \left[(L^*_m f)(x) - \sum_{i=0}^{s-1} \frac{1}{m^i i!} T^*_{m,i}(x) f^{(i)}(x) \right] = \frac{1}{s!} f^{(s)}(x) B_s(x). \tag{3.34}$$

Proof. From Theorem 3.2 we get

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[\left(L_m^* f\right)(x)-\sum_{i=0}^{s-1}\frac{1}{m^i i!}\,T_{m,i}^*(x)f^{(i)}(x)\right]=$$
$$=\frac{1}{s!}\,f^{(s)}(x)\lim_{m\to\infty}\frac{T_{m,s}^*(x)}{m^{\alpha_s}}\,.$$

By (2.3), the conclusion follows. □

Application 3.1. If $a=a'=0$, $b=b'=1$, $x_{m,k}=\dfrac{k}{m}$, $k\in\{0,1,\dots,m\}$, $m\in\mathbb{N}^*$ and $p_{m,k}^*(x)=p_{m,k}(x)$, for any $x\in[0,1]$, $k\in\{0,1,\dots,m\}$, $m\in\mathbb{N}^*$, then L_m^* is the B_m Bernstein operator, $m\in\mathbb{N}^*$. In this case (see[4]), $\alpha_s=\dfrac{s}{2}$, $B_s(x)=[x(1-x)]^{\frac{s}{2}}(s-1)!!$ and (1.10), (1.14), (1.15) takes place.

Application 3.2. If $a=0$, $b=1+p$, $p\in\mathbb{N}$, $a'=0$, $b'=1$, the role of m is played by $m+p$, $x_{m+p,k}=\dfrac{k}{m}$, $k\in\{0,1,\dots,m+p\}$ and $p_{m+p,k}^*(x)=\binom{m+p}{k}x^k(1-x)^{m+p-k}=\widetilde{p}_{m,k}(x)=p_{m+p,k}(x)$, $k\in\{0,1,\dots,m+p\}$, for any $x\in[0,1+p]$. Then the operator L_m^* is the operator $\widetilde{B}_{m,p}$, the Bernstein-Schurer operator.

Theorem 3.9. *Let $f:[0,1+p]\to\mathbb{R}$ be a s times differentiable function at a point $x\in[0,1]$.*

If $s=0$, then

$$\lim_{m\to\infty}\widetilde{B}_{m,p}(x)=f(x), \tag{3.35}$$

if $s\in\mathbb{N}$, then

$$\lim_{m\to\infty}(m+p)^{\frac{s}{2}}\Bigg[\left(\widetilde{B}_{m,p}f\right)(x)- -\sum_{i=0}^{s}\sum_{l=0}^{i}\frac{1}{(m+p)^i i!}\left(\frac{m+p}{m}\right)^i\binom{i}{l}(px)^{i-l}\widetilde{T}_{m,l}(x)f^{(i)}(x)\Bigg]=0 \tag{3.36}$$

and if $s\geq 2$, then

$$\lim_{m\to\infty}(m+p)^{\frac{s}{2}}\Bigg[\left(\widetilde{B}_{m,p}f\right)(x)-\sum_{i=0}^{s-1}\sum_{l=0}^{i}\frac{1}{(m+p)^i i!}\left(\frac{m+p}{m}\right)^i\binom{i}{l}\cdot \cdot(px)^{i-l}\widetilde{T}_{m,l}(x)f^{(i)}(x)\Bigg]=\frac{(s-1)!!}{s!}\,[x(1-x)]^{\frac{s}{2}}f^{(s)}(x). \tag{3.37}$$

Proof. We have

$$\begin{aligned}
T^*_{m+p,i}(x) &= (m+p)^i \sum_{k=0}^{m+p} \left(\frac{k}{m}-x\right)^i \widetilde{p}_{m,k}(x) = \\
&= \left(\frac{m+p}{m}\right)^i \sum_{k=0}^{m+p} (k-mx)^i \ {}_{p_{m+p,k}}(x) = \\
&= \left(\frac{m+p}{m}\right)^i \sum_{k=0}^{m+p} [(k-(m+p)x+px]^i \ {}_{p_{m+p,k}}(x) = \\
&= \left(\frac{m+p}{m}\right)^i \sum_{k=0}^{m+p} \sum_{l=0}^{i} \binom{i}{l} [k-(m+p)x]^l (px)^{i-l} p_{m+p,k}(x) = \\
&= \left(\frac{m+p}{m}\right)^i \sum_{l=0}^{i} \binom{i}{l} (px)^{i-l} \sum_{k=0}^{m+p} [k-(m+p)x]^l p_{m+p,k}(x),
\end{aligned}$$

and so

$$T^*_{m+p,i}(x) = \left(\frac{m+p}{m}\right)^i \sum_{l=0}^{i} \binom{i}{l} (px)^{i-l} T_{m+p,l}(x),\ i \in \{0,1,\dots,s\}. \tag{3.38}$$

In this theorem $\alpha_s = \dfrac{s}{2}$. If $s=0$, because $T_{m+p,0}(x)=1$ and according to (3.18) and (3.4), (3.15) is obtained.

By (1.9), (3.4) and (3.18), (3.16) follows.

If $s \geq 2$, then using (3.18) we have that

$$\begin{aligned}
B_s(x) &= \lim_{m\to\infty} \frac{T^*_{m+p,s}(x)}{(m+p)^{\frac{s}{2}}} = \lim_{m\to\infty} \left(\frac{m+p}{m}\right)^s \cdot \\
&\cdot \left[\sum_{l=0}^{s-1} \binom{s}{l} (px)^{s-l} \frac{T_{m+p,l}(x)}{(m+p)^{\left[\frac{l}{2}\right]}} \frac{1}{(m+p)^{\frac{s}{2}-\left[\frac{l}{2}\right]}} + \frac{T_{m+p,s}(x)}{(m+p)^{\frac{s}{2}}}\right].
\end{aligned}$$

Because s is even, $\dfrac{s}{2} > \left[\dfrac{l}{2}\right]$, for any $l \in \{0,1,\dots,s-1\}$ and considering (1.11), we get

$$B_s(x) = [x(1-x)]^{\frac{s}{2}} (s-1)!!\,. \tag{3.39}$$

By (1.9), (3.14) and (3.19), (3.17) easily follows. □

Remark 3.2. Remind that $T_{m,0}(x)=1$ and $T_{m,1}(x)=0$, for any $x \in [0,1]$, for any $m \in \mathbb{N}$. From (3.18), we have that $\widetilde{T}_{m,0}(x) = T_{m+p,0}(x) = 1$ and $\widetilde{T}_{m,1}(x) = T_{m+p,1}(x) = 0$. For $s=2$, (3.17) becomes

$$\begin{aligned}
&\lim_{m\to\infty}(m+p)\Big[\left(\widetilde{B}_{m,p}f\right)(x) - \widetilde{T}_{m,0}(x) - \frac{m+p}{m}\,px\widetilde{T}_{m,0}(x)f'(x) - \\
&-\frac{1}{m}\,\widetilde{T}_{m,1}(x)f'(x)\Big] = x(1-x)f''(x),
\end{aligned}$$

and thus (1.6) follows.

Application 3.3. If $a = a' = 0$, $b = b' = 1$, $x_{m,k} = \dfrac{k+\alpha}{m+\beta}$, $k \in \{0, 1, \ldots, m\}$ and $p^*_{m,k}(x) = p_{m,k}(x)$, for any $x \in [0,1]$, for any $k \in \{0, 1, \ldots, m\}$, then L^*_m is the Bernstein-Stancu $P_m^{(\alpha,\beta)}$ operator.

Theorem 3.10. *Let $f : [0,1] \to \mathbb{R}$ be a s times differentiable function, at a point $x \in [0,1]$, $0 \le \alpha \le \beta$.*

If $s = 0$, then

$$\lim_{m\to\infty} \left(P_m^{(\alpha,\beta)} f\right)(x) = f(x), \tag{3.40}$$

if $s \in \mathbb{N}$, then

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[\left(P_m^{(\alpha,\beta)} f\right)(x) - \right. \tag{3.41}$$

$$\left. - \sum_{i=0}^{s} \sum_{l=0}^{i} \frac{1}{m^i i!} \left(\frac{m}{m+\beta}\right)^i \binom{i}{l} (\alpha - \beta x)^{i-l} T_{m,l}(x) f^{(i)}(x) \right] = 0$$

and if $s \ge 2$, then

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[\left(P_m^{(\alpha,\beta)} f\right)(x) - \sum_{i=0}^{s-1} \sum_{l=0}^{i} \frac{1}{m^i i!} \left(\frac{m}{m+\beta}\right)^i \binom{i}{l} \cdot \right. \tag{3.42}$$

$$\left. \cdot (\alpha - \beta x)^{i-l} T_{m,l}(x) f^{(i)}(x) \right] = \frac{(s-1)!!}{s!} \left[x(1-x)\right]^{\frac{s}{2}} f^{(s)}(x).$$

Proof. We have

$$\begin{aligned}
T^*_{m,i}(x) &= m^i \sum_{k=0}^{m} (x_{m,k} - x)^i p_{m,k}(x) = \\
&= \left(\frac{m}{m+\beta}\right)^i \sum_{k=0}^{m} [k + \alpha - (m+\beta)x]^i p_{m,k}(x) = \\
&= \left(\frac{m}{m+\beta}\right)^i \sum_{k=0}^{m} \left[(k - mx) + (\alpha - \beta x)\right]^i p_{m,k}(x) = \\
&= \left(\frac{m}{m+\beta}\right)^i \sum_{k=0}^{m} \sum_{l=0}^{i} \binom{i}{l} (k - mx)^l (\alpha - \beta x)^{i-l} p_{m,k}(x) = \\
&= \left(\frac{m}{m+\beta}\right)^i \sum_{l=0}^{i} \binom{i}{l} (\alpha - \beta x)^{i-l} \sum_{k=0}^{m} (k - mx)^l p_{m,k}(x),
\end{aligned}$$

from which

$$T^*_{m,i}(x) = \left(\frac{m}{m+\beta}\right)^i \sum_{l=0}^{i} \binom{i}{l} (\alpha - \beta x)^{i-l} T_{m,l}(x),\ i \in \{0, 1, \ldots, s\}. \tag{3.43}$$

For this theorem $\alpha_s = \dfrac{s}{2}$. If $s = 0$, because $T_{m,0}(x) = 1$, by considering (3.23), (3.4), (3.20) is obtained.

Taking (3.4) and (3.23) into account, (3.21) follows. If $s \geq 2$ then using (3.23), we have

$$B_s(x) = \lim_{m\to\infty} \frac{T^*_{m,s}(x)}{m^{\frac{s}{2}}} =$$

$$= \lim_{m\to\infty} \left(\frac{m}{m+\beta}\right)^s \left[\sum_{l=0}^{s-1} \binom{s}{l} (\alpha - \beta x)^{s-l} \frac{T_{m,l}(x)}{m^{\left[\frac{l}{2}\right]}} \frac{1}{m^{\frac{s}{2}-\left[\frac{l}{2}\right]}} + \frac{T_{m,s}(x)}{m^{\frac{s}{2}}}\right].$$

Because s is even, $\frac{s}{2} > \left[\frac{l}{2}\right]$, for any $l \in \{0, 1, \ldots, s-1\}$ and using (1.11) we obtain

(3.44) $\quad B_s(x) = [x(1-x)]^{\frac{s}{2}} (s-1)!!\,.$

From (3.14) and (3.24), (3.22) follows. □

Remark 3.3. For $s = 2$ in Theorem 3.4, we obtain Theorem 1.3.

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