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On a conjecture for weighted interpolation using Chebyshev polynomials of the third and fourth kinds

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ABSTRACT. A conjecture for the projection norm (or Lebesgue constant) of a weighted interpolation method based on the zeros of Chebyshev polynomials of the third and fourth kinds is resolved. This conjecture was made in a paper by J. C. Mason and G. H. Elliott in 1995. The proof of the conjecture is achieved by relating the projection norm to that of a weighted interpolation method based on zeros of Chebyshev polynomials of the second kind.

1. INTRODUCTION

Suppose $x_0, x_1, \ldots, x_n$ are distinct points (nodes) in $[-1,1]$, let $w \in C[-1,1]$ be a weight function satisfying $w(x) \geq 0$, $w(x_i) \neq 0$, and denote the set of all polynomials of degree no greater than n by P_n. Define an interpolating projection L_n of $C[-1,1]$ on wP_n by

$$(1.1) \qquad (L_n f)(x) = w(x) \sum_{i=0}^{n} \ell_i(x) \frac{f(x_i)}{w(x_i)},$$

where $\ell_i(x)$ is the fundamental Lagrange polynomial

$$(1.2) \qquad \ell_i(x) = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{x - x_k}{x_i - x_k}.$$

(Note that if $w(x) \equiv 1$, then L_n is Lagrange interpolation.) If $\|\cdot\|_\infty$ denotes the uniform norm $\|f\|_\infty = \sup_{-1\leq x\leq 1} |f(x)|$, the projection norm (or Lebesgue constant)

$$\|L_n\| = \sup_{\|f\|_\infty \leq 1} \|L_n f\|_\infty$$

satisfies

$$(1.3) \qquad \|L_n\| = \sup_{x\in[-1,1]} \sum_{i=0}^{n} |\ell_i(x)| \frac{w(x)}{w(x_i)}.$$

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The Chebyshev polynomials of the first, second, third and fourth kinds, of degree $n+1$, are defined by

$$\begin{aligned} T_{n+1}(x) &= \cos(n+1)\theta, \\ U_{n+1}(x) &= [\sin(n+2)\theta]/\sin\theta, \\ V_{n+1}(x) &= [\cos(n+3/2)\theta]/\cos(\theta/2), \\ W_{n+1}(x) &= [\sin(n+3/2)\theta]/\sin(\theta/2), \end{aligned}$$

respectively, where $x = \cos\theta$ and $0 \le \theta \le \pi$. (See [6], for example, for an account of these polynomials and their properties.) The zeros of the Chebyshev polynomials are:

(1.4) zeros of $T_{n+1}(x)$: $x_i = \cos[(i+1/2)\pi/(n+1)]$ $(i = 0, 1, \ldots, n)$,

(1.5) zeros of $U_{n+1}(x)$: $x_i = \cos[(i+1)\pi/(n+2)]$ $(i = 0, 1, \ldots, n)$,

(1.6) zeros of $V_{n+1}(x)$: $x_i = \cos[(i+1/2)\pi/(n+3/2)]$ $(i = 0, 1, \ldots, n)$,

(1.7) zeros of $W_{n+1}(x)$: $x_i = \cos[(i+1)\pi/(n+3/2)]$ $(i = 0, 1, \ldots, n)$.

Now define interpolating projections $L_n^{(1)}$, $L_n^{(2)}$, $L_n^{(3)}$, $L_n^{(4)}$ by (1.1) and (1.2) with weights $w(x) = 1$, $(1-x^2)^{1/2}$, $(1+x)^{1/2}$ and $(1-x)^{1/2}$, respectively, and respective nodes (1.4), (1.5), (1.6) and (1.7).

The projection norm $\|L_n^{(1)}\|$ for (unweighted) Lagrange interpolation on the Chebyshev nodes of the first kind has been studied extensively. For instance, by results of Luttmann and Rivlin [4] and Ehlich and Zeller [2] in the 1960s, $\|L_n^{(1)}\|$ has the asymptotic expansion as $n \to \infty$,

$$\|L_n^{(1)}\| = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{8}{\pi} + \gamma\right) + o(1), \tag{1.8}$$

where $\gamma = 0.577\ldots$ denotes Euler's constant. Discussion of this and other results, including refinements of (1.8), are given in Brutman's survey paper [1, Section 2.2].

In the paper [5], J. C. Mason and G. H. Elliott studied $\|L_n^{(i)}\|$ for $i = 2, 3, 4$. For example, they showed that

$$\|L_n^{(2)}\| = \sup_{0 \le \theta \le \pi/2} F_n(\theta),$$

where

$$F_n(\theta) = \frac{|\sin(n+2)\theta|}{n+2} \sum_{i=0}^{n} \frac{\sin\theta_{i,n}}{|\cos\theta - \cos\theta_{i,n}|} \tag{1.9}$$

and $\theta_{i,n} = (i+1)\pi/(n+2)$. On the basis of numerical computations, the authors made the following conjecture.

Conjecture 1.1. *The supremum of $F_n(\theta)$ occurs at $\pi/2$ if n is odd and at a value that is asymptotic to $\pi(n+1)/[2(n+2)]$ as $n \to \infty$ if n is even.*

Mason and Elliott also showed that $F_n(\pi/2)$ (for odd n) and $F_n(\pi(n+1)/[2(n+2)])$ (for even n) both have the asymptotic expansion

$$\frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) + o(1). \tag{1.10}$$

Therefore, assuming Conjecture 1.1 is correct, it follows that

$$(1.11)\quad \|L_n^{(2)}\| = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{4}{\pi}+\gamma\right) + o(1).$$

Smith [7] later verified (1.11), although the proof did not depend on Conjecture 1.1 (which remains open). The result (1.11) means that not only is $\|L_n^{(2)}\|$ essentially smaller than $\|L_n^{(1)}\|$, but that $L_n^{(2)}$, which is based on a simple node system, has (to within $o(1)$ terms) the same norm as the Lagrange method of minimal norm over all possible choices of nodes. See Brutman [1, Section 3] for a discussion of the optimal choice of nodes for Lagrange interpolation, and Kilgore [3] for some interesting results concerning the projection norms for weighted interpolation with Jacobi weights, and their relation to the minimal norm for Lagrange interpolation.

For the projections $L_n^{(3)}$ and $L_n^{(4)}$, it follows from (1.3) that $\|L_n^{(3)}\| = \|L_n^{(4)}\|$. Based again on numerical results, Mason and Elliott [5, p. 50] made the following conjecture.

Conjecture 1.2. *The norm of the interpolating projection $L_n^{(4)}$ satisfies*

$$\|L_n^{(4)}\| = \|L_n^{(1)}\| + o(1).$$

This conjecture will be proved in the following section, where some observations on the relationship between $\|L_n^{(4)}\|$ and $\|L_n^{(2)}\|$ are also made.

2. Proof of Conjecture 1.2

By (1.3) and (1.7) with $w(x) = (1-x)^{1/2}$,

$$\|L_n^{(4)}\| = \sup_{0\le\phi\le\pi} G_n(\phi),$$

where

$$G_n(\phi) = \frac{|\sin(n+3/2)\phi|}{n+3/2}\sum_{i=0}^{n}\frac{\sin\phi_{i,n}}{|\cos\phi - \cos\phi_{i,n}|}$$

and $\phi_{i,n} = (i+1)\pi/(n+3/2)$. Put $m = 2n+1$ and $\phi = 2\theta$. Thus

$$(2.12)\quad \|L_n^{(4)}\| = \sup_{0\le\theta\le\pi/2} H_m(\theta),$$

where

$$(2.13)\quad H_m(\theta) = \frac{2|\sin(m+2)\theta|}{m+2}\sum_{i=0}^{(m-1)/2}\frac{\sin 2\theta_{i,m}}{|\cos 2\theta - \cos 2\theta_{i,m}|}$$

and $\theta_{i,m} = (i+1)\pi/(m+2)$. The key idea is to compare $H_m(\theta)$ with the function $F_m(\theta)$, defined by (1.9), that was studied in [5] and [7].

For simplicity, write θ_i for $\theta_{i,m}$. Now, $H_m(0) = F_m(0) = 0$ and if $0 \le j \le (m-1)/2$, then $H_m(\theta_j) = F_m(\theta_j) = 1$. Suppose, then, that $\theta \in (0, \pi/2]$ and $\theta_j < \theta < \theta_{j+1}$ for some $j \in \{-1, 0, \dots, (m-1)/2\}$. Thus

$$F_m(\theta) = \frac{|\sin(m+2)\theta|}{m+2}\left[\sum_{i=0}^{j} \frac{\sin\theta_i}{\cos\theta_i - \cos\theta} + \sum_{i=j+1}^{m-j-1} \frac{\sin\theta_i}{\cos\theta - \cos\theta_i} + \sum_{i=m-j}^{m} \frac{\sin\theta_i}{\cos\theta - \cos\theta_i}\right],$$

where the first and last sums vanish if $j = -1$ and the middle sum vanishes if $j = (m-1)/2$. By combining terms using $\theta_{m-i} = \pi - \theta_i$, it follows that

$$F_m(\theta) = \frac{2|\sin(m+2)\theta|}{m+2}\left[\sum_{i=0}^{j} \frac{\sin 2\theta_i}{\cos 2\theta_i - \cos 2\theta} + 2\sum_{i=j+1}^{(m-1)/2} \frac{\sin\theta_i \cos\theta}{\cos 2\theta - \cos 2\theta_i}\right].$$

Therefore, by (2.13),

$$F_m(\theta) - H_m(\theta) = \frac{4|\sin(m+2)\theta|}{m+2}\sum_{i=j+1}^{(m-1)/2} \frac{\sin\theta_i(\cos\theta - \cos\theta_i)}{\cos 2\theta - \cos 2\theta_i}.$$

Observe that all terms in the summation are positive, so $H_m(\theta) \le F_m(\theta)$, with equality if and only if $j = (m-1)/2$ (i.e. when the sum contains no terms).

Now, from the above results,

$$F_m(\pi/2) = H_m(\pi/2) \le \sup_{0\le\theta\le\pi/2} H_m(\theta) \le \sup_{0\le\theta\le\pi/2} F_m(\theta) = \|L_m^{(2)}\|. \tag{2.14}$$

Thus, by the expansion (1.10) for $F_n(\pi/2)$ if n is odd and (1.11), it follows that

$$\sup_{0\le\theta\le\pi/2} H_m(\theta) = \frac{2}{\pi}\log m + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) + o(1).$$

Since $m = 2n+1$ we conclude from (1.8) and (2.12) that

$$\begin{aligned} \|L_n^{(4)}\| &= \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{8}{\pi} + \gamma\right) + o(1) \\ &= \|L_n^{(1)}\| + o(1), \end{aligned} \tag{2.15}$$

which verifies Conjecture 1.2.

To conclude, we remark that (2.15) can be interpreted as

$$\|L_n^{(4)}\| = \|L_{2n+1}^{(2)}\| + o(1). \tag{2.16}$$

However, if Conjecture 1.1 is true, then equality holds throughout (2.14), and so the $o(1)$ term in (2.16) vanishes. Thus we make the following conjecture.

Conjecture 2.3. *The norms of the interpolating projections $L_n^{(2)}$ and $L_n^{(4)}$ are related by*

$$\|L_n^{(4)}\| = \|L_{2n+1}^{(2)}\|.$$

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