CARPATHIAN J. MATH.
25 (2009), No. 2, 203 - 214

Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

A note on nonlinear connections on the cotangent bundle

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ABSTRACT.

In this paper the problem of compatibility between a nonlinear connection and some other geometric structures on the cotangent bundle of a manifold is studied. We prove that the notions of semi-Hamiltonian vector field on cotangent bundle and the metric nonlinear connection on tangent bundle are dual structures, via Legendre transformation.

1. INTRODUCTION

The total space of the cotangent bundle T^*M can be studied using the same methods as in the case of tangent bundle. However, in the case of cotangent bundle there exists some special geometric objects: *Liouville-Hamilton vector field, Liouville 1-form, canonical symplectic structure.* It is well known that the cotangent bundle T^*M of a differentiable manifold M plays a very important role in symplectic geometry and its applications, since this carries a canonical symplectic structure induced by the Liouville form. The Hamiltonian formalism seems to be, in many ways, mathematically more straightforward that the Lagrangian formalism, because on the tangent bundle we do not have a naturally symplectic structure. On the contrary, the tangent bundle has a naturally defined integrable tangent structure and semispray (second order differential equation vector field) which induces a nonlinear connection. In the case of the cotangent bundle we do not have a canonical tangent structure or something similar to a semispray, but there exist some dual objects, as adapted almost tangent structure $\mathcal{J}$ and $\mathcal{J}$-regular vector fields.

In the present paper we study the problem of compatibility between a nonlinear connection and some other geometric structures on the cotangent bundle of a manifold. In the first section the problem of metrizability of a nonlinear connection, dynamical covariant derivative on the tangent bundle [1, 3, 5, 9, 17] and the preliminaries results on the cotangent bundle [7, 12, 16, 18] are presented. The second section deals with the notion of nonlinear connection on T^*M. Vari-ous aspects of this topic were investigated by many authors (see for instance [2, 7, 10, 11, 12, 13, 14, 15, 16, 18]). We present the notion of adapted almost tangent structure $\mathcal{J}$ and $\mathcal{J}$-regular vector field on T^*M (an equivalent definition is given in [16]) and a nonlinear connection $\mathcal{N}$ induced by them is naturally obtained. We define the torsion of a nonlinear connection on T^*M using the Frölicher-Nijenjuis bracket and find its expression in local coordinates. This torsion vanishes for nonlinear connection $\mathcal{N}$ induced by a $\mathcal{J}$-regular vector field. An almost complex structure is introduced and the integrability conditions in terms of torsion and curvature of the connection is given.

In the section three, using a regular Hamiltonian on T^*M and Legendre transformation, we transfer many results between cotangent and tangent spaces (see also [7, 12]). We find, in the other way (see [10, 12]), the expression of a canonical symmetric nonlinear connection induced by a regular Hamiltonian. We prove that a metric nonlinear connection on the tangent bundle and a semi-Hamiltonian vector field of the cotangent bundle are dual structures, via Legendre transformation. Thus, a semispray on TM is transformed into a semi-Hamiltonian vector field on T^*M if the nonlinear connection induced by semispray is metric. Converse is true only with an additionally condition. I have to remark that a more general case involving Lie algebroids is given in [8] and the problem of metrizability for nonlinear connection on the cotangent bundle could be studied.

1.1. **Geometric structures on TM and T^*M.** Let M be a differentiable, n-dimensi-onal manifold and (TM,π,M) its tangent bundle. If (x^i) are the local coordinates on the domain U of a map on M, then the local coordinates on $\pi^{-1}(U)$ are denoted (x^i,y^i), $(i,j=\overline{1,n})$. The vertical vector field $C=y^i\frac{\partial}{\partial y^i}$ on TM is called the *Liouville vector field.* A vector field $S\in\mathcal{X}(TM)$ which satisfies $JS=C$ is called a *semispray* on TM, where $J=\frac{\partial}{\partial y^i}\otimes dx^i$ is the natural tangent structure.
A nonlinear connection N is a horizontal distribution H_uTM which is supplementary to the vertical distribution, that is $T_uTM=V_uTM\oplus H_uTM$. The dynamical covariant derivative [3] that corresponds to a semispray S and a nonlinear connection N is defined on vertical subbundle by $\nabla:\Gamma(VTM)\to\Gamma(VTM)$ through

$$\nabla\left(X^i\frac{\partial}{\partial y^i}\right)=\left(S(X^i)+N^i_jX^j\right)\frac{\partial}{\partial y^i}.$$

Received: 15.02.2009; In revised form: 28.05.2009; Accepted: 27.08.2009
2000 *Mathematics Subject Classification.* 53C05, 70H05.
Key words and phrases. *Nonlinear connection, Hamiltonian formalism, semi-Hamiltonian vector field.*

and the following properties are satisfied

$$\nabla(X+Y) = \nabla X + \nabla Y, \quad \forall X, Y \in \Gamma(VTM),$$
$$\nabla(fX) = S(f)X + f\nabla X, \quad \forall X \in \Gamma(VTM), \quad \forall f \in \mathcal{F}(TM).$$

For a pseudo-Riemannian metric g the dynamical covariant derivative is given by

$$\nabla g(X,Y) = S(g(X,Y)) - g(\nabla X, Y) - g(X, \nabla Y),$$

which in local coordinates leads to

$$\nabla g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = S(g_{ij}) - g_{ik}N_j^k - g_{kj}N_i^k.$$

Let S be a semispray, N a nonlinear connection and ∇ the associated dynamical covariant derivative, then the nonlinear connection is called metric or compatible with the metric tensor g if $\nabla g = 0$, that is

(1.1) $$S(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y), \quad \forall X, Y \in \Gamma(VTM).$$

In local coordinates the previous relation is

(1.2) $$S(g_{ij}) - g_{ik}N_j^k - g_{kj}N_i^k = 0.$$

The vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j},$$

determines a local basis of the horizontal distribution on TM. The vector fields $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, $i = \overline{1,n}$ determine the Berwald basis on TM with

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R_{ji}^k \frac{\partial}{\partial y^k}, \quad R_{ij}^k = \frac{\delta N_i^k}{\delta x^j} - \frac{\delta N_j^k}{\delta x^i}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = \frac{\partial N_i^k}{\partial y^j}\frac{\partial}{\partial y^k}$$

and its dual basis is given by $(dx^i, \delta y^i)$ where $\delta y^i = dy^i + N_j^i dx^j$. We know [4, 6] that if $S = y^i \frac{\partial}{\partial x^i} + S^i \frac{\partial}{\partial y^i}$ is a semispray then the automorphism

$$N = -\mathcal{L}_S J,$$

is a nonlinear connection on TM with the coefficients given by

(1.3) $$N_j^i(x,y) = -\frac{1}{2}\frac{\partial S^i}{\partial y^j}.$$

For every regular Lagrangian on TM there exist the Kern nonlinear connection with the coefficients given by (1.3), where

$$S^i = g^{ij}\left(\frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial x^k \partial y^j} y^k\right),$$

which is a metric nonlinear connection (see [3]).

If (T^*M, τ, M) is the cotangent bundle then the local coordinates on $\tau^{-1}(U)$ are denoted (x^i, p_i), $(i, j = \overline{1,n})$. The natural basis on T^*M is given by $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\right)$. We have the following geometric objects

(1.4) $$C^* = p_i \frac{\partial}{\partial p_i}, \quad \theta = p_i dx^i, \quad \omega = d\theta = dp_i \wedge dx^i,$$

where (dx^i, dp_i) is the dual natural basis. The following properties hold:

1° C^* is a vertical vector field, globally defined on T^*M, called the Liouville-Hamilton vector field.

2° The 1-form θ is globally defined on T^*M and is called the Liouville 1-form.

3° ω is a symplectic structure, called canonical.

The Poisson bracket $\{\cdot,\cdot\}$ on T^*M, is defined by

(1.5) $$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial x^i}, \quad \forall f, g \in \mathcal{F}(T^*M).$$

In the following by a d-tensor field we mean a tensor field on T^*M whose components, under a change of coordinates on T^*M, behave like the components of a tensor field on M.

2. Nonlinear connection on the cotangent bundle

On the cotangent bundle T^*M there exists the integrable vertical distribution V_uT^*M, $u \in T^*M$ generated locally by the basis $\left(\frac{\partial}{\partial p_i}\right)$, $i = \overline{1,n}$. A nonlinear connection $\mathcal{N}$ is a horizontal distribution H_uT^*M which is supplementary to the vertical distribution, that is $T_uT^*M = V_uT^*M \oplus H_uT^*M$. If $\mathcal{N}$ is a nonlinear connection then on the every domain of the local chart $\tau^{-1}(U)$, the adapted basis of the horizontal distribution HT^*M has the form

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \mathcal{N}_{ij}\frac{\partial}{\partial p_j}, \tag{2.1}$$

where $\mathcal{N}_{ij}(x,p)$ are the coefficients of the nonlinear connection $\mathcal{N}$. The dual adapted basis is

$$\delta p_i = dp_i - \mathcal{N}_{ij}dx^j. \tag{2.2}$$

The system of vector fields $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_j}\right)$ defines the local Berwald basis on T^*M. We consider the nonlinear connection $\mathcal{N}$ and denote

$$\tau_{ij} = \frac{1}{2}(\mathcal{N}_{ij} - \mathcal{N}_{ji}). \tag{2.3}$$

Definition 2.1. The nonlinear connection $\mathcal{N}$ on T^*M is called symmetric if

$$\omega(hX, hY) = 0, \quad X, Y \in \mathcal{X}(T^*M),$$

where h is the horizontal projector induced by nonlinear connection.

Locally, we obtain that the nonlinear connection is symmetric if and only if $\tau_{ij} = 0$, that is $\mathcal{N}_{ij} = \mathcal{N}_{ji}$. The following equations hold

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R_{kij}\frac{\partial}{\partial p_k}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_j}\right] = -\frac{\partial \mathcal{N}_{ir}}{\partial p_j}\frac{\partial}{\partial p_r}, \tag{2.4}$$

$$R_{ijk} = \frac{\delta \mathcal{N}_{jk}}{\delta x^i} - \frac{\delta \mathcal{N}_{ik}}{\delta x^j}. \tag{2.5}$$

The curvature of the nonlinear connection $\mathcal{N}$ on T^*M is given by $\Omega = -\mathbf{N}_h$ where h is the horizontal projector induced by $\mathcal{N}$ and $\mathbf{N}_h = \frac{1}{2}[h,h]$ is the Nijenhuis tensor associated to h. In local coordinates we obtain

$$\Omega = -\frac{1}{2}R_{ijk}\frac{\partial}{\partial p_k} \otimes dx^i \wedge dx^j,$$

where R_{ijk} is given by (2.5) and is called the curvature d-tensor of the nonlinear connection $\mathcal{N}$.
The curvature of a nonlinear connection is an obstruction to the integrability of the horizontal distribution. Using (2.4), it results that the horizontal distribution is integrable if and only if the curvature vanishes.

Definition 2.2. An almost tangent structure on the total space T^*M is a morphism $\mathcal{J} : \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$ of rank n such that $\mathcal{J}^2 = 0$. The almost tangent structure is called adapted if

$$Im\mathcal{J} = Ker\mathcal{J} = VT^*M.$$

Locally, an adapted almost tangent structure has the form

$$\mathcal{J} = t_{ij}dx^i \otimes \frac{\partial}{\partial p_j}, \tag{2.6}$$

where $t_{ij}(x,p)$ is a d-tensor field of rank n.

Proposition 2.1. *The adapted almost tangent structure $\mathcal{J}$ is integrable if and only if*

$$\frac{\partial t^{ij}}{\partial p_k} = \frac{\partial t^{kj}}{\partial p_i} \tag{2.7}$$

where $t_{ij}t^{jk} = \delta_i^k$.

Proof. We consider the Nijenhuis tensor associated to the adapted almost tangent structure $\mathcal{J}$

$$\mathbf{N}_{\mathcal{J}}(X,Y) = [\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}[\mathcal{J}X, Y] - \mathcal{J}[X, \mathcal{J}Y] + \mathcal{J}^2[X,Y], \quad \forall X, Y \in \mathcal{X}(T^*M)$$

and locally, we obtain

$$\mathbf{N}_{\mathcal{J}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left(t_{ik}\frac{\partial t_{js}}{\partial p_k} - t_{jk}\frac{\partial t_{is}}{\partial p_k}\right)\frac{\partial}{\partial p_s},$$

$$\mathbf{N}_{\mathcal{J}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_j}\right) = \mathbf{N}_{\mathcal{J}}\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = 0.$$

Thus $\mathcal{J}$ is integrable if and only if

$$t_{ik}\frac{\partial t_{js}}{\partial p_k} = t_{jk}\frac{\partial t_{is}}{\partial p_k}.$$

We multiply this equation with t^{sr} and using the fact that

$$\frac{\partial t_{js}}{\partial p_k}t^{sr} = -t_{js}\frac{\partial t^{sr}}{\partial p_k}, \quad \frac{\partial t_{is}}{\partial p_k}t^{sr} = -t_{is}\frac{\partial t^{sr}}{\partial p_k},$$

we obtain $t_{ik}t_{js}\frac{\partial t^{sr}}{\partial p_k} = t_{jk}t_{is}\frac{\partial t^{sr}}{\partial p_k}$. But $t_{jk}t_{is}\frac{\partial t^{sr}}{\partial p_k} = t_{js}t_{ik}\frac{\partial t^{kr}}{\partial p_s}$ and it results (2.7) □

Definition 2.3. The adapted almost tangent structure $\mathcal{J}$ is called symmetric if

$$\omega(\mathcal{J}X, Y) = \omega(\mathcal{J}Y, X). \tag{2.8}$$

Locally, this relation is equivalent with the symmetry of the tensor $t_{ij}(x,p)$.
If g is a pseudo-Riemannian metric on the vertical subbundle VT^*M, then there exists a unique adapted almost tangent structure $\mathcal{J}$ on T^*M such that

$$g(\mathcal{J}X, \mathcal{J}Y) = -\omega(\mathcal{J}X, Y), \quad X, Y \in \mathcal{X}(T^*M), \tag{2.9}$$

and we say that $\mathcal{J}$ is induced by the metric g.
Locally, if we consider

$$g(x,p) = g^{ij}dp_i \otimes dp_j,$$

then (2.9) implies that $t^{ij} = g^{ij}$.
A symmetric adapted almost tangent structure on T^*M induces a pseudo-Riemannian metric on the vertical subbundle, by (2.9).

Definition 2.4. The torsion of a nonlinear connection $\mathcal{N}$ on T^*M is defined by $\mathcal{T} = [\mathcal{J}, h]$, where h is the horizontal projector and $[\mathcal{J}, h]$ is the Frölicher-Nijenhuis bracket

$$[\mathcal{J}, h](X,Y) = [\mathcal{J}X, hY] + [hX, \mathcal{J}Y] + \mathcal{J}[X,Y] - \mathcal{J}[X, hY] - \\ -\mathcal{J}[hX, Y] - h[X, \mathcal{J}Y] - h[\mathcal{J}X, Y].$$

Locally, we consider

$$\mathcal{T} = \frac{1}{2}\mathcal{T}_{ijk}\frac{\partial}{\partial p_k} \otimes dx^i \wedge dx^j,$$

and by straightforward computation, it results

$$\mathcal{T}_{ijk} = t_{is}\frac{\partial \mathcal{N}_{jk}}{\partial p_s} - t_{js}\frac{\partial \mathcal{N}_{ik}}{\partial p_s} + \frac{\delta t_{jk}}{\delta x^i} - \frac{\delta t_{ik}}{\delta x^j}. \tag{2.10}$$

Let us consider the $\mathcal{F}(T^*M)-$linear application $F : \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$ defined by

$$F(hX) = \mathcal{J}X, \quad F(\mathcal{J}X) = -hX, \quad X \in \mathcal{X}(T^*M). \tag{2.11}$$

We obtain

$$F^2(hX) = F(\mathcal{J}(hX)) = -hX, \quad F^2(\mathcal{J}X) = F(-hX) = -\mathcal{J}X, \\ F\left(\frac{\delta}{\delta x^i}\right) = t_{ij}\frac{\partial}{\partial p_j}, \quad F\left(\frac{\partial}{\partial p_i}\right) = -t^{ij}\frac{\delta}{\delta x^j}.$$

These equations lead to the following results:

Proposition 2.2. *The map F has the properties:*
$i)$ F is an almost complex structure, $F^2 = -Id$.
$ii)$ The local expression of F is given by

$$F = t_{ij}\frac{\partial}{\partial p_i} \otimes dx^j - t^{ij}\frac{\delta}{\delta x^i} \otimes \delta p_j.$$

Proposition 2.3. *The almost complex structure F is integrable if and only if*

$$\mathcal{T}_{ijk} = 0, \quad R_{ijs} = t_{ik}\frac{\partial t_{js}}{\partial p_k} - t_{jk}\frac{\partial t_{is}}{\partial p_k}. \tag{2.12}$$

Proof. Let us consider the Nijenhuis tensor of the almost complex structure $\mathbf{N}_F$. We set

$$\mathbf{N}_F\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \mathbf{N}^k_{ij}\frac{\delta}{\delta x^k} + \mathbf{N}_{ij(k)}\frac{\partial}{\partial p_k},$$

$$\mathbf{N}_F\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_j}\right) = \mathbf{N}^{(j)k}_i\frac{\delta}{\delta x^k} + \mathbf{N}^{(j)}_{i(k)}\frac{\partial}{\partial p_k},$$

$$\mathbf{N}_F\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = -t^{is}t^{jk}\mathbf{N}_F(\frac{\delta}{\delta x^s}, \frac{\delta}{\delta x^k}),$$

These components are given by

$$\mathbf{N}_{ij}^{k} = \mathcal{T}_{ijs}t^{sk}, \quad \mathbf{N}_{ij(s)} = t_{ik}\frac{\partial t_{js}}{\partial p_k} - t_{jk}\frac{\partial t_{is}}{\partial p_k} - R_{ijs},$$

$$\mathbf{N}_{ij(k)} = \mathbf{N}_{i}^{(s)r}t_{sj}t_{kr}, \quad \mathbf{N}_{ij}^{k} = \mathbf{N}_{i(r)}^{(s)}t^{rk}t_{js},$$

hence $\mathbf{N}_F$ vanishes, if and only if (2.12) hold. □

Remark 2.1. A nonlinear connection on T^*M is a morphism $\mathcal{N} : \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$ which satisfies

$$\mathcal{J}\mathcal{N} = \mathcal{J}, \quad \mathcal{N}\mathcal{J} = -\mathcal{J}. \tag{2.13}$$

From [16] we have:

Definition 2.5. Let $\mathcal{J}$ be the adapted almost tangent structure on T^*M. A vector field $X \in \mathcal{X}(T^*M)$ is called $\mathcal{J}$-regular if it satisfies the equation

$$\mathcal{J}[X, \mathcal{J}Y] = -\mathcal{J}Y, \quad \forall Y \in \mathcal{X}(T^*M). \tag{2.14}$$

Locally, a vector field on T^*M given in local coordinates by

$$X = \xi^i(x,p)\frac{\partial}{\partial x^i} + \chi_i(x,p)\frac{\partial}{\partial p_i},$$

is $\mathcal{J}$-regular if and only if

$$t^{ij} = \frac{\partial \xi^j}{\partial p_i}, \tag{2.15}$$

where $t_{ij}t^{jk} = \delta_i^k$.

Remark 2.2. If the equation $\mathcal{J}[X, \mathcal{J}Y] = -\mathcal{J}Y$ is satisfied, for any $Y \in \mathcal{X}(T^*M)$, with the condition $rank[\frac{\partial \xi^j}{\partial p_i}] = n$, then $\mathcal{J}$ is an integrable structure.

Indeed, we have

$$\frac{\partial t^{ij}}{\partial p_k} = \frac{\partial \xi^j}{\partial p_k \partial p_i} = \frac{\partial \xi^j}{\partial p_i \partial p_k} = \frac{\partial t^{kj}}{\partial p_i},$$

and using (2.7) it results that $\mathcal{J}$ is integrable. From [16] we set:

Theorem 2.1. *Let $\mathcal{J}$ be an adapted tangent structure and X a $\mathcal{J}$ -regular vector field on T^*M. Then*

$$\mathcal{N} = -\mathcal{L}_X\mathcal{J}, \tag{2.16}$$

is a nonlinear connection on T^*M.

In local coordinates the coefficients of the above nonlinear connection are given by

$$\mathcal{N}_{ij} = \frac{1}{2}\left(t_{ik}\frac{\partial \chi_j}{\partial p_k} - t_{kj}\frac{\partial \xi^k}{\partial x^i} - X(t_{ij})\right). \tag{2.17}$$

Proposition 2.4. *The torsion of the nonlinear connection $\mathcal{N} = -\mathcal{L}_X\mathcal{J}$ vanishes.*

Proof. From the expression of the horizontal projector $h = \frac{1}{2}(Id - \mathcal{N})$ we obtain

$$\mathcal{T} = [\mathcal{J}, h] = \frac{1}{2}([\mathcal{J}, Id] + [\mathcal{J}, -[X, \mathcal{J}]) = \frac{1}{2}[\mathcal{J}, [\mathcal{J}, X]],$$

and using the Jacobi identity, it results $\mathcal{T} = 0$. □

A vector field X on T^*M is called a Hamiltonian vector field if it is $\mathcal{J}$-regular and

$$\mathcal{L}_X\omega = 0. \tag{2.18}$$

In local coordinates, for $X = \xi^i\frac{\partial}{\partial x^i} + \chi_i\frac{\partial}{\partial p_i}$, then the condition (2.18) is equivalent with [16]

$$a)\ \frac{\partial \xi^j}{\partial p_i} = \frac{\partial \xi^i}{\partial p_j}, \quad b)\ \frac{\partial \chi_i}{\partial p_j} = -\frac{\partial \xi^j}{\partial x^i}, \quad c)\ \frac{\partial \chi_i}{\partial x^j} = \frac{\partial \chi_j}{\partial x^i}. \tag{2.19}$$

Definition 2.6. A vector field $X \in \mathcal{X}(T^*M)$ is a semi-Hamiltonian vector field if it is $\mathcal{J}$-regular and satisfies the relation

$$i_\nu(\mathcal{L}_X\omega) = 0, \quad \forall \nu \in \Gamma(VT^*M).$$

where i_ν is the interior product.

By direct computation, it results that in the case of semi-Hamiltonian vector field, only the conditions (2.19) $a)$ and $b)$ are satisfied.

3. Hamiltonian formalism

A Hamilton space [12] is a pair (M, H) where M is a differentiable, $n-$dimensional manifolds and H is a function on T^*M with the properties:

1° $H : (x, p) \in T^*M \to H(x, p) \in \mathbb{R}$ is differentiable on T^*M and continuous on the null section of the projection $\tau : T^*M \to M$.

2° The Hessian of H with respect to p_i is nondegenerate

$$g^{ij} = \frac{\partial^2 H}{\partial p_i \partial p_j}, \quad rank \left\| g^{ij}(x,p) \right\| = n, \quad on\ \widetilde{T^*M} = T^*M \backslash \{0\}. \tag{3.1}$$

3° d-tensor field $g^{ij}(x, p)$ has constant signature on $\widetilde{T^*M}$.

Every Hamiltonian H on T^*M induces a pseudo-Riemannian metric g_{ij} with $g_{ij}g^{jk} = \delta_i^k$ and g^{jk} given by (3.1) on VT^*M. It induces a unique adapted almost tangent structure, denoted

$$\mathcal{J}_H = g_{ij} dx^i \otimes \frac{\partial}{\partial p_j},$$

such that (2.9) is satisfied. Moreover $\mathcal{J}_H$ is symmetric and integrable, because (2.7) is fulfilled. A $\mathcal{J}$-regular vector field induced by the regular Hamiltonian H is given by

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} + \chi_i \frac{\partial}{\partial p_i}. \tag{3.2}$$

The relation between the symplectic structure ω and the Poisson bracket $\{\cdot, \cdot\}$ can be given using the notion of Hamiltonian system. A Hamiltonian system is a triple (T^*M, ω, H) formed by the cotangent bundle T^*M, the canonical symplectic structure ω and a differentiable Hamiltonian, which satisfies the properties:

1° There exists a unique Hamiltonian vector field $X_H \in \mathcal{X}(T^*M)$ such that

$$i_{X_H}\omega = -dH,$$

($i_{X_H}\omega$ is the interior product) given by

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}.$$

2° The integral curves of the Hamiltonian vector field X_H are given by Hamilton's equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \tag{3.3}$$

or, equivalently

$$\frac{dx^i}{dt} = \{H, x^i\}, \quad \frac{dp_i}{dt} = \{H, p_i\}.$$

The Theorem 2.1 leads to the following result:

Corollary 3.1. *The symmetric nonlinear connection*

$$\mathcal{N} = -\mathcal{L}_{X_H} \mathcal{J}_H, \tag{3.4}$$

has the coefficients given by

$$\mathcal{N}_{ij} = \frac{1}{2} \left(\{g_{ij}, H\} - \left(g_{ik} \frac{\partial^2 H}{\partial p_k \partial x^j} + g_{jk} \frac{\partial^2 H}{\partial p_k \partial x^i} \right) \right), \tag{3.5}$$

and is called the canonical nonlinear connection of the Hamilton space (M, H).

This connection has been introduced by R. Miron [10], using the Legendre transformation defined by H and the canonical nonlinear connection of the Lagrange space, dual to (M, H).

3.1. The duality between Lagrangian and Hamiltonian formalism.

For convenience, we will denote by (x^i, y^i) the coordinates in a local chart on TM and by (q^i, p_i) the coordinates in a local chart on T^*M . Let us consider the regular Hamiltonian $H(q, p)$ on T^*M which induces a local diffeomorphism $\Phi : T^*M \to TM$ given by

$$x^i = q^i, \quad y^i = \xi^i(q, p) = \frac{\partial H}{\partial p_i}. \tag{3.6}$$

and Φ^{-1} has the following components

$$q^i = x^i, \quad p_i = \zeta_i(x, y) = \frac{\partial L}{\partial y^i}, \tag{3.7}$$

where

$$L(x, y) = \zeta_i y^i - H(q, p), \tag{3.8}$$

is the Legendre transformation. From the condition for Φ^{-1} to be the inverse of Φ we obtain [16]

$$\frac{\partial \zeta_i}{\partial y^j} \circ \Phi = g_{ij}, \quad \frac{\partial \zeta_i}{\partial x^j} \circ \Phi = -g_{ik}\frac{\partial \xi^k}{\partial q^j}, \tag{3.9}$$

where

$$g^{ij} = \frac{\partial \xi^j}{\partial p_i} = \frac{\partial^2 H}{\partial p_i \partial p_j}, \quad g_{ij}g^{jk} = \delta_i^k.$$

We have

$$\Phi_* \frac{\partial}{\partial p_i} = (g^{ik} \circ \Phi^{-1})\frac{\partial}{\partial y^k},$$

$$\Phi_* \frac{\partial}{\partial q^i} = \frac{\partial}{\partial x^i} + \left(\frac{\partial \xi^k}{\partial q^i} \circ \Phi^{-1}\right)\frac{\partial}{\partial y^k}, \tag{3.10}$$

$$\Phi_*^{-1}\frac{\partial}{\partial y^i} = g_{ki}\frac{\partial}{\partial p_k}, \quad \Phi_*^{-1}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial q^i} - g_{kh}\frac{\partial \xi^h}{\partial q^i}\frac{\partial}{\partial p_k},$$

where Φ_* is the tangent application of Φ.

Theorem 3.2. *Let X be a $\mathcal{J}$-regular vector field on T^*M and $\Phi : T^*M \to TM$ the diffeomorphism induced by the Hamiltonian H. Then the vector field Φ_*X is a semispray on TM whose induced nonlinear connection N is the image by Φ of the connection $\mathcal{N}$ induced by X on T^*M.*

Proof. We consider $X = \xi^i \frac{\partial}{\partial q^i} + \rho_i \frac{\partial}{\partial p_i}$ a $\mathcal{J}$-regular vector field on T^*M and from (3.10) it results

$$S = \Phi_* X = \xi^i \left(\frac{\partial}{\partial x^i} + \left(\frac{\partial \xi^k}{\partial q^i} \circ \Phi^{-1}\right)\frac{\partial}{\partial y^k} + \rho_i \left(g^{ik} \circ \Phi^{-1}\right)\frac{\partial}{\partial y^k}\right),$$

and using (3.6) we obtain

$$S = y^i \frac{\partial}{\partial x^i} + S^k \frac{\partial}{\partial y^k},$$

where

$$S^k \circ \Phi = \xi^i \frac{\partial \xi^k}{\partial q^i} + \rho_i \frac{\partial \xi^k}{\partial p_i}.$$

We denote by $\widetilde{\Phi}$ the application induced by Φ at the level of tensor fields, and using (3.10) we have

$$\widetilde{\Phi}\mathcal{J} = \left(g_{ij}\Phi^{-1}\right)\Phi_*\frac{\partial}{\partial p_i} \otimes \Phi_*^{-1}(dq^j) = \frac{\partial}{\partial y^i} \otimes dx^i = J,$$

which leads to

$$N = -\mathcal{L}_S J = -\mathcal{L}_{\Phi_* X}\widetilde{\Phi}\mathcal{J} = -\widetilde{\Phi}\left(\mathcal{L}_X \mathcal{J}\right) = \widetilde{\Phi}\mathcal{N}.$$

that is the nonlinear connection N on TM is the image of nonlinear connection $\mathcal{N}$ on T^*M by application $\widetilde{\Phi}$. □

The previous theorem shows that the decomposition $VT^*M \oplus HT^*M$ induced by the nonlinear connection $\mathcal{N}$ on T^*M is mapped by Φ_* into the decomposition $VTM \oplus HTM$ induced by N. It implies (see also [7, 16])

Corollary 3.2. *The following equations hold*

$$\Phi_* \frac{\delta}{\delta q^i} = \frac{\delta}{\delta x^i}, \quad \Phi_*^{-1}\frac{\delta}{\delta x^i} = \frac{\delta}{\delta q^i}, \tag{3.11}$$

$$\mathcal{N}_{ij}(q,p) = -\left(N_i^k(x,y) + \frac{\partial \xi^k}{\partial q^i}\right)g_{jk}, \tag{3.12}$$

$$R_{ijk}g^{is} = R_{jk}^s \circ \Phi, \quad R_{jk}^s \frac{\partial \zeta_i}{\partial y^s} = R_{ijk} \circ \Phi^{-1}, \tag{3.13}$$

$$N_i^j \circ \Phi = -\frac{\delta \xi^j}{\delta q^i}, \quad \mathcal{N}_{ij} \circ \Phi^{-1} = -\frac{\delta \zeta_i}{\delta x^j}. \tag{3.14}$$

Proof. We have

$$\Phi_*^{-1}\left(\frac{\delta}{\delta x^i}\right) = \frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \mathcal{N}_{ij}\frac{\partial}{\partial p_j},$$

and on the other hand

$$\Phi_*^{-1}\left(\frac{\delta}{\delta x^i}\right) = \Phi_*^{-1}\left(\frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}\right) = \frac{\partial}{\partial q^i} - g_{kh}\frac{\partial \xi^h}{\partial q^i}\frac{\partial}{\partial p_k} - N_i^j g_{kj}\frac{\partial}{\partial p_k},$$

and it results (3.12). Next, because Φ^{-1} is a diffeomorphism, we obtain

$$\left[\Phi_*^{-1}(\frac{\delta}{\delta x^i}), \Phi_*^{-1}(\frac{\delta}{\delta x^j})\right] = R^k_{ij}\Phi_*^{-1}(\frac{\partial}{\partial y^k}),$$

$$\left[\Phi_*^{-1}(\frac{\delta}{\delta x^i}), \Phi_*^{-1}(\frac{\delta}{\delta x^j})\right] = \left[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right] = R_{ijk}\frac{\partial}{\partial p_k},$$

and using (3.10) we obtain (3.13). □

The next result shows the relation between the metric nonlinear connection on TM and the semi-Hamiltonian vector field on T^*M, via Legendre transformation.

Theorem 3.3. *Let us consider a semispray S on TM and $\Phi^{-1} : TM \to T^*M$ the diffeomorphism induced by a regular Hamiltonian H. Then $Y = \Phi_*^{-1}S$ is a semi-Hamiltonian vector field on T^*M if and only if the nonlinear connection $N = -\mathcal{L}_S J$ induced by semispray on TM is metric and*

$$\frac{\delta \zeta_i}{\delta x^j} = \frac{\delta \zeta_j}{\delta x^i}, \tag{3.15}$$

with $\zeta_i = \frac{\partial L}{\partial y^i}$.

Proof. We consider a semispray $S = y^i\frac{\partial}{\partial x^i} + S^i\frac{\partial}{\partial y^i}$ on TM and from (3.10) it results

$$\Phi_*^{-1}S = \xi^i\frac{\partial}{\partial q^i} + \left(-\xi^i g_{kj}\frac{\partial \xi^j}{\partial q^i} + S^i g_{ik}\right)\frac{\partial}{\partial p_k}.$$

This, together with the conditions (2.19) b) and (3.10), is equivalent with

$$g^{kj}\left(\frac{\partial \zeta_i}{\partial x^k} - \frac{\partial \zeta_k}{\partial x^i} + \frac{\partial S^l}{\partial y^k}g_{li} + \xi^l\frac{\partial g_{ik}}{\partial x^l} + S^l\frac{\partial g_{ik}}{\partial y^l}\right) = 0,$$

and using (1.3) we obtain

$$\frac{\partial \zeta_i}{\partial x^k} - \frac{\partial \zeta_k}{\partial x^i} = S(g_{ik}) - 2N^l_k g_{li}, \tag{3.16}$$

and it results (interchanging i with k)

$$S(g_{ik}) - N^l_k g_{li} - N^l_i g_{lk} = 0,$$

(which means that N on TM is a metric nonlinear connection), and

$$\frac{\partial \zeta_i}{\partial x^k} - \frac{\partial \zeta_k}{\partial x^i} = N^l_i\frac{\partial \zeta_k}{\partial y^l} - N^l_k\frac{\partial \zeta_i}{\partial y^l},$$

where, $g_{ij} = \partial\zeta_i/\partial y^j$, which leads to

$$\frac{\delta \zeta_i}{\delta x^k} = \frac{\delta \zeta_k}{\delta x^i}.$$

Conversely, if 3.15 is satisfied and N is a metric nonlinear connection, then we obtain 3.16 which ends the proof. □

Acknowledgements. The author wishes to express his thanks to the referee for many useful remarks concerning this paper.

References

[1] Anastasiei, M., *Finsler vector bundles – metrizable connections*, Period. Math. Hung. **48** (2004), 1-2, 83-91
[2] Atanasiu, Gh. and Klepp, F. C., *Nonlinear connection in cotangent bundle*, Publ. Math. Debrecen **39** (1991), 1-2, 107-111
[3] Bucătaru, I., *Metric nonlinear connection*, Diff. Geom. Appl. **25** (2007), 335-343
[4] Crampin, M., *Tangent bundle geometry for Lagrangian dynamics*, J. Phys. A: Math. Gen. **16** (1983), 3755-3772
[5] Crâşmăreanu, M., *Metrizable systems of autonomous second order differential equations*, Carpathian J. Math. **25**, No. 2, 163-176 (2009)
[6] Grifone, J., *Structure presque tangente et connections I* , Ann. Inst. Fourier, **22** (1972), No. 1, 287-334
[7] Hrimiuc, D. and Shimada, H., *On the L-duality between Lagrange and Hamilton Manifold*, Nonlinear World **3** (1996), 613-641
[8] Hrimiuc, D. and Popescu, L., *Nonlinear connections on dual Lie algebroids*, Balkan J. Geom. Appl. **11** (2006), No. 1, 73-80
[9] Martinez E., Carinena J. F. and Sarlet W., *Derivations of differential forms along the tangent bundle projection II*, Diff. Geom. Appl. **3** (1993), No. 1, 1-29
[10] Miron, R., *Hamilton geometry,* An. Stiinţ. Univ. Al. I. Cuza Iaşi, S. I., Math. **35** (1989), 33-67
[11] Miron R., Anastasiei M. and Ianuş, S., *The geometry of the dual of a vector bundle,* Publ. de l'Inst. Math. **46** (1989), 145-162
[12] Miron R., Hrimiuc D., Shimada H. and Sabău, S., *The geometry of Hamilton and Lagrange spaces*, Kluwer Academic Publishers **118**, (2001)
[13] Mitric, G., *Regular 1-forms and connections on the tangent and cotangent bundles*, Tensor **51** (1992), No. 3, 229-234
[14] Mitric, G., *Connections and regularity on the tangent bundle*, An. Stiinţ. Univ. Al. I. Cuza Iaşi. S. I, Math. **44** (1998), No. 2, 373-384
[15] Mitric, G., *Connections and regularity on the cotangent bundle*, Publ. Math. Debrecen **55** (1999), No. 1-2, 141-154
[16] Oproiu, V., *Regular vector fields and connections on cotangent bundle*, An. Stiinţ. Univ. Al. I. Cuza Iaşi. S.I, Math. **37** (1991), No. 1, 87-104
[17] Szilasi, Z. and Muzsnay, Z., *Nonlinear connections and the problem of metrizability*, Publ. Math. Debrecen **42** (1993), No. 1-2, 175-192
[18] Yano, K. and Ishihara, S., *Tangent and cotangent bundles*, M. Dekker Inc., New-York, (1973)

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