CARPATHIAN J. MATH.
Volume **37** (2021), No. 1,
Pages 45 - 51

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2021.01.05

Datko type characterizations for nonuniform polynomial dichotomy

ROVANA BORUGA (TOMA) and MIHAIL MEGAN

ABSTRACT. The aim of the present paper is to give two characterization theorems of Datko type for the nonuniform polynomial dichotomy concept with respect to invariant projection families and also with respect to strongly invariant projection families.

1. INTRODUCTION

In the last decades, one of the most important topics discussed in the field of dynamical systems is the uniform exponential dichotomy behavior of evolution operators in Banach spaces. This concept was mentioned for the first time in the work of O. Perron [14] and refers to the fact that the state space can be decomposed in every moment into a direct sum of two subspaces: the stable subspace and the unstable subspace.

Over the years, the uniform exponential dichotomy notion has been generalized in various forms and researchers dedicated their study to the nonuniform case [8], [9], [12], [13], [16]. The starting point in studying this type of behavior is in the work [2] of the authors L. Barreira and C. Valls who had a notable contribution in developing it.

Also, in this direction it is important to mention an interesting paper of D. Dragičević [5] where the author deals with the nonuniform exponential behavior. More precisely, he obtains some Datko-type characterizations of several classes of a strong nonuniform exponential behavior: contractions, expansions and dichotomies, for both continuous and discrete time cases. Recently, N. Lupa and L.H.Popescu [7], being motivated by the work of Dragičević [5], generalize Datko's theorem on a class of admissible Banach function spaces, using completely different type of techniques from those in [5].

Moreover, in order to generalize the exponential bahavior, L. Barreira and C. Valls [1] introduced in 2009 a more general behavior, namely the nonuniform polynomial dichotomy, which they studied for the continuous case. They were followed by A. J. G. Bento and C. M. Silva [3] who considered the same concept for discrete time systems. Other results in this area were obtained in the papers [4], [10], [11], [15].

In this paper we obtain necessary and sufficient conditions of Datko type for the nonuniform polynomial dichotomy concept of evolution operators regarding two types of projections families: invariant and respectively strongly invariant projection families to the evolution operator.

2. NOTATIONS AND DEFINITIONS

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. We will denote by $\|.\|$ the norm on X and on $\mathcal{B}(X)$ and let I be the identity operator on X. We also consider Δ and T two sets defined by

Received: 16.06.2020. In revised form: 16.12.2020. Accepted: 23.12.2020
2010 *Mathematics Subject Classification.* 34D05, 34D09.
Key words and phrases. *Nonuniform polynomial dichotomy, invariant projections, strongly invariant projections.*
Corresponding author: Rovana Boruga(Toma); rovanaboruga@gmail.com

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}, \qquad T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

Definition 2.1. An application $U : \Delta \to \mathcal{B}(X)$ is called *evolution operator* on X if

(e_1)**:** $U(t,t) = I$ for every $t \geq 0$.
(e_2)**:** $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.

Definition 2.2. An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *strongly measurable* if for all $(s,x) \in \mathbb{R}_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.

Definition 2.3. An application $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be a *projection family* on X if $P^2(t) = P(t)$, for all $t \geq 0$.

Remark 2.1. If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is a projection family on X, then the mapping $Q : \mathbb{R}_+ \to \mathcal{B}(X), Q(t) = I - P(t)$ is also a projection family on X, which is called the complementary projection of P. In what follows, we will denote

$$U_P(t,s) = U(t,s)P(s), \qquad U_Q(t,s) = U(t,s)Q(s), \text{ for all } (t,s) \in \Delta.$$

Remark 2.2. It is easy to see that:

$$U_P(t,t) = P(t), \qquad U_Q(t,t) = Q(t), \text{ for all } (t,s) \in \Delta.$$
$$U_P(t,t_0) = U_P(t,s)U_P(s,t_0), \quad U_Q(t,t_0) = U_Q(t,s)U_Q(s,t_0), \forall (t,s,t_0) \in T.$$

Definition 2.4. A projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is *strongly continuous* if for all $x \in X$ the mapping $t \mapsto P(t)x$ is continuous on $\mathbb{R}_+$.

Definition 2.5. A projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is *invariant* to the evolution operator $U : \Delta \to \mathcal{B}(X)$ if $U(t,s)P(s) = P(t)U(t,s)$, for all $(t,s) \in \Delta$.
If P is an invariant projection family to the evolution operator $U : \Delta \to \mathcal{B}(X)$, we will say that (U,P) is a dichotomic pair.

Definition 2.6. A projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is *strongly invariant* to the evolution operator $U : \Delta \to \mathcal{B}(X)$ if it is invariant to the operator U and for all $(t,s) \in \Delta,\ U(t,s) : KerP(s) = RangeQ(s) \to KerP(t) = RangeQ(t)$ is invertible.

Remark 2.3. If the projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant to the evolution operator $U : \Delta \to \mathcal{B}(X)$ and $Q : \mathbb{R}_+ \to \mathcal{B}(X)$ is the complementary projection family of P, then for all $(t,s) \in \Delta$ there exists the application $V : \Delta \to \mathcal{B}(X),\ V(t,s) : RangeQ(t) \to RangeQ(s)$ that satisfies the following properties:

(v_1)**:** $U(t,s)V(t,s)Q(t)x = Q(t)x$
(v_2)**:** $V(t,s)U(t,s)Q(s)x = Q(s)x$
(v_3)**:** $V(t,t_0) = V(s,t_0)V(t,s)$
(v_4)**:** $V(t,s)Q(t) = Q(s)V(t,s)Q(t)$,

for all $(t,s,t_0) \in T$. We will denote $V_Q(t,s) = V(t,s)Q(t)$.

Proof. See [6]. □

Definition 2.7. We say that the pair (U,P) *has nonuniform polynomial growth (n.p.g.)* if there are a nondecreasing function $M : \mathbb{R}_+ \to [1,\infty)$ and a positive constant $\omega > 0$ such that

(npg_1)**:** $(s+1)^\omega \|U_P(t,s)x\| \leq M(s)(t+1)^\omega \|P(s)x\|$
(npg_2)**:** $(s+1)^\omega \|Q(s)x\| \leq M(t)(t+1)^\omega \|U_Q(t,s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.4. The pair (U,P) has nonuniform polynomial growth if and only if there are a nondecreasing function $M : \mathbb{R}_+ \to [1,\infty)$ and a positive constant $\omega > 0$ such that

(npg_1'): $(s+1)^\omega \|U_P(t,t_0)x_0\| \leq M(s)(t+1)^\omega \|U_P(s,t_0)x_0\|$
(npg_2'): $(s+1)^\omega \|U_Q(s,t_0)x_0\| \leq M(t)(t+1)^\omega \|U_Q(t,t_0)x_0\|$

for all $(t,s,t_0,x_0) \in T \times X$.

Remark 2.5. As particular cases we have:

(1) If $M(t) = M(t+1)^\varepsilon$, then we obtain the polynomial growth property (p.g.).
(2) If $M(t) = M$, then we obtain the uniform polynomial growth property (u.p.g.).

Remark 2.6. It is obvious that $u.p.g. \Rightarrow p.g. \Rightarrow n.p.g.$, but the converse implications are not true.

Definition 2.8. We say that the pair (U,P) *has strongly nonuniform polynomial growth (s.n.p.g.)* if there are a nondecreasing function $M : \mathbb{R}_+ \to [1,\infty)$ and $\omega > 0$ such that

$(snpg_1)$: $(s+1)^\omega \|U_P(t,t_0)x_0\| \leq M(t_0)(t+1)^\omega \|U_P(s,t_0)x_0\|$
$(snpg_2)$: $(s+1)^\omega \|U_Q(s,t_0)x_0\| \leq M(t)(t+1)^\omega \|U_Q(t,t_0)x_0\|$

for all $(t,s,t_0,x_0) \in T \times X$.

Remark 2.7. We have that $s.n.p.g$ implies $n.p.g.$.

Definition 2.9. We say that the pair (U,P) is *nonuniformly polynomially dichotomic (n.p.d.)* if there are a nondecreasing function $N : \mathbb{R}_+ \to [1,\infty)$ and a positive constant $\nu > 0$ such that:

(npd_1): $(t+1)^\nu \|U_P(t,s)x\| \leq N(s)(s+1)^\nu \|P(s)x\|$
(npd_2): $(t+1)^\nu \|Q(s)x\| \leq N(t)(s+1)^\nu \|U_Q(t,s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.8. The pair (U,P) is nonuniformly polynomially dichotomic if and only if there are a nondecreasing function $N : \mathbb{R}_+ \to [1,\infty)$ and $\nu > 0$ such that:

(npd_1'): $(t+1)^\nu \|U_P(t,t_0)x_0\| \leq N(s)(s+1)^\nu \|U_P(s,t_0)x_0\|$
(npd_2'): $(t+1)^\nu \|U_Q(s,t_0)x_0\| \leq N(t)(s+1)^\nu \|U_Q(t,t_0)x_0\|$

for all $(t,s,t_0,x_0) \in \Delta \times X$.

Remark 2.9. As particular cases we have:

(1) If $N(s) = N(s+1)^\varepsilon$ we obtain the polynomial dichotomy concept (p.d.).
(2) If $N(s) = N$ we obtain the uniform polynomial dichotomy concept (u.p.d.).

Remark 2.10. If (U,P) is n.p.d., than (U,P) has n.p.g., but the converse implication is not true.

3. The main result

The main result of the present paper consists of presenting two characterization theorems of Datko type using invariant projection families, respectively strongly invariant projections families.

Theorem 3.1. *Let (U,P) be a dichotomic pair with strongly nonuniform polynomial growth. Then (U,P) is nonuniformly polynomially dichotomic if and only if there exist a nondecreasing function $D : \mathbb{R}_+ \to [1,\infty)$ and a constant $d > 0$ such that*

$$(npD_1): \int_t^\infty \left(\frac{\tau+1}{s+1}\right)^d \cdot \frac{1}{\tau+1} \|U_P(\tau,s)x\| d\tau \leq D(t) \|U_P(t,s)x\|$$

$$(npD_2): \int_s^t \left(\frac{t+1}{\tau+1}\right)^d \cdot \frac{1}{\tau+1} \|U_Q(\tau,s)x\| d\tau \leq D(t) \|U_Q(t,s)x\|$$

for all $(t,s,x)\in\Delta\times X$.

Proof. Necessity. Let $d\in(0,\nu)$.
For (npD_1) we have

$$\int_t^\infty \left(\frac{\tau+1}{s+1}\right)^d\cdot\frac{1}{\tau+1}\|U_P(\tau,s)x\|d\tau\leq N(t)\int_t^\infty \left(\frac{\tau+1}{s+1}\right)^d\cdot\frac{1}{\tau+1}\cdot\left(\frac{\tau+1}{t+1}\right)^{-\nu}\|U_P(t,s)x\|d\tau=$$

$$=N(t)(t+1)^\nu(s+1)^{-d}\|U_P(t,s)x\|\int_t^\infty(\tau+1)^{d-\nu-1}d\tau=$$

$$=\frac{N(t)}{\nu-d}(t+1)^\nu(s+1)^{-d}(t+1)^{d-\nu}\|U_P(t,s)x\|=\frac{N(t)}{\nu-d}\left(\frac{t+1}{s+1}\right)^d\|U_P(t,s)x\|\leq$$

$$\leq\frac{N(t)(t+1)^d}{\nu-d}\|U_P(t,s)x\|\leq D(t)\|U_P(t,s)x\|,\ \text{where } D(t)=1+\frac{N(t)(t+1)^d}{\nu-d}.$$

For $(Dnpd_2)$ we do a similar computation and we obtain:

$$\int_s^t \left(\frac{t+1}{\tau+1}\right)^d\cdot\frac{1}{\tau+1}\|U_Q(\tau,s)x\|d\tau\leq N(t)\int_s^t \left(\frac{t+1}{\tau+1}\right)^d\cdot\frac{1}{\tau+1}\cdot\left(\frac{t+1}{\tau+1}\right)^{-\nu}\|U_Q(t,s)x\|d\tau=$$

$$=N(t)(t+1)^d(t+1)^{-\nu}\|U_Q(t,s)x\|\int_s^t(\tau+1)^{\nu-d-1}d\tau=$$

$$=\frac{N(t)}{\nu-d}(t+1)^d(t+1)^{-\nu}(t+1)^{\nu-d}\|U_Q(t,s)x\|=\frac{N(t)}{\nu-d}\|U_Q(t,s)x\|\leq D(t)\|U_Q(t,s)x\|,$$

because $\dfrac{N(t)}{\nu-d}\leq\dfrac{N(t)}{\nu-d}+1\leq\dfrac{N(t)(t+1)^d}{\nu-d}+1.$

Sufficiency.
We suppose that there exist a nondecreasing function $D:\mathbb{R}_+\to\mathcal{B}(X)$ and a positive constant $d=\nu>0$ such that the relations (npD_1) and (npD_2) are satisfied. Firstly we prove that (npD_1) implies (npd_1). Let $t\geq 2s+1$. Then, we have

$$\left(\frac{t+1}{s+1}\right)^d\|U_P(t,s)x\|=\frac{2}{t+1}\int_{\frac{t-1}{2}}^t\left(\frac{t+1}{s+1}\right)^d\|U_P(t,s)x\|d\tau\leq$$

$$\leq\frac{2}{t+1}M(s)\int_{\frac{t-1}{2}}^t\left(\frac{t+1}{s+1}\right)^d\left(\frac{t+1}{\tau+1}\right)^\omega\|U_P(\tau,s)x\|d\tau=$$

$$=\frac{2}{t+1}M(s)\int_{\frac{t-1}{2}}^t\left(\frac{\tau+1}{s+1}\right)^d\left(\frac{t+1}{\tau+1}\right)^{\omega+d}\|U_P(\tau,s)x\|d\tau\leq$$

$$\leq 2M(s)\int\limits_{\frac{t-1}{2}}^{t}\left(\frac{\tau+1}{s+1}\right)^{d}\frac{1}{\tau+1}\left(\frac{t+1}{\tau+1}\right)^{\omega+d}\|U_P(\tau,s)x\|d\tau\leq$$

$$\leq 2^{\omega+d+1}M(s)\int\limits_{s}^{\infty}\left(\frac{\tau+1}{s+1}\right)^{d}\cdot\frac{1}{\tau+1}\|U_P(\tau,s)x\|d\tau\leq$$

$$\leq 2^{\omega+d+1}M(s)D(s)\|P(s)x\|\leq N(s)\|P(s)x\|,$$

where $N(s)=1+2^{\omega+d+1}\cdot M(s)D(s)$. Now, let $t\in[s,2s+1)$.

$$\left(\frac{t+1}{s+1}\right)^{d}\|U_P(t,s)x\|\leq M(s)\left(\frac{t+1}{s+1}\right)^{d+\omega}\|P(s)x\|\leq 2^{d+\omega}M(s)\|P(s)x\|\leq$$
$$\leq N(s)\|P(s)x\|.$$

Now, we prove that (npD_2) implies (npd_2) and we consider firstly $t\geq 2s+1$. We obtain

$$\left(\frac{t+1}{s+1}\right)^{d}\|Q(s)x\|=\frac{1}{s+1}\int\limits_{s}^{2s+1}\left(\frac{t+1}{s+1}\right)^{d}\|Q(s)x\|d\tau\leq$$

$$\leq\frac{1}{s+1}\int\limits_{s}^{2s+1}\left(\frac{t+1}{s+1}\right)^{d}M(\tau)\left(\frac{\tau+1}{s+1}\right)^{\omega}\|U_Q(\tau,s)x\|d\tau\leq$$

$$\leq\frac{1}{s+1}\int\limits_{s}^{2s+1}M(t)\left(\frac{t+1}{\tau+1}\right)^{d}\left(\frac{\tau+1}{s+1}\right)^{\omega+d}\|U_Q(\tau,s)x\|d\tau\leq$$

$$\leq M(t)\int\limits_{s}^{t}\left(\frac{t+1}{\tau+1}\right)^{d}\cdot\frac{1}{\tau+1}\cdot\left(\frac{\tau+1}{s+1}\right)^{\omega+d+1}\|U_Q(\tau,s)x\|d\tau\leq$$

$$\leq 2^{\omega+d+1}M(t)D(t)\|U_Q(t,s)x\|\leq N(t)\|U_Q(t,s)x\|$$

where $N(t)=1+2^{\omega+d+1}\cdot M(t)D(t)$.
If $t\in[s,2s+1)$ we obtain

$$\left(\frac{t+1}{s+1}\right)^{d}\|Q(s)x\|\leq M(t)\left(\frac{t+1}{s+1}\right)^{d+\omega}\|U_Q(t,s)x\|\leq 2^{d+\omega}M(t)\|U_Q(t,s)x\|.$$

□

Remark 3.11. Similar approaches of this theorem can be found in [16].

Theorem 3.2. *Let $U:\Delta\to\mathcal{B}(X)$ be an evolution operator strongly measurable, $P:\mathbb{R}_+\to\mathcal{B}(X)$ a projection family strongly continuous and strongly invariant to U. If the pair (U,P) has nouniform polynomial growth then (U,P) is nonuniformly polynomially dichotomic if and only if there exist a nondecreasing function $D:\mathbb{R}_+\to[1,\infty)$ and a positive constant $d>0$ such that*

(npD_1'): $$\int\limits_{t}^{\infty}\left(\frac{\tau+1}{s+1}\right)^{d}\cdot\frac{1}{\tau+1}\|U_P(\tau,s)x\|d\tau\leq D(t)\|U_P(t,s)x\|$$

(npD_2'): $$\int\limits_{s}^{t}\left(\frac{t+1}{\tau+1}\right)^{d}\cdot\frac{1}{\tau+1}\|V_Q(t,\tau)x\|d\tau\leq D(t)\|Q(t)x\|$$

for all $(t,s,x)\in\Delta\times X$.

Proof. It is obvious that (npD_1) and (npD_1') coincide. We have to prove that relations (npD_2) and (npD_2') are equivalent. *Necessity.* We suppose that (U,P) is n.p.d. Let $d \in (0,\nu)$.

$$\int_s^t \left(\frac{t+1}{\tau+1}\right)^d \cdot \frac{1}{\tau+1}\|V_Q(t,\tau)x\|d\tau \leq \int_s^t \left(\frac{t+1}{\tau+1}\right)^d \cdot \frac{1}{\tau+1}N(t)\left(\frac{t+1}{\tau+1}\right)^{-\nu}\|Q(t)x\|d\tau =$$

$$= N(t)\int_s^t (t+1)^{d-\nu}(\tau+1)^{\nu-d-1}\|Q(t)x\|d\tau = N(t)(t+1)^{d-\nu}\|Q(t)x\|\int_s^t(\tau+1)^{\nu-d-1}d\tau \leq$$

$$\leq \frac{N(t)}{\nu-d}\|Q(t)x\| \leq D(t)\|Q(t)x\|, \text{where } D(t) = 1+\frac{N(t)}{\nu-d}.$$

Sufficiency. Let $t \geq 2s+1$.

$$(t+1)^d\|V_Q(t,s)x\| = (t+1)^d\|V(t,s)Q(t)x\| =$$

$$= \frac{1}{s+1}\int_s^{2s+1}(t+1)^d\|V(t,s)Q(t)x\|d\tau = \frac{1}{s+1}\int_s^{2s+1}(t+1)^d\|V(\tau,s)V(t,\tau)Q(t)x\|d\tau \leq$$

$$\leq \frac{M(\tau)}{s+1}\int_s^{2s+1}(t+1)^d\left(\frac{\tau+1}{s+1}\right)^\omega\|V(t,\tau)Q(t)x\|d\tau \leq$$

$$\leq M(t)\int_s^{2s+1}\left(\frac{t+1}{\tau+1}\right)^d\left(\frac{\tau+1}{s+1}\right)^{d+\omega+1}\frac{1}{\tau+1}(s+1)^d\|V(t,\tau)Q(t)x\|d\tau \leq$$

$$\leq M(t)\cdot 2^{d+\omega+1}(s+1)^d\|Q(t)x\|.$$

If $t \in [s, 2s+1)$ we obtain

$$(t+1)^d\|V_Q(t,s)x\| \leq (t+1)^d M(t)\left(\frac{t+1}{s+1}\right)^\omega\|Q(t)x\| =$$

$$= M(t)\left(\frac{t+1}{s+1}\right)^{d+\omega}(s+1)^d\|Q(t)x\| =$$

$$= M(t)(s+1)^d\left(\frac{t+1}{s+1}\right)^{d+\omega}\|Q(t)x\| \leq M(t)\cdot 2^{d+\omega}(s+1)^d\|Q(t)x\|.$$

□

Acknowledgments. The authors would like to thank the anonymous referees and handling editor for useful suggestions and comments which have led to an improved final version of the paper and helped to better situate this manuscript in the existing literature.

References

[1] Barreira, L. and Valls, C., *Polynomial growth rates*, Nonlinear Anal., **71** (2009), 5208–5219

[2] Barreira, L. and Valls, C., *Stable manifolds for nonautonomous equations without exponential dichotomy*, J. Differential Equations, **221** (2006), No. 1, 58–90

[3] Bento, A. J. G. and Silva, C. M., *Stable manifolds for nonuniform polynomial dichotomies*, J. Funct. Anal., **257** (2009), 122–148

[4] Dragičević, D., *Admissibility and nonuniform polynomial dichotomies*, Math. Nachr., **293** (2020), No. 2, 226–243

[5] Dragičević, D., *Strong nonuniform behaviour: A Datko type characterization*, J. Math. Anal. Appl., **459** (2018), No. 1, 266–290

[6] Lupa, N. and Megan, M., *Exponential dichotomies of evolution operators in Banach Spaces*, Monatsh. Math., **174** (2014), No. 2, 265–284

[7] Lupa, N. and Popescu, L. H., *Admissible Banach function spaces and nonuniform stabilities*, Mediterr. J. Math., **17** (2020), no. 4, Paper No. 105, 12 pp.

[8] Megan, M., *On (h; k)-dichotomy of evolution operators in Banach spaces*, Dyn. Syst. Appl., **5** (1996), 189–196

[9] Megan, M., Sasu, A. L. and Sasu, B., *On nonuniform exponential dichotomy of evolution operators in Banach spaces*, Integral Equations Operator Theory **44** (2002), No. 1, 71–78

[10] Megan, M., Sasu, A. L. and Sasu, B., *The Asymptotic Behavior of Evolution Fami- lies*, Mirton Publishing House 2003

[11] Megan, M., Stoica, C. S. and Buliga, L., *On asymptotic behaviors for linear skew-evolution semiflows in Banach spaces*, Carpathian J. Math., **23** (2007), No. 1-2, 117–125

[12] Mihiţ, C. L., *On some nonuniform dichotomic behaviors of discrete skew-product semiflows*, Theory Appl. Math. Comput. Sci., **9** (2019), 1–7

[13] Preda, C., Preda, P. and Craciunescu, A., *A version of a theorem of R. Datko for nonuniform exponential contractions*, J. Math. Anal. Appl., **385** (2012), No. 1, 572–581

[14] Perron, O., *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z., **32** (1930), 703–728

[15] Rămneanţu, M. L., Ceauşu, T. and Megan, M., *Polynomial instability of evolution operators in Banach spaces*, Carpathian J. Math., **29** (2013), No. 1, 77–83

[16] Rămneanţu, M. L., Ceauşu, T. and Megan, M., *On nonuniform polynomial dichotomy of evolution operators in Banach spaces*, Int. J. Pure Appl. Math., **75** (2012), No. 3, 305–318

DEPARTMENT OF MATHEMATICS
WEST UNIVERSITY OF TIMIŞOARA
V. PÂRVAN BLV. NO. 4, 300223 TIMIŞOARA, ROMANIA
Email address: rovanaboruga@gmail.com

DEPARTMENT OF MATHEMATICS
ACADEMY OF ROMANIAN SCIENTISTS
INDEPENDENŢEI 54, 050094 BUCHAREST, ROMANIA
Email address: mihail.megan@e-uvt.ro