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On simple normal structure and best proximity points in reflexive Banach space

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ABSTRACT. We introduce the concept of simple normal structure (see Definition **??**) for a pair of subsets in a normed space that is not proximal. Using this concept, we show that if $\mathcal{E}$ is a reflexive Banach space, $\mathcal{A}$ and $\mathcal{B}$ are two nonempty, convex, bounded and closed subsets of $\mathcal{E}$ having a simple normal structure, and $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ is a cyclic relatively nonexpansive map, then $\mathcal{T}^2$ admits a fixed point in $\mathcal{A}$. Moreover, if $\mathcal{T}$ satisfies a min-max condition, then this fixed point of $\mathcal{T}^2$ is also a best proximity point for $\mathcal{T}$.
Using this concept, we obtain the same result for the best proximity point of a cyclic contraction map. We also provide an example of a reflexive Banach space that is strictly convex but not uniformly convex.

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