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Exponential stability for a wave equation with time-varying delay

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ABSTRACT. In this paper, we consider a wave equation with a strong damping and time-varying delay. We shall prove that the solutions decay exponentially to the equilibrium state in the energy norm. The exponential stability estimate in this paper is achieved by imposing appropriate assumptions on the damping and delay weights and constructing suitable Lyapunov functionals. The main objective in this study is to provide a wider range for the delay weight to go beyond the damping weight, under pivotal circumstances, without affecting either the stabilization or the decay rate of the energy of the problem.

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