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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Solving split equality common fixed point problem for infinite families of demicontractive mappings

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ABSTRACT. In this paper, we consider the split equality common fixed point problem of infinite families of demicontractive mappings in Hilbert spaces. We introduce a simultaneous iterative algorithm for solving the split equality common fixed point problem of infinite families of demicontractive mappings and prove a strong convergence of the proposed algorithm under some control conditions.

1. INTRODUCTION

The split feasibility problem (SFP) can also be applied in various disciplines such as image restoration, in radiation therapy treatment planning, in antenna design, in immaterial science and in computerized tomography, etc. (see [2, 4, 5, 6]). The split equality common fixed point problem (SECFP) is a generalization of the split common fixed point problem (SCFP) and the split feasibility problem. Various algorithms were invented to solve problems above (see [3, 7, 8, 9, 17, 21]).

Let $X_i, i = 1, 2, 3$, be a real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let I be the identity mapping. The *split equality fixed point problem* (SEFP) for mappings S and T which was first introduced by Moudafi and Al-Shemas [18] is to find

$$u^* \in Fix(S),\ v^* \in Fix(T) \quad \text{such that} \quad Au^* = Bv^*, \tag{1.1}$$

where $A : X_1 \to X_3, B : X_2 \to X_3$ are two bounded linear operators, $S : X_1 \to X_1$ and $T : X_2 \to X_2$ are two mappings satisfying $Fix(S) \neq \emptyset$ and $Fix(T) \neq \emptyset$, respectively. Note that, if $X_2 = X_3$ and $B = I$, then the SEFPP generalizes the SFPP. To solve problem (1.1) they [18] proposed and proved a weak convergence under some control conditions of the following algorithm:

$$\begin{cases} x_{n+1} = S(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma B^*(Ax_n - By_n)), \qquad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where S and T are firmly quasi-nonexpansive mappings.

Recently, Eslamian [12] considered the following the *split equality common fixed point problem* (SECFP) :

$$\text{Find} \quad u^* \in \cap_{i=1}^{\infty} Fix(S_i),\ v^* \in \cap_{i=1}^{\infty} Fix(T_i) \quad \text{such that} \quad Au^* = Bv^*, \tag{1.3}$$

where $A : X_1 \to X_3,\ B : X_2 \to X_3$ are two bounded linear operators, and $\{S_i : X_1 \to X_1 : i \in \mathbb{N}\}$ and $\{T_i : X_2 \to X_2 : i \in \mathbb{N}\}$ are infinite families of k_1, k_2-demicontractive

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mappings, respectively. They also proposed the following algorithm for solving (1.3) for the class of demicontractive mappings:

$$
\begin{cases}
z_n = x_n - \gamma_n A^*(Ax_n - By_n), \\
u_n = z_n + \sum_{i=1}^{\infty} \alpha_{n,i} \frac{1-k_1}{2}(S_i - I)z_n, \\
x_{n+1} = \theta_n u + (1-\theta_n)u_n, \\
w_n = y_n + \gamma_n B^*(Ax_n - By_n), \\
v_n = w_n + \sum_{i=1}^{\infty} \alpha_{n,i} \frac{1-k_2}{2}(T_i - I)w_n, \\
y_{n+1} = \theta_n v + (1-\theta_n)v_n, \qquad n \in \mathbb{N}.
\end{cases} \tag{1.4}
$$

Using the iterative scheme (1.4), Eslamian obtained a strong convergence results for problem (1.3).

Note that computation of u_n and v_n by algorithm (1.4) are not so easy in practice because they concern the sum of the series in X.

Question. Can we modify algorithm (1.4) to the algorithm which is easy to compute and still obtain its strong convergence to a solution of problem (1.3)?

Throughout this paper, we adopt the following notations.

(i) "$\rightarrow$"and "$\rightharpoonup$"denote the strong and weak convergence, respectively.
(ii) $\omega_\omega(x_n, y_n)$ denote the set of the cluster point of $\{(x_n, y_n)\}$ in the weak topology, that is, there is a subsequence $\{(x_{n_i}, y_{n_i})\}$ of $\{(x_n, y_n)\}$ such that $(x_{n_i}, y_{n_i}) \rightharpoonup (x, y)$.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space X. A mapping $P_C : X \to C$ is said to be *metric projection* of X onto C, if for every $x \in X$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - z\|, \quad \forall z \in C.$$

It is known that P_C is a firmly nonexpansive mapping. Moreover, P_C is characterized by the following properties : $\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in X, y \in C$. In order to establish our convergence theorems, we need the following concepts for single-valued mappings.

Definition 2.1. Let C be a nonempty closed convex subset of a real Hilbert space X. A mapping $T : C \to C$ is said to be

(i) *α-contraction* if there exists $\alpha \in [0, 1)$ such that
$$\|Tu - Tv\| \leq \alpha\|u - v\| \qquad \text{for all } u, v \in C;$$
(ii) *quasi-nonexpansive* if $Fix(T) \neq \emptyset$ and
$$\|Tu - v\| \leq \|u - v\| \qquad \text{for all } u \in C, v \in Fix(T);$$
(iii) *k-strictly pseudo-nonspreading*[19], if there exists $k \in [0, 1)$ such that
$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + k\|u - Tu - (v - Tv)\|^2 + 2\langle u - Tu, v - Tv\rangle \quad \text{for all } u, v \in C;$$
(iv) *k-demicontractive* [10], if $Fix(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that
$$\|Tu - v\|^2 \leq \|u - v\|^2 + k\|u - Tu\|^2 \qquad \text{for all } u \in C, v \in Fix(T).$$

Remark 2.1. It follows from Definition 2.1 that

(1) If T is quasi-nonexpansive, then T is k-demicontractive for any $k \in [0, 1)$.
(2) If T is k-strictly pseudo-nonspreading with $Fix(T) \neq \emptyset$, then T is k-demicontractive.

In 2014, Chang, Kim, Cho and Sim [8] studied the weak ans strong convergence theorems of solution to SCFP for a family k_i-strictly pseudo-nonspreading mapping in a Hilbert space.

Remark 2.2. For negative values of k the class of demicontractive mappings is diminished to a great extent; in [1] such a class (with negative value of k) was considered under the name of *strongly attracting map*. In particular, the mapping T which satisfies Definition 2.1 (iv) with $k = -1$ is called *pseudo-contractive* in [24]. Note also that a mapping T satisfying Definition 2.1 (iv) with $k = 1$ is usually called *hemicontractive* and it was considered by some authors in connection with the strong convergence of the implicit Mann-type iteration (see, for example, [20, 22]).

Definition 2.2. Let C be a nonempty closed convex subset of a real Hilbert space X. Let $T : C \to C$ be a mapping. The mapping $T - I$ is said to be *demiclosed at zero* if for any sequence $\{x_n\}$ in C which $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, then $x \in Fix(T)$.

Lemma 2.1. ([23]) *Let X be a real Hilbert space. Then the following results hold:*

(i) *for all* $t \in [0,1]$ *and* $u, v \in X$, $\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - t(1-t)\|u-v\|^2$;
(ii) $\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \quad \forall u, v \in X$;
(iii) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle \quad \forall u, v \in X$.

Lemma 2.2. ([11]) *Let X be a real Hilbert space. Let $\{x_i, i = 1, 2, ..., n\} \subset X$. For $\alpha_i \in (0,1), i = 1, 2, ..., n$ such that $\sum_{i=1}^n \alpha_i = 1$. Then the following identity holds :*

$$\left\|\sum_{i=1}^n \alpha_i x_i\right\|^2 = \sum_{i=1}^n \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^n \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3. ([25]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation :*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad n \in \mathbb{N},$$

where

(i) $\{\gamma_n\} \subset (0,1)$, $\sum_{n=1}^\infty \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^\infty |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([13]). *Let $\{\kappa_n\}$ be a sequence of real numbers that dose not decrease at infinity, that is there exists at a subsequence $\{\kappa_{n_i}\}$ of $\{\kappa_n\}$ which satisfies $\kappa_{n_i} < \kappa_{n_i+1}$ for all $i \in \mathbb{N}$. For every $n \geq n_o$, define an integer sequence $\{\mu(n)\}$ as follow :*

$$\mu(n) = \max\{l \in \mathbb{N} \,:\, l \leq n, \kappa_l < \kappa_{l+1}\},$$

where $n_o \in \mathbb{N}$ such that $\{l \leq n_o \,:\, \kappa_l < \kappa_{l+1}\} \neq \emptyset$. Then the following hold:

(i) $\mu(n_o) \leq \mu(n_o+1) \leq ...$ *and* $\mu(n) \to \infty$;
(ii) *for all* $n \geq n_o$, $\max\{\kappa_n, \kappa_{\mu(n)}\} \leq \kappa_{\mu(n)+1}$.

3. Main results

In this section, we propose a new algorithm which is a modification of (1.4) and prove its strong convergence under some suitable conditions. We start with the following important lemma :

Lemma 3.5. *For real Hilbert spaces X, let $\{T_i : X \to X : i \in \mathbb{N}\}$ be infinite family of k-demicontractive mappings. Let $\{z_n\}$ and $\{w_n\}$ be sequences in X and let*

$$u_n = z_n + \sum_{i=1}^{n} \alpha_{n,i}\alpha_n (T_i - I)z_n,$$

$$v_n = \beta_{n,0}w_n + \sum_{i=1}^{n} \beta_{n,i}T_i w_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_{n,i}\}, \{\beta_{n,i}\}, \{\alpha_n\}$ are real sequences in $[0,1]$ satisfying $\sum_{i=1}^{n} \alpha_{n,i} = 1$ and $\sum_{i=0}^{n} \beta_{n,i} = 1$ for all $n \in \mathbb{N}$. Then

$$\|u_n - x^*\|^2 \le \|z_n - x^*\|^2 - \sum_{i=1}^{n} \alpha_{n,i}\alpha_n (1 - k - \alpha_n)\|(T_i - I)z_n\|^2, \tag{3.5}$$

$$\|v_n - x^*\|^2 \le \|w_n - x^*\|^2 - \sum_{i=1}^{n} \beta_{n,i}(\beta_{n,0} - k)\|(T_i - I)w_n\|^2, \tag{3.6}$$

for any $x^* \in \bigcap_{i=1}^{\infty} Fix(T_i)$.

Proof. Let $x^* \in \bigcap_{i=1}^{\infty} Fix(T_i)$. Since T_i is k-demicontractive, we obtain

$$\begin{aligned}
\|u_n - x^*\|^2 &\le \sum_{i=1}^{n} \alpha_{n,i}[\|z_n - x^*\|^2 + \alpha_n^2\|T_i z_n - z_n\|^2 + 2\alpha_n\langle z_n - x^*, T_i z_n - z_n\rangle] \\
&= \sum_{i=1}^{n} \alpha_{n,i}[\|z_n - x^*\|^2 + \alpha_n^2\|T_i z_n - z_n\|^2 - 2\alpha_n\|T_i z_n - z_n\|^2 \\
&\quad + 2\alpha_n\langle T_i z_n - x^*, T_i z_n - z_n\rangle] \\
&= \sum_{i=1}^{n} \alpha_{n,i}[\|z_n - x^*\|^2 + \alpha_n^2\|T_i z_n - z_n\|^2 - 2\alpha_n\|T_i z_n - z_n\|^2 \\
&\quad + \alpha_n\|T_i z_n - z_n\|^2 + \alpha_n\|T_i z_n - x^*\|^2 - \alpha_n\|z_n - x^*\|^2] \\
&\le \sum_{i=1}^{n} \alpha_{n,i}[\|z_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|T_i z_n - z_n\|^2 + \alpha_n k\|T_i z_n - z_n\|^2] \\
&= \|z_n - x^*\|^2 - \sum_{i=1}^{n} \alpha_{n,i}\alpha_n(1 - k - \alpha_n)\|T_i z_n - z_n\|^2.
\end{aligned}$$

Since T_i is k-demicontractive and by Lemma 2.2, we obtain

$$\begin{aligned}
\|v_n - x^*\|^2 &= \left\|\sum_{i=0}^{n} \beta_{n,i}(T_i w_n - x^*)\right\|^2 \\
&\le \beta_{n,0}\|w_n - x^*\|^2 + \sum_{i=1}^{n} \beta_{n,i}\|T_i w_n - x^*\|^2 - \sum_{i=1}^{n} \beta_{n,0}\beta_{n,i}\|w_n - T_i w_n\|^2 \\
&\le \beta_{n,0}\|w_n - x^*\|^2 + \sum_{i=1}^{n} \beta_{n,i}[\|w_n - x^*\|^2 + k\|(T_i - I)w_n\|^2] \\
&\quad - \sum_{i=1}^{n} \beta_{n,0}\beta_{n,i}\|(T_i - I)w_n\|^2
\end{aligned}$$

$$= \|w_n - x^*\|^2 - \sum_{i=1}^{n} \beta_{n,i}(\beta_{n,0} - k)\|(T_i - I)w_n\|^2.$$

This completes the proof. □

Now, we introduce a new algorithm for solving the split equality problem for infinite families of demicontractive mappings and then prove its strong convergence.

Theorem 3.1. *Let X_1, X_2 and X_3 be real Hilbert spaces, let $A : X_1 \to X_3$ and $B : X_2 \to X_3$ be two bounded linear operators with their adjoint operators A^* and B^*, respectively. Let $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ be two contraction mappings with constants $\rho_1, \rho_2 \in [0, 1)$. Let $\{S_i : X_1 \to X_1 : i \in \mathbb{N}\}$ and $\{T_i : X_2 \to X_2 : i \in \mathbb{N}\}$ be infinite families of k_1, k_2-demicontractive mappings such that $S_i - I$ and $T_i - I$, are demiclosed at zero. Suppose that $\Omega = \{(u^*, v^*) \in \bigcap_{i=1}^{\infty} Fix(S_i) \times \bigcap_{i=1}^{\infty} Fix(T_i) : Au^* = Bv^*\} \neq \emptyset$. Let $(x_1, y_1) \in X_1 \times X_2$ arbitrarily, let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$(3.7)\qquad \begin{cases} z_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ u_n = z_n + \sum_{i=1}^{n} \alpha_{n,i}\alpha_n(S_i - I)z_n, \\ x_{n+1} = \theta_n f_1(x_n) + (1 - \theta_n)u_n, \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ v_n = \beta_{n,0}w_n + \sum_{i=1}^{n} \beta_{n,i}T_i w_n, \\ y_{n+1} = \theta_n f_2(y_n) + (1 - \theta_n)v_n, \qquad n \in \mathbb{N}, \end{cases}$$

where $\{\gamma_n\}, \{\alpha_n\}, \{\theta_n\}, \{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ are sequences in $[0, 1]$ satisfying the following conditions :

(C1) $\sum_{i=1}^{n} \alpha_{n,i} = \sum_{i=0}^{n} \beta_{n,i} = 1$ *and* $\beta_{n,0} > k_2$ *for all* $n \in \mathbb{N}$;
(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$, *and* $\liminf_{n\to\infty}(\beta_{n,0} - k_2)\beta_{n,i} > 0$ *for all* $i \in \mathbb{N}$;
(C3) $\lim_{n\to\infty} \theta_n = 0$ *and* $\sum_{n=1}^{\infty} \theta_n = \infty$;
(C4) $0 < b_1 \le \gamma_n \le b_2 < \frac{2}{\|A\|^2+\|B\|^2}$ *for all* $n \in \mathbb{N}$;
(C5) $0 < a_1 \le \alpha_n \le a_2 < 1 - k_1$ *for all* $n \in \mathbb{N}$,

for some positive real number b_1, b_2, a_1 and a_2. Then the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$ which solves the variational inequality problem

$$(3.8)\qquad \langle (I_{X_1\times X_2} - f)(x^*, y^*), (u, v) - (x^*, y^*)\rangle_{X_1\times X_2} \ge 0, \quad (u, v) \in \Omega,$$

where $I_{X_1\times X_2}$ is identity map on $X_1 \times X_2$ and $f(x, y) = (f_1(x), f_2(y))$ for all $(x, y) \in X_1 \times X_2$.

Proof. Since $P_\Omega \circ f$ is a contraction mapping on $X_1 \times X_2$, there is a unique $(x^*, y^*) \in \Omega$ Then $(x^*, y^*) \in \cap_{i=1}^{\infty} Fix(S_i) \times \cap_{i=1}^{\infty} Fix(T_i)$ such that $Ax^* = By^*$. By (3.7) we get

$$\begin{aligned} \|z_n - x^*\|^2 &\le \|x_n - x^*\|^2 - 2\gamma_n\langle x_n - x^*, A^*(Ax_n - By_n)\rangle + \gamma_n^2\|A\|^2\|Ax_n - By_n\|^2 \\ &= \|x_n - x^*\|^2 - \gamma_n\|Ax_n - Ax^*\|^2 - \gamma_n\|Ax_n - By_n\|^2 \\ &\quad + \gamma_n\|Ax^* - By_n\|^2 + \gamma_n^2\|A\|^2\|Ax_n - By_n\|^2 \\ &= \|x_n - x^*\|^2 - \gamma_n\|Ax_n - Ax^*\|^2 + \gamma_n\|Ax^* - By_n\|^2 \\ (3.9)\qquad &\quad - \gamma_n(1 - \gamma_n\|A\|^2)\|Ax_n - By_n\|^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned}\|w_n - y^*\|^2 \leq \|y_n - y^*\|^2 - \gamma_n\|By_n - By^*\|^2 + \gamma_n\|Ax_n - By^*\|^2 \\ - \gamma_n(1-\gamma_n\|B\|^2)\|Ax_n - By_n\|^2. \end{aligned} \tag{3.10}$$

From (3.9), (3.10), (C4) and by taking into account the fact that $Ax^* = By^*$, we have

$$\begin{aligned}\|z_n - x^*\|^2 + \|w_n - y^*\|^2 &\leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ &\quad - \gamma_n(2-\gamma_n(\|A\|^2 + \|B\|^2))\|Ax_n - By_n\|^2 \\ &\leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2. \end{aligned} \tag{3.11}$$

By Lemma 3.5 we obtain

$$\|u_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \sum_{i=1}^{n} \alpha_{n,i}\alpha_n(1-k_1-\alpha_n)\|(S_i - I)z_n\|^2, \tag{3.12}$$

$$\|v_n - y^*\|^2 \leq \|w_n - y^*\|^2 - \sum_{i=1}^{n} \beta_{n,i}(\beta_{n,0}-k_2)\|(T_i - I)w_n\|^2. \tag{3.13}$$

From (3.12) and Lemma 2.1(i), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \theta_n\|f_1(x_n) - x^*\|^2 + (1-\theta_n)\|u_n - x^*\|^2 \\ &\leq \theta_n[\|f_1(x_n) - f_1(x^*)\|^2 + \|f_1(x^*) - x^*\|^2] \\ &\quad + 2\theta_n\|f_1(x_n) - f_1(x^*)\|\|f_1(x^*) - x^*\| + (1-\theta_n)\|u_n - x^*\|^2 \\ &\leq \theta_n[\rho_1\|x_n - x^*\|^2 + \|f_1(x^*) - x^*\|^2] \\ &\quad + 2\theta_n\rho_1\|x_n - x^*\|\|f_1(x^*) - x^*\| + (1-\theta_n)\|z_n - x^*\|^2 \\ &\quad - (1-\theta_n)\sum_{i=1}^{n} \alpha_{n,i}\alpha_n(1-k_1-\alpha_n)\|(S_i - I)z_n\|^2. \end{aligned} \tag{3.14}$$

Using (3.13), we obtain

$$\begin{aligned}\|y_{n+1} - y^*\|^2 &\leq \theta_n[\rho_1\|y_n - y^*\|^2 + \|f_2(y^*) - y^*\|^2] \\ &\quad + 2\theta_n\rho_2\|y_n - y^*\|\|f_2(y^*) - y^*\| + (1-\theta_n)\|w_n - y^*\|^2 \\ &\quad - (1-\theta_n)\sum_{i=1}^{n} \beta_{n,i}(\beta_{n,0}-k_2)\|(T_i - I)w_n\|^2. \end{aligned} \tag{3.15}$$

Next, set $\rho = \max\{\rho_1, \rho_2\}$ and $\kappa_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$. By (3.11), (3.14) and (3.15), we obtain

$$\begin{aligned}\kappa_{n+1} &\leq \theta_n\rho\kappa_n + \theta_n[\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2] \\ &\quad + 2\theta_n\rho[\|x_n - x^*\|\|f_1(x^*) - x^*\| + \|y_n - y^*\|\|f_2(y^*) - y^*\|] \\ &\quad + (1-\theta_n)[\|z_n - x^*\|^2 + \|w_n - y^*\|^2] \\ &\leq \theta_n\rho\kappa_n + \theta_n[\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2] \\ &\quad + 2\theta_n\rho[\|x_n - x^*\|\|f_1(x^*) - x^*\| + \|y_n - y^*\|\|f_2(y^*) - y^*\|] + (1-\theta_n)\kappa_n \end{aligned}$$

$$= (1-\theta_n(1-\rho))\kappa_n + \theta_n[\|f_1(x^*)-x^*\|^2 + \|f_2(y^*)-y^*\|^2]$$
$$+ 2\theta_n\rho[\|x_n-x^*\|\|f_1(x^*)-x^*\| + \|y_n-y^*\|\|f_2(y^*)-y^*\|]$$
$$\leq \max\left\{\kappa_n, \frac{\vartheta_n}{1-\rho}\right\}, \tag{3.16}$$

where $\vartheta_n = \|f_1(x^*)-x^*\|^2+\|f_2(y^*)-y^*\|^2+2\rho[\|x_n-x^*\|\|f_1(x^*)-x^*\|+\|y_n-y^*\|\|f_2(y^*)-y^*\|]$. It follows from induction that

$$\kappa_n \leq \max\left\{\kappa_1, \frac{\vartheta_1}{1-\rho}\right\}, \quad n \in \mathbb{N},$$

which implies that $\{\kappa_n\}$ is bounded. Therefore $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, $\{z_n\}$, $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded. By (3.11), (3.12) and (3.13), we get

$$\kappa_{n+1} \leq \kappa_n + \theta_n[\|f_1(x^*)-x^*\|^2 + \|f_2(y^*)-y^*\|^2]$$
$$+ 2\theta_n\rho[\|x_n-x^*\|\|f_1(x^*)-x^*\| + \|y_n-y^*\|\|f_2(y^*)-y^*\|]$$
$$- \gamma_n(2-\gamma_n(\|A\|^2+\|B\|^2))\|Ax_n-By_n\|^2$$
$$- \sum_{i=1}^{n}\alpha_{n,i}\alpha_n(1-k_1-\alpha_n)\|(S_i-I)z_n\|^2 - \sum_{i=1}^{n}\beta_{n,i}(\beta_{n,0}-k_2)\|(T_i-I)w_n\|^2$$
$$\leq \kappa_n + \theta_n M - \gamma_n(2-\gamma_n(\|A\|^2+\|B\|^2))\|Ax_n-By_n\|^2$$
$$- \sum_{i=1}^{n}\alpha_{n,i}\alpha_n(1-k_1-\alpha_n)\|(S_i-I)z_n\|^2 - \sum_{i=1}^{n}\beta_{n,i}(\beta_{n,0}-k_2)\|(T_i-I)w_n\|^2, \tag{3.17}$$

where $M = \sup_n\{\vartheta_n\}$. This implies for $j = 1, 2, ...n$,

$$\alpha_{n,j}\alpha_n(1-k_1-\alpha_n)\|(S_i-I)z_n\|^2 \leq \kappa_n - \kappa_{n+1} + \theta_n M, \tag{3.18}$$

and

$$\beta_{n,j}(\beta_{n,0}-k_2)\|(T_i-I)w_n\|^2 \leq \kappa_n - \kappa_{n+1} + \theta_n M. \tag{3.19}$$

Using (3.17), we obtain

$$\gamma_n(2-\gamma_n(\|A\|^2+\|B\|^2))\|Ax_n-By_n\|^2 \leq \kappa_n - \kappa_{n+1} + \theta_n M. \tag{3.20}$$

To this end, we consider the following two cases.

Case 1. Suppose that $\{\kappa_n\}_{n\geq n_o}$ is non-increasing for some $n_o \in \mathbb{N}$. Then we get $\lim_{n\to\infty}\kappa_n$ exists. By (3.18), (3.19), (3.20) and (C2)-(C5), we have $\lim_{n\to\infty}\|(S_i-I)z_n\| = 0 = \lim_{n\to\infty}\|(T_i-I)w_n\|$, and $\lim_{n\to\infty}\|Ax_n-By_n\| = 0$. It implies that

$$\lim_{n\to\infty}\|z_n-x_n\| = \lim_{n\to\infty}\|w_n-y_n\| = 0. \tag{3.21}$$

Since the sequence $\{x_n\}$ and $\{y_n\}$ are bounded we have $\omega_\omega(x_n,y_n)$ is nonempty. Let $(\bar{u},\bar{v}) \in \omega_\omega(x_n,y_n)$. From (3.21), we have $(\bar{u},\bar{v}) \in \omega_\omega(z_n,w_n)$. By demiclosedness principle of S_i-I and T_i-I at zero, we obtain $\bar{u} \in \cap_{i=1}^{\infty}Fix(S_i)$ and $\bar{v} \in \cap_{i=1}^{\infty}Fix(T_i)$. On the other hand, we have $A\bar{u}-B\bar{v} \in \omega_\omega(Ax_n-By_n)$, so there is a subsequence $\{(x_{n_k},y_{n_k})\}$ of $\{(x_n,y_n)\}$ such that $Ax_{n_k}-By_{n_k} \rightharpoonup A\bar{u}-B\bar{v}$. By lower semicontinuity of the norm, we get

$$\|A\bar{u}-B\bar{v}\| \leq \liminf_{k\to\infty}\|Ax_{n_k}-By_{n_k}\| = 0.$$

Therefore $(\bar{u},\bar{v}) \in \Omega$. So $\omega_\omega(x_n,y_n) \subset \Omega$. Choose a subsequence $\{(x_{n_p},y_{n_p})\}$ of $\{(x_n,y_n)\}$ such that $\limsup_{n\to\infty}\langle f_1(x^*)-x^*, x_n-x^*\rangle + \langle f_2(y^*)-y^*, y_n-y^*\rangle = \lim_{p\to\infty}\langle f_1(x^*)-$

$x^*, x_{n_p} - x^*\rangle + \langle f_2(y^*) - y^*, y_{n_p} - y^*\rangle$. We may assume that $(x_{n_p}, y_{n_p}) \rightharpoonup (\bar{x}, \bar{y})$ as $p \to \infty$. Since $\omega_\omega(x_n, y_n) \subset \Omega$ and (x^*, y^*) be the solution of a variational inequality problem (3.8), we get

$$\limsup_{n\to\infty} \langle f_1(x^*) - x^*, x_n - x^*\rangle + \langle f_2(y^*) - y^*, y_n - y^*\rangle \leq 0. \tag{3.22}$$

Using Lemma 2.1(iii) and (3.12), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\theta_n)\|u_n - x^*\|^2 + 2\theta_n\langle f_1(x_n) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\theta_n)\|z_n - x^*\|^2 + \rho_1\theta_n[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2] \\ &\quad + 2\theta_n\langle f_1(x^*) - x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{3.23}$$

Similarly, we obtain

$$\begin{aligned} \|y_{n+1} - y^*\|^2 &\leq (1-\theta_n)\|w_n - x^*\|^2 + \rho_2\theta_n[\|y_n - y^*\|^2 + \|y_{n+1} - y^*\|^2] \\ &\quad + 2\theta_n\langle f_2(y^*) - y^*, y_{n+1} - y^*\rangle. \end{aligned} \tag{3.24}$$

From (3.11), (3.23) and (3.24), we obtain

$$\begin{aligned} \kappa_{n+1} &\leq [1 - \frac{\theta_n(1-\rho)}{1-\theta_n\rho}]\kappa_n \\ &\quad + \frac{2\theta_n}{1-\theta_n\rho}[\langle f_1(x^*) - x^*, x_{n+1} - x^*\rangle + \langle f_2(y^*) - y^*, y_{n+1} - y^*\rangle]. \end{aligned} \tag{3.25}$$

By (3.22), (3.25), (C3) and Lemma 2.3, we can conclude that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. That is $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$.

Case 2. Suppose that there exists an integer m_o such that

$$\|x_{m_o} - x^*\|^2 + \|y_{m_o} - y^*\|^2 \leq \|x_{m_o+1} - x^*\|^2 + \|y_{m_o+1} - y^*\|^2.$$

Then we have $\kappa_{m_o} \leq \kappa_{m_o+1}$. Let $\{\mu(n)\}$ be a sequence defined by

$$\mu(n) = \max\{l \in \mathbb{N} : l \leq n, \kappa_l \leq \kappa_{l+1}\},$$

for all $n \geq m_o$. By Lemma 2.4, we obtain that $\{\mu(n)\}$ is a nondecreasing sequence such that

$$\lim_{n\to\infty} \mu(n) = \infty \quad \text{and} \quad \kappa_{\mu(n)} \leq \kappa_{\mu(n)+1}, \quad \text{for all} \quad n \geq m_o.$$

By the same argument as in the case 1, we obtain

$$\limsup_{n\to\infty} \langle f_1(x^*) - x^*, x_{\mu(n)} - x^*\rangle + \langle f_2(y^*) - y^*, y_{\mu(n)} - y^*\rangle \leq 0,$$

and

$$\begin{aligned} \kappa_{\mu(n)+1} &\leq \left[1 - \frac{\theta_{\mu(n)}(1-\rho)}{1-\theta_{\mu(n)}\rho}\right]\kappa_{\mu(n)} \\ &\quad + \frac{2\theta_{\mu(n)}}{1-\theta_{\mu(n)}\rho}[\langle f_1(x^*) - x^*, x_{\mu(n)+1} - x^*\rangle + \langle f_2(y^*) - y^*, y_{\mu(n)} - y^*\rangle]. \end{aligned}$$

So, we get $\lim_{n\to\infty} \kappa_{\mu(n)} = 0$. This implies $\lim_{n\to\infty} \kappa_{\mu(n)+1} = 0$. By Lemma 2.4, we have

$$0 \leq \kappa_n \leq \max\{\kappa_n, \kappa_{\mu(n)}\} \leq \kappa_{\mu(n)+1},$$

so $\kappa_n \to 0$, which implies $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. That is $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$.

Therefore, the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$ which solves the variational inequality problem (3.8). This completes the proof. $\square$

Remark 3.3.

(i) Theorem 3.1 can be used for infinite families of quasi-nonexpansive mappings because the class of quasi-nonexpansive mappings is included in that of demicontractive mappings.
(ii) Theorem 3.1 can be used for infinite families of strictly pseudo-nonspreading mappings because the class of strictly pseudo-nonspreading mappings is included in that of demicontractive mappings.
(iii) Putting $B = I$ and $X_2 = X_3$, in Theorem 3.1, we have a new algorithm for solving SCFP and we obtain that the sequence $\{(x_n, y_n)\}$ generated by (3.7) converges strongly to $(x^*, y^*) \in \Omega$ which solves the variational inequality problem (3.8).

4. Numerical example for the main result

We now give some numerical example to support our main result. Let $X_1 = X_2 = \mathbb{R}$ with the usual norm. Define the mappings $S_i : \mathbb{R} \to \mathbb{R}$ and $T_i : \mathbb{R} \to \mathbb{R}$ by

$$S_i(x) = \frac{-3x}{i}, \qquad i \in \mathbb{N},$$

and

$$T_i(x) = \begin{cases} \dfrac{i}{i+1}\sqrt{x} & \text{if } x \geq 1, \\ \dfrac{-2i}{i+1}x & \text{otherwise} \end{cases} \qquad i \in \mathbb{N},$$

for all $x \in \mathbb{R}$. Then we have S_i and T_i are $\frac{2}{3}$ and $\frac{3}{4}$−demicontractive mappings for all $i \in \mathbb{N}$ and $\bigcap_{i=1}^{\infty} F(S_i) = \{0\} = \bigcap_{i=1}^{\infty} F(T_i)$. Next, we define the mappings $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ by

$$f_1(x) = \frac{x}{4} \quad \text{and} \quad f_2(x) = \frac{x}{8} \quad \text{for all } x \in \mathbb{R}.$$

Let bounded linear operators $A : \mathbb{R} \to \mathbb{R}$ and $B : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = 3x$ and $Bx = -\frac{x}{5}$ for all $x \in \mathbb{R}$. Define the real sequence $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ as follows:

$$\alpha_{n,i} = \begin{cases} 1 & \text{if } n = i = 1, \\ \dfrac{1}{3^i}\left(\dfrac{n}{n+1}\right) & \text{if } n > i, \\ 1 - \sum_{i=1}^{n-1} \dfrac{1}{3^i}\left(\dfrac{n}{n+1}\right) & \text{if } n = i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

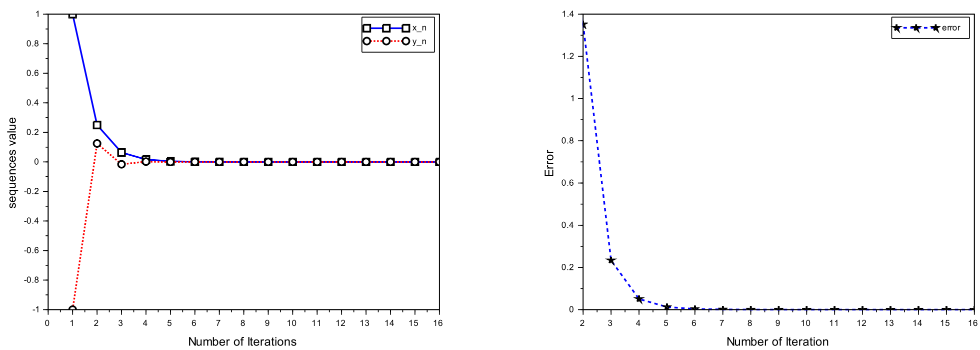
and

$$\beta_{n,i} = \begin{cases} \dfrac{1}{2^i}\left(\dfrac{n}{n+1}\right) & \text{if } n > i-1, \\ 1 - \left(\dfrac{n}{n+1}\right)\sum_{i=1}^{n} \dfrac{1}{2^i} & \text{if } n = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\gamma_n = 0.001, \alpha_n = 0.002$ and $\theta_n = \frac{1}{n^{0.01}}$ for all $n \in \mathbb{N}$. Now, we start with the initial point $(x_1, y_1) = (1, -1)$ and the criterion for stopping our testing method is taken as : $\|(x_n, y_n) - (x_{n-1}, y_{n-1})\|_2 < 10^{-5}$. Then the sequence $\{(x_n, y_n)\}$ generated by (3.7) and $\varepsilon_n = \|(x_n, y_n) - (x_{n-1}, y_{n-1})\|_2$ are shown in the following table:

Table 1: Numerical example of algorithm (3.7)

No. of Iterations	x_n	y_n	ε_n
1	1.000000	-1.0000000	–
2	0.250000	0.1250000	1.35208173
3	0.063775	-0.0159483	0.23355155
4	0.016459	0.0020509	0.05062376
5	0.004282	-0.0002643	0.01239465
6	0.001121	0.0000341	0.00317511
7	0.000295	-0.0000044	0.00082699
8	0.000078	0.0000006	0.00021709
9	0.000021	-0.0000001	0.00005728
10	0.0000055	0.000000009	0.00001517
11	0.0000015	-0.000000001	0.00000403
12	0.00000039	0.0000000002	0.00000108



(A) (B)

We observe from Table 1 that $(x_n, y_n) \to (0,0) \in \Omega$. We also note that the error bounded of $\|(x_{12}, y_{12}) - (x_{11}, y_{11})\|_2 < 10^{-5}$ and we can use $(x_{12}, y_{12}) = (0.00000039, 0.0000000002)$, to approximate the solution of (1.3) with accuracy at least 5 D.P.

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