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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Convergence of inexact orbits of monotone nonexpansive mappings

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ABSTRACT. We study monotone nonexpansive self-mappings of a closed and convex cone in an ordered Banach space with particular emphasis on the asymptotic behavior of their inexact iterates.

1. INTRODUCTION AND MAIN RESULTS

During more than fifty years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [13, 14, 15, 20, 21] and the references cited therein. This activity stems from Banach's classical theorem [4] regarding the existence of a unique fixed point for a strict contraction. Since that seminal result, many developments have taken place in this area. We mention, for instance, existence results for fixed points of nonexpansive mappings which are not strictly contractive [13, 14, 18, 19]. Such results were obtained for general nonexpansive mappings in special Banach spaces, while for self-mappings of general complete metric spaces existence results were established for, the so-called, contractive mappings [17]. For general nonexpansive mappings in general Banach spaces the existence of a unique fixed point was established in the generic sense by using the Baire category approach [6, 7, 20, 21].

Another important topic in metric fixed point theory is the convergence of (inexact) iterates of a mapping to one of its fixed points.

For example, let (X, ρ) be a metric space and let $A : X \to X$ be nonexpansive. In [5] (see also Section 2.21 of [21]), under the assumption that for every $x \in X$, the sequence $\{A^i(x)\}_{i=1}^{\infty}$ converges, it was shown that for each given sequence of computational errors $\{r_i\}_{i=1}^{\infty} \subset (0, \infty)$ satisfying

$$\sum_{i=1}^{\infty} r_i < \infty,$$

each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$\rho(x_{i+1}, A(x_i)) \le r_{i+1},\ i = 1, 2, \dots,$$

converges to a fixed point of A. This result has found several interesting applications. It is, for instance, an important ingredient in the superiorization methodology and in the study of perturbation resilience of algorithms. See, for example, [8, 9, 10, 11] and the references mentioned therein.

In the present paper we study the asymptotic behavior of inexact iterates of monotone nonexpansive mappings – a class of nonlinear mappings which is the subject of a rapidly growing area of research [1, 12].

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Let $(X, \|\cdot\|)$ be a Banach space ordered by a closed and convex cone $X_+ \subset X$ satisfying

$$X_+ \cap (-X_+) = \{0\}.$$

Note that for all $x, y \in X$,

$$x \le y \text{ if and only if } y - x \in X_+.$$

For each $x \in X$ and each $r > 0$, set

$$B(x, r) := \{y \in X : \ \|x - y\| \le r\}.$$

Suppose that

(1.1) $$\|x\| \le \|y\| \text{ for all } x, y \in X_+ \text{ satisfying } x \le y.$$

In this case the cone X_+ is called *normal*.

We assume that the normal cone X_+ has an interior point $x_* \in X_+$. Then there exists $r_* > 0$ such that

(1.2) $$B(x_*, r_*) \subset X_+.$$

For each $x \in X$, set

(1.3) $$\|x\|_* := \inf\{\lambda \in [0, \infty) : \ -\lambda x_* \le x \le \lambda x_*\}.$$

It is clear that for any $x \in X$, $\|x\|_*$ is well defined and finite, $\|\cdot\|_*$ is a norm on X and

(1.4) $$\{x \in X : \ \|x\|_* \le 1\} = \{x \in X : \ -x_* \le x \le x_*\}.$$

It is well known that the norms $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent. Indeed, let $x \in X$. In view of (1.1) and (1.3),

$$-\|x\|_* x_* \le x \le \|x\|_* x_*$$

and

$$\|x\| \le \|x + \|x\|_* x_*\| + \|\|x\|_* x_*\| \ \le \ 2\|x\|_* \|x_*\| + \|x\|_* \|x_*\| \le 3\|x\|_* \|x_*\|.$$

On the other hand, by (1.2),

$$B(0, r_*) + x_* \subset X_+,$$

$$B(0, r_*) \subset \{z \in X : \ -x_* \le z \le x_*\} = \{z \in X : \ \|z\|_* \le 1\}$$

and for all $z \in X \setminus \{0\}$,

$$\|r_* \|z\|^{-1} z\|_* \le 1, \ \|z\|_* \le r_*^{-1} \|z\|.$$

In the sequel we assume that

$$\|\cdot\| = \|\cdot\|_*.$$

Let a mapping $T : X_+ \to X_+$ satisfy

(1.5) $$T(x) \le T(y) \text{ for all } x, y \in X_+ \text{ satisfying } x \le y$$

and

(1.6) $$\|T(x) - T(y)\| \le \|x - y\|$$

for all $x, y \in X_+$ satisfying $x \le y$. Such a mapping T is said to be *monotone nonexpansive*.

In the present paper we establish the following two results.

Theorem 1.1. *Assume that for each $x \in X_+$, the sequence $\{T^i(x)\}_{i=1}^\infty$ converges and that a sequence $\{\gamma_i\}_{i=1}^\infty \subset (0, \infty)$ satisfies*

(1.7) $$\sum_{i=1}^\infty \gamma_i < \infty.$$

Then each sequence $\{x_i\}_{i=0}^\infty \subset X_+$ such that

$$\|x_{i+1} - T(x_i)\| \le \gamma_{i+1} \text{ for all integers } i \ge 0$$

also converges.

Theorem 1.2. *The mapping T is continuous.*

Note that an analog of Theorem 1.1 for nonexpansive mappings was obtained in [5]. The convergence property established in Theorem 1.1 is close (but not equivalent) to the shadowing property which is of interest in the qualitative study of dynamical systems [2, 16]. Applications of monotone nonexpansive mappings to the study of certain classes of matrix equations, differential equations and integral equations are discussed in [3].

2. Proof of Theorem 1.1

Let a sequence $\{x_i\}_{i=0}^{\infty} \subset X_+$ satisfy

$$\|x_{i+1} - T(x_i)\| \leq \gamma_{i+1} \text{ for all integers } i \geq 0. \tag{2.8}$$

It is sufficient to show that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence.

Let $\epsilon > 0$ be given. In view of (1.7), there exists a natural number n_0 such that

$$\sum_{i=n_0}^{\infty} \gamma_i < \epsilon/4. \tag{2.9}$$

Set

$$y_{n_0} := x_{n_0}. \tag{2.10}$$

By (1.4) and (2.8),

$$-\gamma_{n_0+1}x_* \leq x_{n_0+1} - T(x_{n_0}) \leq \gamma_{n_0+1}x_*. \tag{2.11}$$

Set

$$y_{n_0+1} := T(x_{n_0}) + \gamma_{n_0+1}x_*. \tag{2.12}$$

It follows from (2.10) and (2.12) that

$$y_{n_0+1} = T(y_{n_0}) + \gamma_{n_0+1}x_*. \tag{2.13}$$

Equations (2.11) and (2.12) imply that

$$x_{n_0+1} \leq y_{n_0+1}. \tag{2.14}$$

By (2.11) and (2.12), we have

$$y_{n_0+1} - x_{n_0+1} \leq T(x_{n_0}) + \gamma_{n_0+1}x_* - T(x_{n_0}) + \gamma_{n_0+1}x_* \leq 2\gamma_{n_0+1}x_*. \tag{2.15}$$

For each integer $k \geq n_0 + 1$, set

$$y_{k+1} := T(y_k) + \gamma_{k+1}x_*. \tag{2.16}$$

We claim that for each integer $k \geq n_0 + 1$,

$$y_k \geq T^{k-n_0}(y_{n_0}) \tag{2.17}$$

and

$$y_k - T^{k-n_0}(y_{n_0}) \leq \sum_{i=n_0+1}^{k} \gamma_i x_*. \tag{2.18}$$

In view of (2.13), inequalities (2.17) and (2.18) are valid for $k = n_0 + 1$. Assume now that $k \geq n_0 + 1$ is an integer and that (2.17) and (2.18) are true. By (2.16) and (2.17),

$$y_{k+1} = T(y_k) + \gamma_{k+1}x_* \geq T(T^{k-n_0}(y_{n_0})) = T^{k+1-n_0}(y_{n_0}). \tag{2.19}$$

It follows from (1.3), (1.6), (2.17) and (2.18) that

$$\|T(y_k) - T(T^{k-n_0}(y_{n_0}))\| \leq \|y_k - T^{k-n_0}(y_{n_0})\| \leq \sum_{i=n_0+1}^{k} \gamma_i$$

and

$$T(y_k) \leq T^{k+1-n_0}(y_{n_0}) + \sum_{i=n_0+1}^{k} \gamma_i x_*. \tag{2.20}$$

Equations (2.16) and (2.20) imply that

$$\begin{aligned} y_{k+1} &= T(y_k) + \gamma_{k+1}x_* \leq T^{k+1-n_0}(y_{n_0}) + \sum_{i=n_0+1}^{k} \gamma_i x_* + \gamma_{k+1}x_* \\ &= T^{k+1-n_0}(y_{n_0}) + \sum_{i=n_0+1}^{k+1} \gamma_i x_*. \end{aligned} \tag{2.21}$$

In view of (2.19) and (2.21), inequalities (2.17) and (2.18) hold for $k+1$ too. Thus we have shown by induction that (2.17) and (2.18) indeed hold for all integers $k \geq n_0 + 1$.

By (1.3), (2.9), (2.17) and (2.18), for all integers $k \geq n_0 + 1$, we have

$$\|y_k - T^{k-n_0}(y_{n_0})\| \leq \sum_{i=n_0+1}^{\infty} \gamma_i < \epsilon/4. \tag{2.22}$$

Next we claim that for each integer $k \geq n_0 + 1$, we have

$$x_k \leq y_k \tag{2.23}$$

and

$$y_k - x_k \leq 2 \sum_{i=n_0+1}^{k} \gamma_i x_*. \tag{2.24}$$

In view of (2.11) and (2.12), inequalities (2.23) and (2.24) do hold for $k = n_0 + 1$.

Assume that $k \geq n_0 + 1$ is an integer and that (2.23) and (2.24) hold. By (1.3) and (2.8),

$$-\gamma_{k+1}x_* \leq x_{k+1} - T(x_k) \leq \gamma_{k+1}x_*. \tag{2.25}$$

It follows from (2.16), (2.23) and (2.25) that

$$x_{k+1} \leq T(x_k) + \gamma_{k+1}x_* \leq T(y_k) + \gamma_{k+1}x_* = y_{k+1}. \tag{2.26}$$

Equations (2.16) and (2.25) imply that

$$\begin{aligned} y_{k+1} - x_{k+1} &= T(y_k) + \gamma_{k+1}x_* - x_{k+1} \\ &\leq T(y_k) + \gamma_{k+1}x_* - T(x_k) + \gamma_{k+1}x_*. \end{aligned} \tag{2.27}$$

By (1.3), (1.6), (2.23) and (2.24),

$$\|T(y_k) - T(x_k)\| \leq \|y_k - x_k\| \leq 2 \sum_{i=n_0+1}^{k} \gamma_i$$

and

$$T(y_k) - T(x_k) \leq 2 \sum_{i=n_0+1}^{k} \gamma_i x_*. \tag{2.28}$$

It follows from (2.27) and (2.28) that

$$y_{k+1} - x_{k+1} \le 2 \sum_{i=n_0+1}^{k} \gamma_i x_* + 2\gamma_{k+1} x_*. \tag{2.29}$$

In view of (2.26) and (2.29), inequalities (2.23) and (2.24) are valid for $k+1$ too. Thus we have shown by induction that (2.23) and (2.24) are indeed valid for all integers $k \ge n_0 + 1$.

By (1.3), (2.9), (2.23) and (2.24), for all integers $k \ge n_0 + 1$, we have

$$\|y_k - x_k\| \le 2 \sum_{i=n_0+1}^{k} \gamma_i < \epsilon/2. \tag{2.30}$$

It now follows from (2.22) and (2.30) that for all integers $k \ge n_0 + 1$,

$$\|x_k - T^{k-n_0}(y_{n_0})\| < 3\epsilon/4. \tag{2.31}$$

The assumptions of the theorem imply that the limit

$$\lim_{k\to\infty} T^k(y_{n_0})$$

exists. When combined with (2.31), this implies that for all sufficiently large natural numbers k,

$$\|x_k - \lim_{i\to\infty} T^i(y_{n_0})\| < \epsilon.$$

Since ϵ is an arbitrary positive number, we conclude that $\{x_k\}_{k=0}^{\infty}$ is a Cauchy sequence. Thus the sequence $\{x_k\}_{k=0}^{\infty}$ indeed converges, as asserted. Theorem 1.1 is proved.

3. Proof of Theorem 1.2

Let $x_0 \in X_+$ and $\epsilon > 0$ be given. Assume that a point $x \in X_+$ satisfies

$$\|x - x_0\| \le \epsilon/4. \tag{3.32}$$

By (1.3) and (3.32), we have

$$-(\epsilon/4)x_* \le x - x_0 \le (\epsilon/4)x_*$$

and

$$x_0 - (\epsilon/4)x_* \le x \le x_0 + (\epsilon/4)x_*. \tag{3.33}$$

It follows from (1.3), (1.5), (1.6), (3.32) and (3.33) that

$$T(x) \le T(x_0 + 4^{-1}\epsilon x_*)$$

and

$$\begin{aligned} \|T(x) - T(x_0 + 4^{-1}\epsilon x_*)\| &\le \|x_0 + 4^{-1}\epsilon x_* - x\| \\ &\le 4^{-1}\epsilon + \|x_0 - x\| \le 2^{-1}\epsilon. \end{aligned} \tag{3.34}$$

By (1.3) and (1.6), we have

$$\|T(x_0) - T(x_0 + 4^{-1}\epsilon x_*)\| \le \|4^{-1}\epsilon x_*\| = 4^{-1}\epsilon. \tag{3.35}$$

In view of (3.34) and (3.35), we conclude that

$$\|T(x) - T(x_0)\| \le \epsilon.$$

This completes the proof of Theorem 1.2.

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