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A population model with pseudo exponential survival

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ABSTRACT. This paper considers a model for population dynamics with age structure. Following Dufresne (2006) and Beghriche et all (2022) the probability of survival is assumed to be a linear combination of exponentials, and a product of a polynomial and an exponential. The number of births in unit time is characterized through a system of ordinary differential equations. This is solved explicitly in special cases, which leads to closed form expressions for the population size. The later allows an assymptotic analysis with three cases; the population goes extinct, explodes, or converges to a finite number depending of the interplay between model parameters. From a practical standpoint our modelling approach leads to a better fit of population data when compared to the exponential survival, and it also allows for more shapes of population as a function of time.

1. INTRODUCTION

This work analyzes a population model with age structure. There are many papers on this topic, we only refer a few [2, 3, 4, 10, 11, 14, 16, 17].

The pioneer work of [14] was among the first to model the population dynamics, or more precisely, its density, as map which depends on time and age. The fertility function and mortality intensity are exogenous in his model. The total population at a given time is obtained by integrating the density map along its age dependency. Thus, the density map is key in studying the population model.

A system of integral and differential equations can be employed to characterize the density map. A Volterra integral equation turns out to be equivalent to this system. It well known by now that fixed point arguments yield existence to such Volterra equations, but their numerical implementation is burdensome.

Let us turn now to the contributions of our paper. In order to make the problem more tractable, and inspired by the works of [1] and [6] we assumed that the mortality intensity is a combination of exponentials or an exponential multiplied by a polynomial. In this paradigm we manage to reduce the Volterra integral equation to a system of ordinary differential equations which can be solved easily. Moreover, in special settings closed form solutions were obtained by means of Laplace transform. In the special case of exponential mortality intensity our closed form solutions coincide with the ones obtained in such a setting by [2].

Our methodology can be extend naturally if one approximated a general mortality intensity function by linear combinations of exponentials.

The closed form solutions we obtained for the total population allowed us to perform an asymptotic analysis. This revealed that in the limit the population goes extinct, explodes, or converges to a finite number.

We provide the survival and fertility parameters interplay which leads to one of these situations.

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Let us comment now on the real world applicability of our work. Our explicit solutions show that the population is a sum of three or four exponentials (depending on the model considered) and this will have a profound impact when compared to the exponential survival model where the population is exponential. Not only our model allows for more shapes of the population function of time but will also provide a better fit to population data. There is a rich literature by now on the fitting of real world data with a linear combination of exponentials; we mentioned [5] and [9] only. The later shows the superiority of the fit when more exponentials are employed (the comparison is done with one, two, and three exponentials).

The reminder of this paper is organized as follows: Section 2 presents the model; Sections 3 deals with the case when the mortality intensity is a linear combination of exponentials; Section 4 treats the case when the mortality intensity is a polynomial times an exponential; Section 4.1 treats a new fertility model; The conclusion can be found in Section 5.

2. The model

Let us introduce the model. The function $\rho(a,t)$ denotes the density of people aged a years at time t, which will be assumed a differentiable in both variables. The number of people aged between

$$a \text{ and } a + \Delta a$$

at time t, where Δa is the *small time increment*, can be approximated by

$$\rho(a,t)\Delta a.$$

Consequently, the total population at time t is

$$\sum_a \rho(a,t)\Delta a.$$

By passing $\Delta a \to 0$ one gets

$$\int_0^\infty \rho(a,t)\, da$$

the total size of population at time t, which will be denoted by

$$P(t) = \int_0^\infty \rho(a,t)\, da. \tag{2.1}$$

In this model people can die and death rate, then the *mortality intensity* or *age-specific mortality* or *death modulus* is denoted by the nonnegative function $\mu(a)$.

As such, during the time interval from t to $t + \Delta t$ a fraction

$$\mu(a)\Delta t$$

of people aged between a and $a + \Delta a$ at time t die.

The number of people

$$\rho(a,t)\Delta a$$

at time t have ages between a and $a + \Delta a$.

The number of deaths from this age cohort between during the times t and $t + \Delta t$ is

$$\rho(a,t)\Delta a \cdot \mu(a)\Delta t.$$

The number of people surviving at time $t + \Delta t$, with ages between

$$a + \Delta t \text{ and } a + \Delta t + \Delta a$$

can be thus approximated by

$$\rho(a+\Delta t, t+\Delta t)\Delta a \approx \rho(a,t)\Delta a - \rho(a,t)\mu(a)\Delta a \Delta t,$$

or, equivalently,

$$\frac{\rho(a+\Delta t, t+\Delta t) - \rho(a,t)}{\Delta t} + \mu(a)\rho(a,t) \approx 0. \tag{2.2}$$

Let $\Delta t \to 0$ to get

(2.3)
$$\begin{aligned}\lim_{\Delta t\to 0} \frac{\rho(a+\Delta t,t+\Delta t)-\rho(a,t)}{\Delta t} &= \lim_{\Delta t\to 0} \frac{\rho(a+\Delta t,t+\Delta t)-\rho(a,t+\Delta t)}{\Delta t} + \lim_{\Delta t\to 0} \frac{\rho(a,t+\Delta t)-\rho(a,t)}{\Delta t} \\ &= \lim_{\Delta t\to 0}(\rho_a(a,t+\Delta t) + \rho_t(a,t)) \\ &= \rho_a(a,t) + \rho_t(a,t),\end{aligned}$$

where $\rho_a(a,t)$ (respectively $\rho_t(a,t)$) is the partial derivative of $\rho(a,t)$ with respect to a (respectively t).

Coupling (2.2) and (2.3) together we obtain the Lotka-McKendrick equation

$$\rho_a(a,t) + \rho_t(a,t) + \mu(a)\rho(a,t) = 0, \tag{2.4}$$

(for more on this see [2, p. 274] or [8, Section 2, pp. 128-130]).

The probability that a subject of age a_0 will survive to age a is then given by

$$e^{-\int_{a_0}^{a} \mu(\alpha)\, d\alpha}.$$

Consequently, the probability of survival from age 0 (birth) to age a is

$$\pi(a) = e^{-\int_0^a \mu(\alpha)\, d\alpha}. \tag{2.5}$$

Following [7] and [6] we may assume that this probability is a linear combination of exponentials.

The birth process in our model is given by a function called the *birth modulus* or *natality rates* or *fecundity function*, denoted by $\beta(a)$, which will be positive. Then

$$\beta(a)\Delta t$$

is the number of people born from parents with ages between a and $a+\Delta a$ in the time interval from t to $t+\Delta t$, and in turn the total number of births occurring during t and $t+\Delta t$ is

$$\Delta t \sum_a \beta(a)\rho(a,t)\Delta a \to \Delta t \int_0^\infty \beta(a)\rho(a,t)\, da.$$

Since this quantity equals

$$\rho(0,t)\Delta t$$

one obtains *renewal condition* or *the total birth rate* or *fertility rate*, at the time t given by

$$B(t) = \rho(0,t) = \int_0^\infty \beta(a)\rho(a,t)\, da, \tag{2.6}$$

which it should depend on the total population and so in practice $\rho(a,t)$ is zero for large age. The time 0 age distribution is exogenously given

$$\rho(a,0) = \Phi(a), \tag{2.7}$$

is assumed to be known and it is called the initial population distribution.

Finally, (2.4), (2.6) and (2.7) leads to the following PDE characterization of ρ

$$\begin{cases} \rho_a(a,t) + \rho_t(a,t) + \mu(a)\rho(a,t) = 0 & \text{for} \quad a, t \geq 0, \\ \rho(0,t) = \int\limits_0^\infty \beta(a)\rho(a,t)\,da & \text{for} \quad t > 0, \\ \rho(a,0) = \Phi(a) & \text{for} \quad a > 0. \end{cases} \tag{2.8}$$

The problem (2.8) discovered first by McKendrick [14] is the starting point of many population models analyzed by many other authors.

Obviously, to solve (2.8) we can rewrite the equation

$$\rho_a(a,t) + \rho_t(a,t) + \mu(a)\rho(a,t) = 0$$

as the well know transport equation

$$\left(e^{\int\limits_0^a \mu(\alpha)\,d\alpha}\rho(a,t)\right)'_{|a} + \left(e^{\int\limits_0^a \mu(\alpha)\,d\alpha}\rho(a,t)\right)'_{|t} = 0,$$

which has the general solution

$$\rho(a,t) = e^{-\int\limits_0^a \mu(\alpha)\,d\alpha} h\,(t-a) \tag{2.9}$$

for some function $h : \mathbb{R} \to \mathbb{R}$. Next, we simply plug the general solution (2.9) into the boundary conditions and solve for the yet to be determined function h

$$\begin{aligned} \rho(0,t) = h\,(t) = B(t) &\Longrightarrow \rho(a,t) = e^{-\int\limits_0^a \mu(\alpha)\,d\alpha} B\,(t-a) && \text{for} \quad t \geq a, \\ \rho(a,0) = \Phi(a) = e^{-\int\limits_0^a \mu(\alpha)\,d\alpha} h\,(-a) &\Longrightarrow \rho(a,t) = \Phi(a-t) e^{-\int\limits_{a-t}^a \mu(\alpha)\,d\alpha} && \text{for} \quad t < a. \end{aligned}$$

Therefore, the solution $\rho(a,t)$ is given by an implicit formula

$$\rho(a,t) = \begin{cases} B(t-a)e^{-\int\limits_0^a \mu(\alpha)\,d\alpha} & \text{for} \quad t \geq a \\ \Phi(a-t)e^{-\int\limits_{a-t}^a \mu(\alpha)\,d\alpha} & \text{for} \quad t < a \end{cases} \tag{2.10}$$

(see [13, Section 2, pp. 4-6] for more details). From this one gets the following Volterra integral equation on the birth rate

$$\begin{aligned} \rho(0,t) &= \int\limits_0^\infty \beta(a)\rho(a,t)\,da \\ &= \int\limits_0^t \beta(a)\rho(0,t-a)e^{-\int\limits_0^a \mu(\alpha)\,d\alpha}\,da + \int\limits_t^\infty \beta(a)\Phi(a-t)e^{-\int\limits_{a-t}^a \mu(\alpha)\,d\alpha}\,da \\ &= \int\limits_0^t \beta(a)\,B(t-a)e^{-\int\limits_0^a \mu(\alpha)\,d\alpha}\,da + \int\limits_t^\infty \beta(a)\Phi(a-t)e^{-\int\limits_{a-t}^a \mu(\alpha)\,d\alpha}\,da. \end{aligned}$$

We let $\Psi(t)$ be the rate of births from members who were present in the population at time 0, and in terms of the initial age distribution and the *birth and mortality intensity*

$$\Psi(t) = \int_t^\infty \beta(a)\Phi(a-t)e^{-\int_{a-t}^{a}\mu(\alpha)\,d\alpha}\,da = \int_0^\infty \beta(t+s)\Phi(s)e^{-\int_{s}^{s+t}\mu(\alpha)\,d\alpha}\,ds. \tag{2.11}$$

With notation (2.5) and (2.11), the *renewal equation* characterizes $B(t)$ is given by

$$B(t) = \Psi(t) + \int_0^t \beta(a)\pi(a)B(t-a)\,da, \tag{2.12}$$

(see [12] for the authors that developed this theory). Moreover, if $B(t)$ is a solution of the *renewal equation*, then a solution of the PDE of ρ is (2.10).

However, until now the explicit solution for (2.12) is obtained in some very special cases. This means that numerical method is the most used alternative for the researchers to obtain new results. On the other hand in many real world phenomena it is useful to know an explicit solution for (2.12). Then, our objective of the paper is to present an explicit formula for the McKendrick model in a special framework that appears in the natural phenomena.

To introduce our results we assume as in [2], that the expected number of offspring for each individual over a lifetime, being the sum over all ages a of probability of survival to age a multiplied by the number of offspring at age a is finite, i.e.

$$R = \int_0^\infty \beta\,(a)\,\pi\,(a)\,da < \infty.$$

3. The Case of Pseudo Exponential Mortality Intensity

In the following, inspired by [6] and [7], we consider a model with $\pi(a)$ pseudo exponential, i.e.

$$\pi(a) = \sum_{i=1}^n c_i e^{-\mu_i a}, \tag{3.13}$$

for positive constants c_i, μ_i with

$$c_1 \neq 0, \sum_{i=1}^n \mu_i^2 \neq 0 \text{ and } \sum_{i=1}^n c_i = 1. \tag{3.14}$$

The last assumption is imposed for tractability and is natural as we can approximate the complement of a distribution function by a combination of exponentials according with [6, Section 3].

Next, we assume that the function $\beta(a)$ can be expressed in exponential form as follows

$$\beta(a) = \beta e^{-\mu a}, \tag{3.15}$$

and with a notational change one can let $\beta = 1$. We should mention that, if $\mu > 0$, the function in (3.15) describes behavior analogous to that of mammals (see [15, p. 404], or [12, 19] for more details). On the other hand for $a = 0$, this function $\beta(a)$ is of a form that could be appropriate for many species of fish where fecundity increases with age (size).

Our first main result can be expressed in the following.

Theorem 3.1. *Assume the following: (3.13), (3.14) and (3.15). Under these assumptions, the integral equation (2.12) has a unique solution. Moreover, the solution and the population (2.1) can be expressed exactly when $n = 2$, $c_1 \neq 0$, $c_2 \neq 0$ and $\Phi\,(a)$ is the Dirac function.*

Proof. In the first part, we establish the existence and uniqueness of the solution. Returning to our objective, in our current context

$$B(t) = \Psi(t) + \int_0^t \beta(a)\pi(a)B(t-a)\,da,$$

becomes

$$B(t) = \Psi(t) + \sum_{i=1}^{n} c_i B_i(t),$$

where

$$B_i(t) = \int_0^t e^{-(\mu+\mu_i)a} B(t-a)da = \int_0^t e^{-(\mu+\mu_i)(t-z)} B(z)dz. \tag{3.16}$$

It follows that

$$B_i'(t) = B(t) - (\mu+\mu_i)\int_0^t e^{-(\mu+\mu_i)(t-z)} B(z)dz = B(t) - (\mu+\mu_i)B_i(t).$$

Thus we get the following ODE system for $B_i(t)$ in the form

$$B_i'(t) = \Psi(t) + (c_i - (\mu+\mu_i))B_i(t) + \sum_{j\neq i}^{n} c_j B_j(t), \qquad B_i(0) = 0.$$

We can rewrite the system as

$$B_i'(t) = \psi(t) + \sum_{j=1}^{n} \alpha_{ij} B_j(t),$$

with

$$\alpha_{ij} = c_j \quad i \neq j, \quad \alpha_{ii} = c_i - (\mu+\mu_i).$$

In canonical form the system can be written

$$b'(t) = Ab(t) + u(t), \tag{3.17}$$

where

$$b = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}, \; u = \begin{pmatrix} \psi \\ \vdots \\ \psi \end{pmatrix}$$

and

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}.$$

From the general theory of linear ODE systems, (3.17) has a unique solution, which, given the boundary condition is

$$b(t) = \int_0^t e^{A(t-s)} u(s)ds.$$

Let us notice that

$$\Psi(t) = \beta(t)\pi(t) = \sum_{i=1}^{n} c_i e^{-(\mu+\mu_i)t},$$

and

$$R = \int_0^\infty \beta(t)\pi(t)dt = \sum_{i=1}^{n} \frac{c_i}{\mu + \mu_i}. \tag{3.18}$$

We can compute $B(t)$ via Laplace transform (see [7]).

We assume for simplicity that $n = 2$. In the remaining of the paper the Laplace transform is denoted by a hat

$$\hat{B}(p) = \widehat{\Phi}\,(p) + \widehat{F}\,(p)\,\hat{B}(p). \tag{3.19}$$

Here $\widehat{\Phi}\,(p)$ is the Laplace transform of Ψ

$$\begin{aligned}
\widehat{\Phi}\,(p) &= \int_0^\infty \Psi\,(a)\,e^{-pa}da \\
&= \int_0^\infty \sum_{i=1}^{2} c_i e^{-(\mu+\mu_i)a} e^{-pa} da \\
&= \sum_{i=1}^{2} c_i \int_0^\infty e^{-(p+\mu+\mu_i)a} da \\
&= \sum_{i=1}^{2} \frac{c_i}{p+\mu+\mu_i},
\end{aligned}$$

$\widehat{F}\,(p)$ is the Laplace transform of $\beta\pi$

$$\widehat{F}\,(p) = \int_0^\infty \beta(a)\pi(a)e^{-pa}da = \frac{c_1}{p+\mu+\mu_1} + \frac{c_2}{p+\mu+\mu_2}.$$

Observing that

$$\widehat{F}\,(0) = R, \ \lim_{p\longrightarrow\infty} \widehat{F}\,(p) = 0 \text{ and } \widehat{F}\,(p) \text{ is decreasing in } p$$

we conclude that the equation

$$\widehat{F}\,(p) = 1$$

has a unique real solution, denoted by p_0. Moreover, we distinguish the cases

1. if $R > 1$ then $p_0 > 0$;
2. if $R < 1$ then $p_0 < 0$;
3. if $R = 1$ then $p_0 = 0$.

Indeed, let us prove for the case 1. It is enough to assume by contradiction that $p_0 < 0$ to see

$$1 = \widehat{F}\,(p_0) > \widehat{F}\,(0) = R,$$

a contradiction with $R > 1$, concluding the case 1.

Let us return to the Laplace transform of B, which is given by

$$\hat{B}(p) = \sum_{i=1}^{2} \frac{c_i}{p+\mu+\mu_i} + \sum_{i=1}^{2} \frac{c_i}{p+\mu+\mu_i}\hat{B}(p).$$

Then, if $p \neq p_0$ we have

$$\hat{B}(p) = \frac{\frac{c_1}{p+\mu+\mu_1} + \frac{c_2}{p+\mu+\mu_2}}{1 - \frac{c_1}{p+\mu+\mu_1} - \frac{c_2}{p+\mu+\mu_2}},$$

or equivalently

$$\hat{B}(p) = \frac{c_1 (p+\mu+\mu_2) + c_2 (p+\mu+\mu_1)}{(p+\mu+\mu_1)(p+\mu+\mu_2) - c_1 (p+\mu+\mu_2) - c_2 (p+\mu+\mu_1)},$$

which is well defined for all $p \neq p_0$.

Let p_1, p_2 the roots of

$$p^2 + (2\mu+\mu_1+\mu_2-1)p + (\mu+\mu_1)(\mu+\mu_2) - \mu - c_1\mu_2 - c_2\mu_1 = 0.$$

This quadratic has real roots since the discriminant is positive

$$\Delta = (\mu_1 - \mu_2 + c_2 - c_1)^2 + 4c_1(1-c_1) \geq 0,$$

and the roots are

$$p_1 = \frac{-(2\mu+\mu_1+\mu_2-1)+\sqrt{\Delta}}{2}, \; p_2 = \frac{-(2\mu+\mu_1+\mu_2-1)-\sqrt{\Delta}}{2}.$$

Moreover, we observe that $\Delta > 0$ (this is the case when $c_1 \neq 0$, $c_2 \neq 0$). As such

$$\hat{B}(p) = \left(\frac{p_1+\mu+c_2\mu_1+c_1\mu_2}{p_1-p_2}\right)\frac{1}{p-p_1} + \left(\frac{p_2+\mu+c_2\mu_1+c_1\mu_2}{p_2-p_1}\right)\frac{1}{p-p_2}.$$

Therefore

$$B(t) = \left(\frac{p_1+\mu+c_2\mu_1+c_1\mu_2}{p_1-p_2}\right)e^{p_1 t} + \left(\frac{p_2+\mu+c_2\mu_1+c_1\mu_2}{p_2-p_1}\right)e^{p_2 t}.$$

To calculate the total population, let us remember that

$$\begin{aligned} P(t) &= \int_0^\infty \rho(a,t)\,da = \int_0^t \rho(a,t)\,da + \int_t^\infty \rho(a,t)\,da \\ &= \int_0^t B(t-a)e^{-\int_0^a \mu(\alpha)\,d\alpha}\,da + \int_t^\infty \Phi(a-t)e^{-\int_{a-t}^a \mu(\alpha)\,d\alpha}\,da, \end{aligned}$$

so the population size will be different from the previous model studied. Here

$$e^{-\int_{a-t}^a \mu(\alpha)d\alpha} = e^{-\int_{a-t}^a \mu(\alpha)\,d\alpha} = e^{-\int_0^a \mu(\alpha)\,d\alpha} e^{\int_0^{a-t} \mu(\alpha)\,d\alpha} = \frac{\pi(a)}{\pi(a-t)}.$$

Using the definition of π

$$\pi(a) = c_1 e^{-\mu_1 a} + c_2 e^{-\mu_2 a}$$

yields

$$\frac{\pi(a)}{\pi(a-t)} = \frac{c_1 e^{-\mu_1 a} + c_2 e^{-\mu_2 a}}{c_1 e^{-\mu_1 (a-t)} + c_2 e^{-\mu_2 (a-t)}}.$$

In the final part of the proof theorem, we concentrate on a particular scenario of our model by supposing that $\Phi(a)$ is the Dirac function. Under this assumption

$$\Phi(a-t)\frac{\pi(a)}{\pi(a-t)} = \Phi(a-t)\frac{\pi(t)}{\pi(t-t)},$$

which implies

$$\int_t^\infty \Phi(a-t)\frac{\pi(a)}{\pi(a-t)}da = \frac{\pi(t)}{\pi(0)}\int_t^\infty \Phi(a-t)da = c_1 e^{-\mu_1 t} + c_2 e^{-\mu_2 t}.$$

In order to ease notation

$$B(t) = K_1 e^{p_1 t} + K_2 e^{p_2 t},$$

with

$$K_1 = \left(\frac{p_1 + \mu + c_2\mu_1 + c_1\mu_2}{p_1 - p_2}\right), \quad K_2 = \left(\frac{p_2 + \mu + c_2\mu_1 + c_1\mu_2}{p_2 - p_1}\right).$$

Thus,

$$\begin{aligned} & B(t-a)e^{-\int\limits_0^a \mu(\alpha)\,d\alpha} \\ = \;& \left(K_1 e^{p_1(t-a)} + K_2 e^{p_2(t-a)}\right)\left(c_1 e^{-\mu_1 a} + c_2 e^{-\mu_2 a}\right) \\ = \;& K_1 c_1 e^{p_1 t - (\mu_1+p_1)a} + K_1 c_2 e^{p_1 t - (\mu_2+p_1)a} + K_2 c_1 e^{p_2 t - (\mu_1+p_2)a} + K_2 c_2 e^{p_2 t - (\mu_2+p_2)a} \\ = \;& K_1 e^{p_1 t}\left(c_1 e^{-(\mu_1+p_1)a} + c_2 e^{-(\mu_2+p_1)a}\right) + K_2 e^{p_2 t}\left(c_1 e^{-a(\mu_1+p_2)} + c_2 e^{-(\mu_2+p_2)a}\right) \end{aligned}$$

from where

$$\begin{aligned} & \int\limits_0^t B(t-a)e^{-\int\limits_0^a \mu(\alpha)\,d\alpha}\,da \\ & = K_1\left(c_1 \tfrac{1-e^{-(\mu_1+p_1)t}}{\mu_1+p_1} + c_2 \tfrac{1-e^{-(\mu_2+p_1)t}}{\mu_2+p_1}\right)e^{tp_1} + K_2\left(c_1 \tfrac{1-e^{-(\mu_1+p_2)t}}{\mu_1+p_2} + c_2 \tfrac{1-e^{-(\mu_2+p_2)t}}{\mu_2+p_2}\right)e^{p_2 t} \\ & = K_1\left(c_1 \tfrac{e^{p_1 t}-e^{-\mu_1 t}}{\mu_1+p_1} + c_2 \tfrac{e^{p_1 t}-e^{-\mu_2 t}}{\mu_2+p_1}\right) + K_2\left(c_1 \tfrac{e^{p_2 t}-e^{-\mu_1 t}}{\mu_1+p_2} + c_2 \tfrac{e^{p_2 t}-e^{-\mu_2 t}}{\mu_2+p_2}\right). \end{aligned}$$

Therefore

$$\begin{aligned} P(t) &= K_1\left(c_1 \frac{e^{p_1 t} - e^{-\mu_1 t}}{\mu_1 + p_1} + c_2 \frac{e^{p_1 t} - e^{-\mu_2 t}}{\mu_2 + p_1}\right) \\ & \quad + K_2\left(c_1 \frac{e^{p_2 t} - e^{-\mu_1 t}}{\mu_1 + p_2} + c_2 \frac{e^{p_2 t} - e^{-\mu_2 t}}{\mu_2 + p_2}\right) + c_1 e^{-\mu_1 t} + c_2 e^{-\mu_2 t}. \\ &= \left(\frac{c_1 K_1}{\mu_1 + p_1} + \frac{c_2 K_1}{\mu_2 + p_1}\right)e^{p_1 t} + \left(\frac{c_1 K_2}{\mu_1 + p_2} + \frac{c_2 K_2}{\mu_2 + p_2}\right)e^{p_2 t} \\ & \quad - \left(\frac{c_1 K_1}{\mu_1 + p_1} + \frac{c_1 K_2}{\mu_1 + p_2} - c_1\right)e^{-\mu_1 t} - \left(\frac{c_2 K_1}{\mu_2 + p_1} + \frac{c_2 K_2}{\mu_2 + p_2} - c_2\right)e^{-\mu_2 t}. \end{aligned}$$

Having solved for $P(t)$ in closed form we can perform an asymptotic analysis, which clearly is as $t \longrightarrow \infty$

$$\text{either } P(t) \to 0 \text{ either } P(t) \to \infty \text{ either } P(t) \to l \in (0, \infty).$$

The asymptotic analysis shows the trade-off between fertility parameters and survival parameters and which ones prevail in the limit. Our first main theorem has now been proven. □

To conclude this section, we observe that in the special case of $c_2 = 0$ (take $\mu_2 = \mu_1$ so $\Delta > 0$) we obtain

$$B(t) = e^{(1-(\mu+\mu_1))t},$$

which is the result obtained in the genesis model by [2] (page 279, example 1). Indeed this is the case since, with these parameters,

$$p_2 = -\mu - \mu_1, p_1 = 1 - \mu - \mu_1,$$

and

$$p_2 + \mu + c_2\mu_1 + c_1\mu_2 = 0.$$

4. Another pseudo exponential survival

In the following, inspired by [1] we consider another pseudo exponential survival as follows

$$\pi(a) = \left(\sum_{i=0}^{n} c_i a^i\right) e^{-\mu_1 a}, \tag{4.20}$$

and

$$\beta(a) = e^{-\mu a}, \tag{4.21a}$$

with the appropriate conditions on the parameters defined in (3.14).

We are now able to express our second main result as follows.

Theorem 4.2. *Suppose that the following assumptions hold: (4.20) and (4.21a). Under these assumptions, the integral equation (2.12) has a unique solution. Additionally, when $n = 1$ and $\Phi(a)$ is the Dirac function, the solution and the population (2.1) can be expressed exactly.*

Proof. We follow the same approach as in Theorem 3.1, starting with the existence and uniqueness of the solution (2.12) becomes

$$B(t) = \Psi(t) + \sum_{i=0}^{n} c_i B_i(t),$$

where

$$B_i(t) = \int_0^t e^{-\overline{\mu}a} a^i B(t-a)da = \int_0^t e^{-\overline{\mu}(t-z)}(t-z)^i B(z)dz, \tag{4.22}$$

and $\overline{\mu} = \mu + \mu_1$.

By differentiating we get the following ODE system for $B_i(t)$ in the form

$$\begin{aligned} B_0'(t) &= B(t) - \overline{\mu} B_0(t), \\ B_i'(t) &= -\overline{\mu} B_i(t) + i B_{i-1}(t), \quad i \geq 1. \end{aligned}$$

In canonical form the system can be written

$$b'(t) = Ab(t) + u(t), \tag{4.23}$$

where

$$b = \begin{pmatrix} B_0 \\ \vdots \\ B_n \end{pmatrix}, \; u = \begin{pmatrix} \psi \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} c_0 - \overline{\mu} & c_1 & c_2 & c_3 & \cdots & c_n \\ 1 & -\overline{\mu} & 0 & 0 & \cdots & 0 \\ 0 & 2 & -\overline{\mu} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & -\overline{\mu} \end{pmatrix}.$$

From the general theory of linear ODE systems, (4.23) has a unique solution, which, given the boundary condition is

$$b(t) = \int_0^t e^{A(t-s)} u(s) ds.$$

Our next step is to find the population in exact form. As in the case of (3.18) we have

$$R = \int_0^\infty \beta(t)\pi(t)dt = \sum_{i=0}^n c_i \int_0^\infty t^i e^{-(\mu_1+\mu)t}dt = \sum_{i=0}^n \frac{c_i i!}{(\mu_1+\mu)^{i+1}}. \tag{4.24}$$

As in Section 3 we denote by p_0 the solution of

$$\widehat{F}(p) = 1$$

where $\widehat{F}(p)$ is the Laplace transform of $\beta\pi$. Then, the Laplace transform of B for $n = 1$ is given by

$$\hat{B}(p) = \frac{c_0}{p+\bar{\mu}} + \frac{c_1}{(p+\bar{\mu})^2} + \left[\frac{c_0}{p+\bar{\mu}} + \frac{c_1}{(p+\bar{\mu})^2}\right]\hat{B}(p).$$

Consequently

$$\hat{B}(p) = \frac{c_0(p+\bar{\mu}) + c_1}{(p+\bar{\mu})^2 - c_0(p+\bar{\mu}) - c_1},$$

is well defined for all $p \neq p_0$. Let's point that

$$\frac{c_0(p+\bar{\mu}) + c_1}{(p+\bar{\mu})^2 - c_0(p+\bar{\mu}) - c_1} = \frac{A\,(p-p_2+\bar{\mu})}{(p-p_1+\bar{\mu})\,(p-p_2+\bar{\mu})} + \frac{B\,(p-p_1+\bar{\mu})}{(p-p_1+\bar{\mu})\,(p-p_2+\bar{\mu})},$$

where p_1, p_2 are the roots of the quadratic

$$x^2 - c_0x - c_1 = 0.$$

They are real since $\Delta = c_0^2 + 4c_1$ and

$$p_1 = \frac{c_0+\sqrt{\Delta}}{2}, p_2 = \frac{c_0-\sqrt{\Delta}}{2}.$$

Moreover

$$\begin{cases} A + B = c_0 \\ A\,(-p_2+\bar{\mu}) + B\,(-p_1+\bar{\mu}) = c_0\bar{\mu} + c_1. \end{cases}$$

This system has the solution

$$A = \frac{c_1+c_0p_1}{p_1-p_2} \text{ and } B = -\frac{c_1+c_0p_2}{p_1-p_2}.$$

Thus,

$$\hat{B}(p) = \frac{c_1+c_0p_1}{p_1-p_2}\frac{1}{p-p_1+\bar{\mu}} - \frac{c_1+c_0p_2}{p_1-p_2}\frac{1}{p-p_2+\bar{\mu}}.$$

Therefore

$$B(t) = \frac{c_1+c_0p_1}{p_1-p_2}e^{(p_1-\bar{\mu})t} - \frac{c_1+c_0p_2}{p_1-p_2}e^{(p_2-\bar{\mu})t}.$$

The total population can now be calculated, as previously mentioned

$$\begin{aligned} P(t) &= \int_0^\infty \rho(a,t)\,da = \int_0^t \rho(a,t)\,da + \int_t^\infty \rho(a,t)\,da \\ &= \int_0^t B(t-a)e^{-\int_0^a \mu(\alpha)\,d\alpha}da + \int_t^\infty \Phi(a-t)e^{-\int_{a-t}^a \mu(\alpha)\,d\alpha}da. \end{aligned}$$

We can make an initial observation that the population size will be distinct from the previous model. Here

$$e^{-\int_{a-t}^a \mu(\alpha)d\alpha} = \frac{\pi(a)}{\pi(a-t)} = \frac{(c_0+c_1a)e^{-\mu_1 a}}{[c_0+c_1\,(a-t)]\,e^{-\mu_1(a-t)}},$$

since

$$\pi(a) = (c_0 + c_1 a)e^{-\mu_1 a}.$$

Let us notice that

$$\begin{aligned}
\int_t^\infty \Phi(a-t)\frac{\pi(a)}{\pi(a-t)}da &= \int_t^\infty \Phi(a-t)\frac{\pi(t)}{\pi(t-t)}da \\
&= \frac{\pi(t)}{\pi(0)}\int_t^\infty \Phi(a-t) = \frac{\pi(t)}{c_0} = \frac{(c_0+c_1 t)e^{-\mu_1 t}}{c_0}.
\end{aligned}$$

In order to ease notation

$$B(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t},$$

with

$$K_1 = \frac{c_1 + c_0 p_1}{p_1 - p_2}, \quad K_2 = -\frac{c_1 + c_0 p_2}{p_1 - p_2}, \quad \lambda_1 = p_1 - \bar{\mu}, \quad \lambda_2 = p_2 - \bar{\mu}.$$

Thus,

$$\begin{aligned}
& B(t-a)e^{-\int_0^a \mu(\alpha)\,d\alpha} \\
={}& \left(K_1 e^{\lambda_1(t-a)} + K_2 e^{\lambda_2(t-a)}\right)\frac{1}{2}(c_0+c_1 t)e^{-\mu_1 t} \\
={}& K_1\frac{c_0}{e^{a\lambda_1}}e^{(\lambda_1-\mu_1)t} + tK_1\frac{c_1}{e^{a\lambda_1}}e^{(\lambda_1-\mu_1)t} + tK_2\frac{c_1}{e^{a\lambda_2}}e^{(\lambda_2-\mu_1)t} + K_2\frac{c_0}{e^{a\lambda_2}}e^{(\lambda_2-\mu_1)t} \\
={}& K_1 e^{(\lambda_1-\mu_1)t}e^{-\lambda_1 a}(c_0+tc_1) + K_2 e^{(\lambda_2-\mu_1)t}e^{-\lambda_2 a}(tc_1+c_0) \\
={}& (tc_1+c_0)\left(K_1 e^{(\lambda_1-\mu_1)t}e^{-\lambda_1 a} + K_2 e^{(\lambda_2-\mu_1)t}e^{-\lambda_2 a}\right)
\end{aligned}$$

from where

$$\begin{aligned}
\int_0^t B(t-a)e^{-\int_0^a \mu(\alpha)\,d\alpha}da &= (tc_1+c_0)\int_0^t \left(K_1 e^{(\lambda_1-\mu_1)t}e^{-a\lambda_1} + K_2 e^{(\lambda_2-\mu_1)t}e^{-a\lambda_2}\right)da \\
&= (tc_1+c_0)\int_0^t \left(K_1 e^{(\lambda_1-\mu_1)t}e^{-\lambda_1 a} + K_2 e^{(\lambda_2-\mu_1)t}e^{-\lambda_2 a}\right)da \\
&= (tc_1+c_0)\left(\int_0^t K_1 e^{(\lambda_1-\mu_1)t}e^{-\lambda_1 a}da + \int_0^t K_2 e^{(\lambda_2-\mu_1)t}e^{-\lambda_2 a}da\right) \\
&= (tc_1+c_0)\left(K_1 e^{(\lambda_1-\mu_1)t}\int_0^t e^{-\lambda_1 a}da + K_2 e^{(\lambda_2-\mu_1)t}\int_0^t e^{-\lambda_2 a}da\right) \\
&= \left(K_1 e^{(\lambda_1-\mu_1)t}\frac{1-e^{-\lambda_1 t}}{\lambda_1} + K_2 e^{(\lambda_2-\mu_1)t}\frac{1-e^{-\lambda_2 t}}{\lambda_2}\right)(tc_1+c_0)
\end{aligned}$$

Thus,

$$
\begin{aligned}
P(t) &= \left(K_1 e^{(\lambda_1-\mu_1)t}\frac{1-e^{-\lambda_1 t}}{\lambda_1} + K_2 e^{(\lambda_2-\mu_1)t}\frac{1-e^{-\lambda_2 t}}{\lambda_2}\right)(tc_1+c_0) + \frac{c_0+c_1 t}{c_0}e^{-\mu_1 t} \\
&= \left(K_1 e^{\lambda_1 t}\frac{1-e^{-\lambda_1 t}}{\lambda_1} + K_2 e^{\lambda_2 t}\frac{1-e^{-\lambda_2 t}}{\lambda_2}\right)(c_0+c_1 t)e^{-\mu_1 t} + \frac{c_0+c_1 t}{c_0}e^{-\mu_1 t} \\
&= \left(K_1 \frac{e^{\lambda_1 t}-1}{\lambda_1} + K_2 \frac{e^{\lambda_2 t}-1}{\lambda_2}\right)(c_0+c_1 t)e^{-\mu_1 t} + \frac{c_0+c_1 t}{c_0}e^{-\mu_1 t} \\
&= \left(\frac{K_1 e^{\lambda_1 t}}{\lambda_1} - \frac{K_1}{\lambda_1} + \frac{K_2 e^{\lambda_2 t}}{\lambda_2} - \frac{K_2}{\lambda_2}\right)(c_0+c_1 t)e^{-\mu_1 t} + \frac{c_0+c_1 t}{c_0}e^{-\mu_1 t} \\
&= \left[\frac{K_1}{\lambda_1}e^{(\lambda_1-\mu_1)t} + \frac{K_2}{\lambda_2}e^{(\lambda_2-\mu_1)t} + \left(-\frac{K_1}{\lambda_1} - \frac{K_2}{\lambda_2} + \frac{1}{c_0}\right)e^{-\mu_1 t}\right](c_0+c_1 t).
\end{aligned}
$$

Having solved for $P(t)$ in closed form we can perform an asymptotic analysis as in the previous section. We have now completed the proof for our second main theorem. □

To conclude this section, we observe that in the special case of $c_0 = 1$ and $c_1 = 0$, we have

$$p_1 = 1, p_2 = 0$$

leading to

$$B(t) = e^{(1-\bar{\mu})t} = e^{(1-(\mu+\mu_1))t},$$

which is the result obtained in the genesis model by [2] (page 279, example 1).

Remark 4.1. As per [6], the pseudo exponential functions that we consider can approximate any continuous survival probability function, thereby making our approach in Theorems 3.1-4.2 applicable to all such functions.

We consider a fertility model at the end of the paper, using the same arguments as in our Sections 3-4 and inspired by [18].

4.1. **A Fertility Model .** We take the fertility function to be

$$\beta(a) = \left(\sum_{i=0}^{n} c_i a^i\right) e^{-\mu a}.$$

Moreover, the survival function is assumed exponential

$$\pi(a) = e^{-\mu_1 a}.$$

Let us notice that we get the same equation for $B(t)$ as in the previous model since the product $\beta(a)\pi(a)$ is the same. As such, we obtain the same formula for $B(t)$ and his Laplace transform. Next, we will compute the total population $P(t)$ in the special case of $n = 1$. Recall that in this case

$$B(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t},$$

with

$$K_1 = \frac{c_1 + c_0 p_1}{p_1 - p_2}, \quad K_2 = -\frac{c_1 + c_0 p_2}{p_1 - p_2}, \ \lambda_1 = p_1 - \bar{\mu}, \quad \lambda_2 = p_2 - \bar{\mu}.$$

The total population $P(t)$ is given by

$$\begin{aligned} P(t) &= \int_0^\infty \rho(a,t)\,da = \int_0^t \rho(a,t)\,da + \int_t^\infty \rho(a,t)\,da \\ &= \int_0^t B(t-a)e^{-\int_0^a \mu(\alpha)\,d\alpha}\,da + \int_t^\infty \Phi(a-t)e^{-\int_{a-t}^a \mu(\alpha)\,d\alpha}\,da, \end{aligned}$$

so the population size will be different from the previous model. Here $\mu(\alpha) = \mu_1$. Let us notice that

$$\int_t^\infty \Phi(a-t)e^{-\int_{a-t}^a \mu(\alpha)\,d\alpha}\,da = \frac{\pi(t)}{\pi(0)} \int_t^\infty \Phi(a-t)da = e^{-\mu_1 t}.$$

Moreover,

$$\begin{aligned} \int_0^t B(t-a)e^{-\int_0^a \mu(\alpha)\,d\alpha}\,da &= \int_0^t B(t-a)e^{-\mu_1 a}\,da \\ &= \int_0^t \left(K_1 e^{\lambda_1(t-a)} + K_2 e^{\lambda_2(t-a)}\right) e^{-\mu_1 a}\,da \\ &= \int_0^t \left(K_1 e^{\lambda_1 t-(\lambda_1+\mu_1)a} + K_2 e^{\lambda_2 t-(\lambda_2+\mu_1)a}\right)\,da \\ &= \frac{K_1}{\lambda_1+\mu_1}e^{\lambda_1 t} + \frac{K_2}{\lambda_2+\mu_1}e^{\lambda_2 t} - \left(\frac{K_1}{\lambda_1+\mu_1} + \frac{K_2}{\lambda_2+\mu_1}\right)e^{-\mu_1 t} \\ &= \frac{K_1}{\lambda_1+\mu_1}\left(e^{\lambda_1 t} - e^{-\mu_1 t}\right) + \frac{K_2}{\lambda_2+\mu_1}\left(e^{\lambda_2 t} - e^{-\mu_1 t}\right). \end{aligned}$$

Therefore

$$\begin{aligned} P(t) &= \frac{K_1}{\lambda_1+\mu_1}\left(e^{\lambda_1 t} - e^{-\mu_1 t}\right) + \frac{K_2}{\lambda_2+\mu_1}\left(e^{\lambda_2 t} - e^{-\mu_1 t}\right) + e^{-\mu_1 t} \\ &= \frac{K_1}{\lambda_1+\mu_1}e^{\lambda_1 t} + \frac{K_2}{\lambda_2+\mu_1}e^{\lambda_2 t} + \left(1 - \frac{K_1}{\lambda_1+\mu_1} - \frac{K_2}{\lambda_2+\mu_1}\right)e^{-\mu_1 t}. \end{aligned}$$

Having solved for $P(t)$ in closed form we can perform an asymptotic analysis as before.

5. Conclusion

Our results show that the population is a sum of three or four exponentials (depending on the model considered), which is a significant departure from the classical exponential survival model where the population is exponential. Our model not only allows for more shapes of the population function of time but also provides a better fit to population data. There is a rich literature on the fitting of real-world data with a linear combination of exponentials, and we have mentioned [5] and [9] as examples. The latter shows the superiority of the fit when more exponentials are employed (the comparison is done with one, two, and three exponentials). Consequently, our population models will perform much better than the classical exponential survival model on data fitting. While more general models for population with numerical solutions are available, the lack of explicit solutions will render them not applicable to data fitting.

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