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Constructing DNA Codes using Double Cyclic Codes of Odd and Even Lengths over $\mathbb{F}_2 + u\mathbb{F}_2$

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ABSTRACT. In this paper, we investigate the algebraic structure and theorems for constructing DNA codes of double cyclic codes of length (α, β) over the finite commutative chain ring $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$, where α and β are odd and even positive integers, respectively. Our main objectives include studying the structural properties of these codes, determining their generator polynomials and generating DNA codes from these codes. We investigate theorems for two types of double cyclic codes of length (α, β) over $\mathbb{F}_2 + u\mathbb{F}_2$, ideal for generating DNA codes. These codes utilise non-separable structures and satisfy reverse and reverse-complement constraints. Additionally, we propose DNA codes derived from our results.

1. INTRODUCTION

Deoxyribonucleic acid, or DNA, is a molecule that contains the necessary genetic materials for all living things to form and maintain an organism. DNA is a linear molecule consisting of four different nucleotide bases: Adenine (A), Thymine (T), Guanine (G) and Cytosine (C). DNA strands can be viewed as sequences of these nucleotide bases. Each strand of DNA is an ordered quaternary sequence of the letters A, T, G and C with two distinct ends known as the $5'$ and $3'$ ends. According to the Watson-Crick complement rule (WCC), the two strands are connected by chemical bonds between the bases, i.e. A bonds with T and C bonds with G. The complement of a nucleotide base x will be denoted by $\overline{x}$, that is, $\overline{A} = T, \overline{T} = A, \overline{G} = C$ and $\overline{C} = G$. This pairing is done in reverse order with opposite directions. For example, a DNA strand $5' - TGAGCAA - 3'$ pairs with a DNA strand $3' - \overline{T}\,\overline{G}\,\overline{A}\,\overline{G}\,\overline{C}\,\overline{A}\,\overline{A} - 5'$ and its pair can be written as $5' - TTGCTCA - 3'$.

The concept of DNA computing was started in 1994. In [3], the author used DNA molecules to solve the directed Hamiltonian path problem. Since then, DNA codes have been used to improve work and solve problems in computer science and mathematics. Since the DNA alphabet consists of four letters, early studies of DNA computing that employed algebraic coding theory techniques worked on an error-correcting code over four-element sets with an algebraic structure. The construction of DNA codes using additive and linear codes over a finite field with four elements was studied in [6]. The authors provided four combinatorial constraints for a good DNA code: Hamming distance, reverse constraint, reverse-complement constraint and fixed GC-constraint. The first three constraints are designed to reduce the possibility of unwanted hybridizations between DNA strands and the last constraint aims to create similar melting temperatures. A code that satisfies some or all of the constraints is called a DNA code. With these constraints, studies on DNA codes employing linear and cyclic codes over different algebraic structures were investigated. For instance, in [1], the authors studied the theory of constructing linear and cyclic codes of odd length over $GF(4)$ suited for DNA computing, with a focus on

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the reverse-complement constraint. Later, in [11], the authors investigated the construction of DNA codes using cyclic codes over $\mathbb{F}_2[u]/\langle u^2-1\rangle$ based on CG-constraint. Shortly after, the investigation of constructing cyclic codes of odd length and even length over the finite ring $\mathbb{F}_2+u\mathbb{F}_2$, where $u^2=0$ satisfy reverse and reverse-complement constrains was studied in [7] and [9]. Constructing DNA codes in the works mentioned above has been studied using a code over an alphabet derived from either fields or rings. Recently, in [5], the authors introduced a new concept for constructing DNA codes using a new type of linear codes. The new type of codes is called $\mathbb{F}_4RS$-cyclic codes of length (α,β,γ), where $R=\mathbb{F}_4+u\mathbb{F}_4$, with $u^2=u$ and $S=\mathbb{F}_4+u\mathbb{F}_4+v\mathbb{F}_4$, with $u^2=u, v^2=v, uv=vu=0$. These codes are made up of 3 alphabet sets and are a generalisation of cyclic codes over the rings $\mathbb{F}_4$, R and S. The authors gave necessary and sufficient conditions for $\mathbb{F}_4RS$-cyclic codes to be reversible and reverse-complement codes. However, the concept of this work is based on separable codes. In general, we will refer to this type of codes as cyclic codes over mixed alphabets.

A new type of linear codes known as double cyclic codes was recently introduced in [4] and these codes are a generalisation of cyclic codes. The authors studied the algebraic structure of double cyclic codes over $\mathbb{Z}_2$. Furthermore, they showed that double cyclic codes of length (α,β) over $\mathbb{Z}_2$ can be viewed as $\mathbb{Z}_2[x]$-submodules of $\frac{\mathbb{Z}_2[x]}{\langle x^\alpha-1\rangle}\times\frac{\mathbb{Z}_2[x]}{\langle x^\beta-1\rangle}$, where α and β are positive integers. Shortly after, the study of the algebraic structure of double cyclic codes and their dual codes over various finite fields and finite rings gained the attention of many researchers. Since the structure of double cyclic codes is similar to the structure of cyclic codes over mixed alphabets, in [8], we applied the idea of construction DNA codes from $\mathbb{F}_4RS$-cyclic codes to double cyclic codes. We proposed theorems for generating DNA codes using non-separable codes of double cyclic codes of length (α,β) over $\mathbb{F}_2+u\mathbb{F}_2$. However, the values of α and β are odd positive integers. Based on the recent work, in this paper, we are motivated to study the construction of DNA codes using non-separable double cyclic codes over the ring $\mathbb{F}_2+u\mathbb{F}_2$, where $u^2=0$, but we will concentrate on the various length values. The values of the lengths α and β are positive integers with odd and even values, respectively. We begin by investigating the structure of double cyclic codes of length (α,β) over $\mathbb{F}_2+u\mathbb{F}_2$. Following that, we study the necessary and sufficient conditions for the double cyclic codes to generate DNA codes.

This paper is organised as follows: In section 2, we give some definitions and the algebraic structure of cyclic codes over $\mathbb{F}_2+u\mathbb{F}_2$. In section 3, we discuss the structure of double cyclic codes of length (α,β) over $\mathbb{F}_2+u\mathbb{F}_2$ and determine their generator polynomials, where α is an odd positive integer and β is an even positive integer. In section 4, we investigate the necessary and sufficient conditions for non-separable double cyclic codes to be reversible and reversible-complement codes. Furthermore, we provide examples of DNA codes generated by our results.

2. Preliminaries

Let $(\Re,+,\cdot)$ be a finite commutative ring with identity. An ideal I of $\Re$ is called principal if it is generated by one element. A ring $\Re$ is a principal ideal ring if its ideals are principal. The ring $\Re$ is called a local ring if $\Re$ has a unique maximal ideal. Furthermore, the ring $\Re$ is called a chain ring if the set of all ideals of $\Re$ is linearly ordered under set-theoretic inclusion. Let $a,b\in\Re$ and $a\neq 0$. If there is an element $c\in\Re$ such that $b=a\cdot c$, we say that a divides b in $\Re$ and write $a|b$ in $\Re$. In this case, a is called a divisor of b. Let $\Re[x]$ be a polynomial ring over the ring $\Re$. For any $a(x)\in\Re[x]$, $a(x)$ is regular if $a(x)$ is not a zero divisor.

Proposition 2.1. [10] *Suppose that $\Re$ is a finite local commutative ring with maximal ideal $\langle m\rangle$. Let $a(x)$ and $b(x)$ be non-zero polynomials in $\Re[x]$. If $a(x)$ is regular, then there exist polynomials $q(x)$ and $r(x)$ in $\Re[x]$ such that $b(x) = a(x)q(x) + r(x)$ and $\deg(r(x)) \leq \deg(a(x)) - 1$.*

Throughout the paper, R denotes the commutative ring $\mathbb{F}_2 + u\mathbb{F}_2 = \{0, 1, u, 1+u\}$ with $u^2 = 0$. The ring R is a finite commutative chain ring with the maximal ideal $\langle u\rangle$ and the characteristic 2. Let $f(x) = f_0 + f_1x + \cdots + f_kx^k$ be a polynomial over R with degree k. Then, the reciprocal polynomial of $f(x)$ is defined as

$$f^*(x) = x^k f(\tfrac{1}{x}) = f_k + f_{k-1}x + \cdots + f_0x^k.$$

Then, if $f_0 \neq 0$, then $\deg(f^*(x)) = \deg(f(x))$ otherwise $\deg(f^*(x)) \leq \deg(f(x))$. Furthermore, $f(x)$ is called self-reciprocal if $f^*(x) = f(x)$.

A non-empty subset C of R^n is called a linear code of length n over R if C is an R-submodule of R^n. A linear code C of length n over R is called cyclic code if for any $(a_0, a_1, ..., a_{n-1}) \in C$, $(a_{n-1}, a_0, ..., a_{n-2}) \in C$. For any $a = (a_0, a_1, ..., a_{n-1})$ in R^n, we can identify a with a polynomial $a(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ in $\frac{R[x]}{\langle x^n-1\rangle}$. Let the quatient ring $\frac{R[x]}{\langle x^n-1\rangle}$ be denoted by R_n. Then, a linear code C of length n over R is a cyclic code if and only if C is an ideal of the ring R_n.

The structure of cyclic codes of length n over R is as follows:

Theorem 2.1. [2] *Let C be a cyclic code of length n over R. Then,*

(1) If n is an odd number, then R_n is a principal ideal ring and $C = \langle f_0(x), uf_1(x)\rangle = \langle f_0(x) + uf_1(x)\rangle$, where $f_0(x), f_1(x) \in \mathbb{F}_2[x]$ and $f_1(x)|f_0(x)|(x^n - 1)$.

(2) If n is an even number, then

(a) $C = \langle g(x) + up(x)\rangle$, where $g(x), p(x) \in \mathbb{F}_2[x]$ with $g(x)|(x^n - 1)$, $(g(x) + up(x))|(x^n - 1)$ and $g(x)|p(x)(\frac{x^n-1}{g(x)})$. Or,

(b) $C = \langle g(x) + up(x), ua(x)\rangle$, where $g(x), p(x), a(x) \in \mathbb{F}_2[x]$ with $a(x)|g(x)|(x^n - 1)$, $a(x)|p(x)(\frac{x^n-1}{g(x)})$ and $\deg(a(x)) > \deg(p(x))$.

3. Double cyclic codes over R

In this section, we will consider the structure and generator polynomials of double cyclic codes of length (α, β) over R, where α is an odd positive integer and β is an even positive integer.

Now, we will present the concept of double cyclic codes over R. Let α and β be two non-negative integers and $n = \alpha + \beta$. Then, R^n is an R-submodule of $R^\alpha \times R^\beta$ and any linear code C of length n over R is an R-submodule of $R^\alpha \times R^\beta$. For any element $\boldsymbol{v} = (a_0, a_1, ..., a_{\alpha-1}; b_0, b_1, ..., b_{\beta-1})$ in $R^\alpha \times R^\beta$, the double cyclic shift of $\boldsymbol{v}$ is defined as follows:

$$\tau(\boldsymbol{v}) = (a_{\alpha-1}, a_0, ..., a_{\alpha-2}; b_{\beta-1}, b_0, ..., b_{\beta-2}).$$

Definition 3.1. [8] *A linear code C of length $n = \alpha + \beta$ over R is called a double cyclic code of length (α, β) over R if $\tau(\boldsymbol{v}) \in C$ for all $\boldsymbol{v} \in C$.*

The code C is called separable if $C = C_\alpha \times C_\beta$, where C_α and C_β are the canonical projections of C on the first α and the last β coordinates , respectively.

Let $\boldsymbol{v} = (a_0, a_1, ..., a_{\alpha-1}; b_0, b_1, ..., b_{\beta-1}) \in R^\alpha \times R^\beta$. This element can be identified with an element in $\frac{R[x]}{\langle x^\alpha-1\rangle} \times \frac{R[x]}{\langle x^\beta-1\rangle}$ as follows:

$$\boldsymbol{v}(x) = (a_0 + a_1x + \cdots + a_{\alpha-1}x^{\alpha-1}; b_0 + b_1x + \cdots + b_{\beta-1}x^{\beta-1}) = (a(x); b(x)).$$

We denote the set $\frac{R[x]}{\langle x^\alpha-1\rangle} \times \frac{R[x]}{\langle x^\beta-1\rangle}$ by $R_{\alpha,\beta}$. This identification provides a one-to-one correspondence between $R^\alpha \times R^\beta$d and $R_{\alpha,\beta}$. For any polynomial $k(x)$ in $R[x]$ and polynomial $\boldsymbol{v}(x) = (a(x); b(x))$ in $R_{\alpha,\beta}$, the multiplication of $k(x)$ and $\boldsymbol{v}(x)$ is defined as follows:

$$k(x) * \boldsymbol{v}(x) = (k(x)a(x) \pmod{x^\alpha - 1}; k(x)b(x) \pmod{x^\beta - 1}).$$

Therefore, the ring $R_{\alpha,\beta}$ is an $R[x]$-module with respect to the usual addition and multiplication $*$. Throughout the paper, we will use $(k(x)a(x); k(x)b(x))$ instead of $(k(x)a(x) \pmod{x^\alpha - 1}; k(x)b(x) \pmod{x^\beta - 1})$. Moreover, for any $\boldsymbol{v}(x) = (a_0 + a_1x + \cdots + a_{\alpha-1}x^{\alpha-1}; b_0 + b_1x + \cdots + b_{\beta-1}x^{\beta-1})$ in $R_{\alpha,\beta}$, we obtain that $x * \boldsymbol{v}(x)$ represents the element $(a_{\alpha-1}, a_0, ..., a_{\alpha-2}; b_{\beta-1}, b_0, ..., b_{\beta-2})$ in $R^\alpha \times R^\beta$. This implies that any double cyclic code C of length (α, β) over R is an $R[x]$-submodule of $R_{\alpha,\beta}$. This reasoning leads to the following theorem.

Theorem 3.2. [8] *A linear code C of length $n = \alpha + \beta$ over R is a double cyclic code of length (α, β) over R if and only if C is an $R[x]$-submodule of $R_{\alpha,\beta}$.*

Let $f(x)$ be a non-zero polynomial over R and n be a positive integer. Then, by Proposition 2.1, there exist polynomials $q(x), r(x)$ in $R[x]$ such that $f(x) = (x^n - 1)q(x) + r(x)$, where $\deg(r(x)) \leq n - 1$. We denote the remainder $r(x)$ by $[f(x)]_{(x^n-1)}$.

Throughout the paper, α is an odd positive integer and β is an even positive integer. The following theorem is the algebraic structure of double cyclic codes of length (α, β) over R.

Theorem 3.3. *Let C be a double cyclic code of length (α, β) over R. Then, we can classify C into two types:*

Type 1: $C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x)) \rangle$, *where $f_0(x), f_1(x), g(x)$ and $p(x)$ are polynomials in $\mathbb{F}_2[x]$ with $f_1(x)|f_0(x)|(x^\alpha - 1)$, $g(x)|(x^\beta - 1), (g(x) + up(x))|(x^\beta - 1)$, $g(x)|p(x)(\frac{x^\beta-1}{g(x)})$ and $l(x)$ is a polynomial in $R[x]$.*

Type 2: $C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle$, *where $f_0(x), f_1(x), g(x), p(x), a(x)$ are polynomials in $\mathbb{F}_2[x]$ with $f_1(x)|f_0(x)|(x^\alpha - 1)$, $a(x)|g(x)|(x^\beta - 1)$, $a(x)|p(x)(\frac{x^\beta-1}{g(x)})$ and $l_1(x), l_2(x)$ are polynomials in $R[x]$.*

Proof. Let C be a double cyclic code of length (α, β) over R. We define $\varphi_\beta : C \to R_\beta$ by $\varphi_\beta(a(x); b(x)) = b(x) \pmod{x^\beta - 1}$ for all $(a(x); b(x)) \in C$. Then, φ_β is an $R[x]$-module homomorphism. It is obvious that $\varphi_\beta(C)$ is an ideal of R_β. By Theorem 2.1(2), we obtain that $\varphi_\beta(C)$ can be represented as 2 types:

1. $\varphi_\beta(C) = \langle g(x) + up(x) \rangle$, where the polynomials $g(x), p(x)$ are in $\mathbb{F}_2[x]$ with $g(x)|(x^\beta - 1), (g(x) + up(x))|(x^\beta - 1)$ and $g(x)|p(x)(\frac{x^\beta-1}{g(x)})$. Or,
2. $\varphi_\beta(C) = \langle g(x) + up(x), ua(x) \rangle$, where the polynomials $g(x), p(x), a(x)$ are in $\mathbb{F}_2[x]$ with $a(x)|g(x)|(x^\beta - 1)$ and $a(x)|p(x)(\frac{x^\beta-1}{g(x)})$.

Note that $\ker \varphi_\beta = \{(a(x); 0) : (a(x); 0) \in C\}$. Let $I = \{a(x) \in R_\alpha : (a(x); 0) \in \ker \varphi_\beta\}$. Then, I is an ideal of R_α. By Theorem 2.1(1), we obtain that $I = \langle f_0(x) + uf_1(x) \rangle$, where $f_0(x), f_1(x) \in \mathbb{F}_2[x]$ with $f_1(x)|f_0(x)|(x^\alpha - 1)$. Hence, $\ker \varphi_\beta = \langle (f_0(x) + uf_1(x); 0) \rangle$. By 1^{st}-isomorphism, $C/\ker \varphi_\beta \cong \varphi_\beta(C)$. Since $\varphi_\beta(C)$ has two types, we will divide the consideration of the double cyclic code C into two parts.

Firstly, suppose that $\varphi_\beta(C) = \langle g(x) + up(x), ua(x) \rangle$. Then, there exist polynomials $l_1(x), l_2(x) \in R[x]$ such that $(l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \in C$. This implies that $\langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle \subseteq C$. For any $\boldsymbol{v}(x) = (b(x); d(x))$

in C, we obtain that $d(x) \in \varphi_\beta(C)$. Hence, there exist polynomials $k_1(x), k_2(x) \in R[x]$ such that $d(x) = k_1(x)(g(x) + up(x)) + uk_2(x)a(x)$. Consider,

$$\begin{aligned}
&(b(x); d(x)) - [k_1(x) * (l_1(x); g(x) + up(x))] - [k_2(x) * (l_2(x); ua(x))]\\
&= (b(x) - k_1(x)l_1(x) - k_2(x)l_2(x); d(x) - k_1(x)(g(x) + up(x)) - uk_2(x)a(x))\\
&= (b(x) - k_1(x)l_1(x) - k_2(x)l_2(x); 0) \in \ker \varphi_\beta.
\end{aligned}$$

Then, $(b(x) - k_1(x)l_1(x) - k_2(x)l_2(x); 0) = k_3(x) * (f_0(x) + uf_1(x); 0)$, where $k_3(x) \in R[x]$. So, $[b(x) - k_1(x)l_1(x) - k_2(x)l_2(x)]_{(x^\alpha - 1)} = [k_3(x)(f_0(x) + uf_1(x))]_{(x^\alpha-1)}$. This implies that

$$\begin{aligned}
(b(x); d(x)) =& (k_3(x)(f_0(x) + uf_1(x)) + k_1(x)l_1(x) + k_2(x)l_2(x);\\
&\quad k_1(x)(g(x) + up(x)) + uk_2(x)a(x))\\
=& k_3(x) * (f_0(x) + uf_1(x); 0) + k_1(x) * (l_1(x); g(x) + up(x))\\
&+ k_2(x) * (l_2(x); ua(x))\\
\in& \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x))\rangle.
\end{aligned}$$

Therefore, $C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x))\rangle$.

Lastly, suppose that $\varphi_\beta(C) = \langle g(x) + up(x)\rangle$, there is a polynomial $l(x)$ in $R[x]$ such that $(l(x); g(x) + up(x)) \in C$. Then, $\langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x))\rangle \subseteq C$. Similarly to the above case, we obtain that $C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x))\rangle$. As a result, there are two different ways to write the double cyclic code C, that is,

$$\begin{aligned}
C &= \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x))\rangle \text{ or}\\
C &= \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x))\rangle.
\end{aligned}$$

□

From Theorem 3.3, it is clear that the code C is separable if and only if $l(x), l_1(x)$ and $l_2(x)$ are zero. Next, we will investigate the conditions of the polynomials $l(x), l_1(x)$ and $l_2(x)$ derived from Theorem 3.3.

Proposition 3.2. *Let $C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x))\rangle$ be a double cyclic code of length (α, β) over R, where the polynomials $f_0(x), f_1(x), l_1(x), l_2(x), g(x), p(x), a(x)$ satisfy Theorem 3.3 (Type 2). Then, we can assume that the degrees of $l_1(x)$ and $l_2(x)$ are less than the degree of $f_0(x) + uf_1(x)$.*

Furthermore, $f_0(x) + uf_1(x)$ divides $ul_2(x), ul_1(x)\frac{x^\beta-1}{g(x)}, l_2(x)\frac{x^\beta-1}{a(x)}, ul_1(x) + \frac{g(x)}{a(x)}l_2(x)$ and $\frac{x^\beta-1}{g(x)}l_1(x) + p(x)\frac{x^\beta-1}{g(x)a(x)}l_2(x)$ in R_α.

Proof. Assume that $\deg(l_1(x)) \geq \deg(f_0(x)+uf_1(x))$ and $\deg(l_2(x)) \geq \deg(f_0(x)+uf_1(x))$. Let $i = \deg(l_1(x)) - \deg(f_0(x)+uf_1(x))$ and $j = \deg(l_2(x)) - \deg(f_0(x)+uf_1(x))$. Suppose that $C' = \langle (f_0(x)+uf_1(x); 0), (l_1(x)-ax^i(f_0(x)+uf_1(x)); g(x)+up(x)), (l_2(x)-a'x^j(f_0(x)+uf_1(x)); ua(x))\rangle$ is a double cyclic code of length (α, β) over R, where a and a' are the leading coefficients of $l_1(x)$ and $l_2(x)$, respectively. Hence, we have $C' \subseteq C$. Consider

$$\begin{aligned}
(l_1(x); g(x) + up(x)) &= (l_1(x) - ax^i(f_0(x) + uf_1(x)); g(x) + up(x))\\
&\quad + ax^i * (f_0(x) + uf_1(x); 0) \text{ and}\\
(l_2(x); ua(x)) &= (l_2(x) - a'x^j(f_0(x) + uf_1(x)); ua(x))\\
&\quad + a'x^j * (f_0(x) + uf_1(x); 0).
\end{aligned}$$

We obtain that $(l_1(x); g(x)+up(x))$ and $(l_2(x); ua(x))$ are in C'. Thus, $C \subseteq C'$. This implies that $C = C'$ and hence, the degree of $l_1(x)$ and $l_2(x)$ can be reduced in C and so we can assume that $\deg(l_1(x)) < \deg(f_0(x) + uf_1(x))$ and $\deg(l_2(x)) < \deg(f_0(x) + uf_1(x))$.

Furthermore, we consider $u * (l_2(x); ua(x)) = (ul_2(x); 0)$. Then, $(ul_2(x); 0) \in \ker \varphi_\beta$. This implies that $f_0(x) + uf_1(x)$ divides $ul_2(x)$ in R_α. For the other polynomials, we can apply the same technique. □

Proposition 3.3. *Let $C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x)) \rangle$ be a double cyclic code of length (α, β) over R, where the polynomials $f_0(x), f_1(x), l(x), g(x), p(x)$ satisfy Theorem 3.3 (Type 1). Then, we can assume that the degree of $l(x)$ is less than the degree of $f_0(x) + uf_1(x)$.*

Furthermore, $f_0(x) + uf_1(x)$ divides $l(x)\frac{x^\beta - 1}{g(x)+up(x)}$ and $ul(x)\frac{x^\beta - 1}{g(x)}$ in R_α.

Proof. The proof is analogous to the proof of Proposition 3.2. □

4. DNA CODES

In this section, we will discuss the necessary and sufficient conditions for double cyclic codes to be reversible and reversible-complement codes. Then, we will begin with the fundamental notations of DNA codes.

Let $S_{\mathcal{D}_4} = \{A, T, G, C\}$ be a set of DNA alphabets. For any $y \in S_{\mathcal{D}_4}$, we denote the complement of y as $\overline{y}$, i.e., $\overline{A} = T, \overline{T} = A, \overline{G} = C$ and $\overline{C} = G$. A DNA code of length n is a set of sequences $(a_0, a_1, ..., a_{n-1})$, where $a_i \in S_{\mathcal{D}_4}$ for $i = 0, 1, ..., n-1$. In [7] and [9], the authors gave a one-to-one correspondence ϕ between the elements of R and $S_{\mathcal{D}_4}$ as

$$\phi(0) = A, \phi(1) = G, \phi(u) = T \text{ and } \phi(1+u) = C.$$

According to the linking in the DNA bases, we can see that $\overline{0} = u, \overline{1} = 1+u, \overline{u} = 0$ and $\overline{1+u} = 1$. They expanded the map ϕ so that $\phi(C)$ can be viewed as a DNA code for some code C of length n over R. Let $a = (a_0, a_1, ..., a_{n-1})$ be a codeword in a code C of length n over R. The reverse (respectively, complement, reverse-complement) of a is denoted by $a^r = (a_{n-1}, a_{n-2}, ..., a_0)$ (respectively, $a^c = (\bar{a}_0, \bar{a}_1, ..., \bar{a}_{n-1})$, $a^{rc} = (\bar{a}_{n-1}, \bar{a}_{n-2}, ..., \bar{a}_0)$). A code C of length n over R is called reversible (respectively, complement, reversible-complement) if $a^r \in C$ ($a^c \in C, a^{rc} \in C$) for all $a \in C$.

In [8], we have studied the construction of DNA codes by using double cyclic codes of length (α, β) over R when α and β are odd positive integers. In our study, we employed the mapping ϕ and extended it for some code C of length (α, β) over R. For codes of length (α, β) over R, the definitions of reversible, complement and reversible-complement are as follows:

Let α, β be positive integers and $\boldsymbol{v} = (a; b) = (a_0, a_1, ..., a_{\alpha-1}; b_0, b_1, ..., b_{\beta-1})$ be a codeword in a code C of length (α, β) over R. Then, the reverse (respectively, complement, reverse-complement) of $\boldsymbol{v}$ is defined as $\boldsymbol{v}^r = (a_{\alpha-1}, a_{\alpha-2}, ..., a_0; b_{\beta-1}, b_{\beta-2}, ..., b_0)$, (respectively, $\boldsymbol{v}^c = (\overline{a_0}, \overline{a_1}, ..., \overline{a_{\alpha-1}}; \overline{b_0}, \overline{b_1}, ..., \overline{b_{\beta-1}})$, $\boldsymbol{v}^{rc} = (\overline{a_{\alpha-1}}, \overline{a_{\alpha-2}}, ..., \overline{a_0}; \overline{b_{\beta-1}}, \overline{b_{\beta-2}}, ..., \overline{b_0})$). Hence, we can see that $\boldsymbol{v}^r = (a^r; b^r)$, $\boldsymbol{v}^c = (a^c; b^c)$ and $\boldsymbol{v}^{rc} = (a^{rc}; b^{rc})$.

Definition 4.2. [8] *A double cyclic code C of length (α, β) over R is called reversible (respectively, complement and reversible-complement) if $\boldsymbol{v}^r \in C$ (respectively, $\boldsymbol{v}^c \in C$ and $\boldsymbol{v}^{rc} \in C$) for all $\boldsymbol{v} \in C$.*

Furthermore, if $\alpha = 0$, C is a reversible (complement, reversible-complement) cyclic code of length β over R. If $\beta = 0$, C is a reversible (complement, reversible-complement) cyclic code of length α over R.

Definition 4.3. [8] *Let C be a linear code of length $n = \alpha + \beta$ over R. Then, C is called a double cyclic DNA code if C is a double cyclic code of length (α, β) over R and $\boldsymbol{v}^{rc} \in C$ for all $\boldsymbol{v} \in C$ and $\boldsymbol{v}^{rc} \neq \boldsymbol{v}$.*

The study in this section will be separated into two subsections. In the first subsection, we will investigate the necessary and sufficient conditions for double cyclic codes to be reversible codes. Additionally, the last subsection examines the necessary and sufficient conditions for double cyclic codes to be reversible-complement codes. It also includes examples of DNA codes generated from our results.

4.1. **Reversible codes.** Let $\boldsymbol{v} = (a_0, a_1, ..., a_{\alpha-1}; b_0, b_1, ..., b_{\beta-1}) \in R^\alpha \times R^\beta$. Then, we obtain that $\boldsymbol{v}^r = (a_{\alpha-1}, a_{\alpha-2}, ..., a_0; b_{\beta-1}, b_{\beta-2}, ..., b_0)$. Therefore, the reverse of $\boldsymbol{v}$ can be identified with an element in $R_{\alpha,\beta}$ as follows:

$$\begin{aligned}\boldsymbol{v}^r(x) &= (a_{\alpha-1} + a_{\alpha-2}x + \cdots + a_0 x^{\alpha-1}; b_{\beta-1} + b_{\beta-2}x + \cdots + b_0 x^{\beta-1}) \\ &= (x^{\alpha-1-\deg(a(x))} a^*(x); x^{\beta-1-\deg(b(x))} b^*(x)).\end{aligned}$$

Furthermore, the reverse of the polynomial $\boldsymbol{v}(x)$ can be written as $[\boldsymbol{v}(x)]^r = \boldsymbol{v}^r(x)$.

First, we will give propositions from [8] that are important for the study in this subsection.

Proposition 4.4. [8] *Let $\boldsymbol{v}, \boldsymbol{w} \in R^\alpha \times R^\beta$ and $k \in R$.*

Suppose that $\boldsymbol{v}(x)$ and $\boldsymbol{w}(x)$ are polynomials in $R_{\alpha,\beta}$ that correspond to $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively. Then,

(1) $[\boldsymbol{v} + \boldsymbol{w}]^r = \boldsymbol{v}^r + \boldsymbol{w}^r$,
(2) $[k * \boldsymbol{v}]^r = k * \boldsymbol{v}^r$,
(3) $[\boldsymbol{v}(x) + \boldsymbol{w}(x)]^r = \boldsymbol{v}^r(x) + \boldsymbol{w}^r(x)$.

Proposition 4.5. [8] *Let $(a(x); b(x))$ be an element in $R_{\alpha,\beta}$ and k be a positive integer. Then,*

$$(x^k a(x); x^k b(x))^r = x^{(m+1)\alpha-1-\deg(x^k a(x))} * (a^*(x); 0) + x^{(n+1)\beta-1-\deg(x^k b(x))} * (0; b^*(x)),$$

where $m = n = 0$ or m, n are the smallest positive integers such that
$m\alpha - \deg(x^k a(x)) + \deg([x^k a(x)]_{(x^\alpha-1)}) \geq 0$ and $n\beta - \deg(x^k b(x)) + \deg([x^k b(x)]_{(x^\beta-1)}) \geq 0$.

Corollary 4.1. *Let $(a(x); b(x)) \in R_{\alpha,\beta}$ and $k \in \mathbb{Z}^+$.*

Suppose that $\beta = 2\gamma\alpha$ and $\deg(b(x)) = (2\gamma - 1)\alpha + \deg(a(x))$, where γ is a non-negative integer. Then,

$$(x^k a(x); x^k b(x))^r = x^{(2M\gamma+1)\alpha-1-\deg(x^k a(x))} * (a^*(x); b^*(x)),$$

where M is 0 or the smallest positive integer such that
$M\alpha - \deg(x^k a(x)) + \deg([x^k a(x)]_{(x^\alpha-1)}) \geq 0$ and $M\beta - \deg(x^k b(x)) + \deg([x^k b(x)]_{(x^\beta-1)}) \geq 0$.

Proof. Suppose that $\beta = 2\gamma\alpha$ and $\deg(b(x)) = (2\gamma - 1)\alpha + \deg(a(x))$. By Proposition 4.5, we obtain that

$$(x^k a(x); x^k b(x))^r = x^{(m+1)\alpha-1-\deg(x^k a(x))} * (a^*(x); 0) + x^{(n+1)\beta-1-\deg(x^k b(x))} * (0; b^*(x)),$$

where $m = n = 0$ or m, n are the smallest positive integers such that
$m\alpha - \deg(x^k a(x)) + \deg([x^k a(x)]_{(x^\alpha-1)}) \geq 0$ and $n\beta - \deg(x^k b(x)) + \deg([x^k b(x)]_{(x^\beta-1)}) \geq 0$.
Let $M = \max\{m, n\}$. Then, M is zero or the smallest positive integer such that

$$\begin{aligned}&M\alpha - \deg(x^k a(x)) + \deg([x^k a(x)]_{(x^\alpha-1)}) \geq 0 \text{ and} \\ &M\beta - \deg(x^k b(x)) + \deg([x^k b(x)]_{(x^\beta-1)}) \geq 0.\end{aligned}$$

Without loss of generality, we suppose that $M = m$. Then, $M = n + n'$ for some non-negative integer n'. Since $\deg(x^k b(x)) = \deg(x^k a(x)) + (2\gamma - 1)\alpha$,

$$\begin{aligned}(2M\gamma + 1)\alpha - 1 - \deg(x^k a(x)) &= 2n\gamma\alpha + 2n'\gamma\alpha + \alpha - 1 - \deg(x^k a(x) \\ &= n\beta + n'\beta + \alpha - 1 - \deg(x^k a(x)) \\ &= n\beta + n'\beta + \alpha - 1 - \deg(x^k b(x)) + 2\gamma\alpha \\ &= (n+1)\beta - 1 - \deg(x^k b(x)) + n'\beta.\end{aligned}$$

Hence,

$$\begin{aligned}&x^{(2M\gamma+1)\alpha-1-\deg(x^k a(x))} * (a^*(x); b^*(x)) \\ &= x^{m(2\gamma)\alpha+\alpha-1-\deg(x^k a(x))} * (a^*(x); 0) + x^{(n+1)\beta-1-\deg(x^k a(x))+n'\beta} * (0; b^*(x)) \\ &= x^{(m+1)\alpha-1-\deg(x^k a(x))+m(2\gamma-1)\alpha} * (a^*(x); 0) + x^{(n+1)\beta-1-\deg(x^k a(x))+n'\beta} * (0; b^*(x)) \\ &= x^{(m+1)\alpha-1-\deg(x^k a(x))} * (a^*(x); 0) + x^{(n+1)\beta-1-\deg(x^k b(x))} * (0; b^*(x)) \\ &= (x^k a(x); x^k b(x))^r.\end{aligned}$$

Therefore,

$$(x^k a(x); x^k b(x))^r = x^{(2M\gamma+1)\alpha-1-\deg(x^k a(x))} * (a^*(x); b^*(x)),$$

where M is 0 or the smallest positive integer such that
$M\alpha - \deg(x^k a(x)) + \deg([x^k a(x)]_{(x^\alpha-1)}) \geq 0$ and $M\beta - \deg(x^k b(x)) + \deg([x^k b(x)]_{(x^\beta-1)}) \geq 0$. □

Next, we will investigate the necessary and sufficient conditions for reversible double cyclic codes of length (α, β) over R. In [7] and [9], the authors have investigated the necessary and sufficient conditions for reversible cyclic codes of length n over R as follows:

Theorem 4.4. *Let C be a cyclic code of length n over R. Then,*

- *(1) If n is an odd number, then $C = \langle f_0(x) + uf_1(x)\rangle$ is reversible if and only if $f_0(x)$ and $f_1(x)$ are self-reciprocal.*
- *(2) If n is an even number and $i = \deg(g(x)) - \deg(p(x))$, then*
  - *(a) $C = \langle g(x) + up(x)\rangle$ is reversible if and only if*
    - *(i) $g(x)$ is self-reciprocal,*
    - *(ii) $x^i p^*(x) = p(x)$ or $g(x) = x^i p^*(x) + p(x)$.*
  - *(b) $C = \langle g(x) + up(x), ua(x)\rangle$ is reversible if and only if*
    - *(i) $g(x)$ and $a(x)$ are self-reciprocal,*
    - *(ii) $a(x)|(x^i p^*(x) + p(x))$.*

Lemma 4.1. *Let $C = \langle (f_0(x), f_1(x); 0), (l(x); g(x) + up(x))\rangle$ be a double cyclic code of length (α, β) over R, where the polynomials $f_0(x), f_1(x), g(x), p(x), l(x)$ satisfy Theorem 3.3 (Type 1) and Proposition 3.3. If C is a reversible code, then*

- *(1) $f_0(x), f_1(x)$ and $g(x)$ are self-reciprocal;*
- *(2) $x^i p^*(x) = p(x)$ or $g(x) = x^i p^*(x) + p(x)$, where $i = \deg(g(x)) - \deg(p(x))$.*

Proof. For any $(a(x); b(x)) \in C$, we define

$$\varphi_\beta : C \to R_\beta \text{ by } \varphi_\beta(a(x); b(x)) = b(x) \pmod{x^\beta - 1}.$$

Then, φ_β is an $R[x]$-module homomorphism. Suppose that C is a reversible code. Then, $\varphi_\beta(C)$ is a reversible cyclic code of length β over R. By Theorem 4.4, we obtain that

- (1) $g(x)$ is self-reciprocal,
- (2) $x^i p^*(x) = p(x)$ or $g(x) = x^i p^*(x) + p(x)$, where $i = \deg(g(x)) - \deg(p(x))$.

Let $I = \langle (f_0(x) + uf_1(x); 0)\rangle \subseteq C$. We claim that I is a reversible code. It is easy to see that I is a double cyclic code of length (α, β) over R. Assume that I is not a reversible code. Then, there exists a polynomial $\boldsymbol{v}(x) \in I$ such that $\boldsymbol{v}^r(x) \in C - I$. Hence,

$$\boldsymbol{v}(x) = \lambda(x) * (f_0(x) + uf_1(x); 0) \text{ for some } \lambda(x) \in R[x] \text{ and}$$
$$\boldsymbol{v}^r(x) = (x^{\alpha-1-\deg(h(x))}h^*(x); 0), \text{ where } h(x) = [\lambda(x)(f_0(x) + uf_1(x))]_{(x^\alpha-1)}.$$

This implies that
$\boldsymbol{v}^r(x) = (x^{\alpha-1-\deg(h(x))}h^*(x); 0) = \lambda_1(x) * (f_0(x) + uf_1(x); 0) + \lambda_2(x) * (l(x); g(x) + up(x))$ for some $\lambda_1(x), \lambda_2(x) \in R[x]$ such that $\lambda_2(x) \neq 0$. Thus,

$$[\lambda_1(x)(f_0(x) + uf_1(x)) + \lambda_2(x)l(x)]_{(x^\alpha-1)} = x^{\alpha-1-\deg(h(x))}h^*(x) \text{ and} \tag{4.1}$$
$$[\lambda_2(x)(g(x) + up(x))]_{(x^\beta-1)} = 0. \tag{4.2}$$

By (4.2), $\lambda_2(x) = \frac{x^\beta-1}{g(x)+up(x)}k(x)$, where $k(x) \in R[x]$. By Proposition 3.3, we have

$$(f_0(x) + uf_1(x))| \left(\frac{x^\beta - 1}{g(x) + up(x)}k(x)l(x)\right) \text{ in } R_\alpha.$$

Then, $\lambda_2(x)l(x) = \lambda_3(x)(f_0(x) + uf_1(x)) + q_1(x)(x^\alpha - 1)$ for some $\lambda_3(x), q_1(x) \in R[x]$. By (4.1), there exists $q_2(x) \in R[x]$ such that

$$(\lambda_1(x) + \lambda_3(x))(f_0(x) + uf_1(x)) = q_2(x)(x^\alpha - 1) + x^{\alpha-1-\deg(h(x))}h^*(x).$$

This means that $(x^{\alpha-1-\deg(h(x))}h^*(x); 0) \in I$, which is a contradiction. Hence, I is a reversible code. By Theorem 4.4, $f_0(x)$ and $f_1(x)$ are self-reciprocal. □

Lemma 4.2. *Let γ be a non-negative integer and*

$$C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x))\rangle$$

be a double cyclic code of length $(\alpha, \beta = 2\gamma\alpha)$ over R, where the polynomials $f_0(x), f_1(x), g(x), p(x), l(x)$ satisfy Theorem 3.3 (Type 1) and Proposition 3.3.

Suppose that $\deg(g(x) + up(x)) = (2\gamma - 1)\alpha + \deg(l(x))$. If C is a reversible code, then

(1) $(f_0(x) + uf_1(x))|\big(l^*(x) - l(x)\big)$ in R_α. Or
(2) $(f_0(x) + uf_1(x))|\big((1 + u)l(x) - l^*(x)\big)$ in R_α.

Proof. Suppose that C is a reversible code. Since $(l(x); g(x) + up(x)) \in C$, we obtain that

$$(l(x); g(x) + up(x))^r = (x^{\alpha-1-\deg(l(x))}l^*(x); x^{\beta-1-\deg(g(x)+up(x))}(g(x) + up(x))^*) \in C.$$

This implies that

$$(l^*(x); (g(x) + up(x))^*) = x^{(2\gamma-1)\alpha+\deg(l(x))+1} * (l(x); g(x) + up(x))^r \in C.$$

Hence, there exist $\lambda_1(x), \lambda_2(x) \in R[x]$ such that

$$(l^*(x); (g(x) + up(x))^*) = \lambda_1(x) * (f_0(x) + uf_1(x); 0) + \lambda_2(x) * (l(x); g(x) + up(x)).$$

This means that

$$[\lambda_1(x)(f_0(x) + uf_1(x)) + \lambda_2(x)l(x)]_{(x^\alpha-1)} = l^*(x) \text{ and} \tag{4.3}$$
$$[\lambda_2(x)(g(x) + up(x))]_{(x^\beta-1)} = g(x) + ux^ip^*(x), \tag{4.4}$$

where $i = \deg(g(x)) - \deg(p(x))$. By Lemma 4.1, we have $x^ip^*(x) = p(x)$ or $g(x) = x^ip^*(x) + p(x)$.

Firstly, assume that $x^ip^*(x) = p(x)$. From (4.4), we obtain that

$$(\lambda_2(x) + 1)(g(x) + up(x)) \equiv 0 \pmod{x^\beta - 1}.$$

Hence, $\lambda_2(x) = 1 + \frac{x^\beta-1}{g(x)+up(x)}h_1(x)$ for some $h_1(x) \in R[x]$. By (4.3), there is $q_1(x) \in R[x]$ such that

$$\lambda_1(x)(f_0(x) + uf_1(x)) + l(x) + \frac{x^\beta - 1}{g(x) + up(x)}h_1(x)l(x) = q_1(x)(x^\alpha - 1) + l^*(x).$$

By Proposition 3.3, we have $(f_0(x) + uf_1(x))|l(x)\frac{x^\beta-1}{g(x)+up(x)}$ in R_α. This implies that $(f_0(x) + uf_1(x))|\big(l^*(x) - l(x)\big)$ in R_α.

Lastly, we will assume that $g(x) = x^ip^*(x) + p(x)$. From (4.4), we obtain that

$$[\lambda_2(x)(g(x) + up(x))]_{(x^\beta-1)} = (1 + u)(g(x) + up(x)).$$

This implies that $(\lambda_2(x) + 1 + u)(g(x) + up(x)) \equiv 0 \pmod{x^\beta - 1}$. Then, we obtain that $\lambda_2(x) = (1+u) + \frac{x^\beta-1}{g(x)+up(x)}h_2(x)$ for some $h_2(x) \in R[x]$. From (4.3), there exists a polynomial $q_2(x) \in R[x]$ such that

$$\lambda_1(x)(f_0(x) + uf_1(x)) + \left((1 + u) + \frac{x^\beta - 1}{g(x) + up(x)}h_2(x)\right) l(x) = q_2(x)(x^\alpha - 1) + l^*(x).$$

By Proposition 3.3, we have $(f_0(x) + uf_1(x))|l(x)\frac{x^\beta-1}{g(x)+up(x)}$ in R_α. Therefore, we obtain that $(f_0(x) + uf_1(x))|\big((1 + u)l(x) - l^*(x)\big)$ in R_α. □

By using Lemma 4.1 and Lemma 4.2, we can obtain the following Theorem.

Theorem 4.5. *Let γ be a non-negative integer and*

$$C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x)) \rangle$$

be a double cyclic code of length $(\alpha, \beta = 2\gamma\alpha)$ over R, where the polynomials $f_0(x), f_1(x), g(x), p(x), l(x)$ satisfy Theorem 3.3 (Type 1) and Proposition 3.3.

Suppose that $\deg(g(x) + up(x)) = (2\gamma - 1)\alpha + \deg(l(x))$. Then, C is a reversible code if and only if

(1) $f_0(x), f_1(x)$ and $g(x)$ are self-reciprocal;

(2) (a) $x^ip^*(x) = p(x)$ and $(f_0(x) + uf_1(x))|\big(l^*(x) - l(x)\big)$ in R_α. Or

(b) $g(x) = x^ip^*(x) + p(x)$ and $(f_0(x) + uf_1(x))|\big((1 + u)l(x) - l^*(x)\big)$ in R_α, where $i = \deg(g(x)) - \deg(p(x))$.

Proof. Suppose that C is a reversible code. We obtain conditions 1 and 2 from Lemmas 4.1 and 4.2.

On the other hand, we suppose that conditions 1 and 2 are true. We obtain that $(l^*(x); g^*(x) + ux^ip^*(x)) \in C$. Since $\gcd(f_0(x), \frac{x^\alpha-1}{f_0(x)}) = 1$, there exist $a'(x), b'(x) \in \mathbb{F}_2[x]$ such that $a'(x)f_0(x) + b'(x)\frac{x^\alpha-1}{f_0(x)} = 1$. Then, we obtain that

$$\begin{aligned}&\left[\left(1 + (1 + x^j)ua'(x)f_1(x) + (1 + x^j)b'(x)\frac{x^\alpha - 1}{f_0(x)}\right)(f_0(x) + uf_1(x))\right]_{(x^\alpha-1)}\\&= f_0(x) + ux^jf_1(x), \text{ where } j = \deg(f_0(x)) - \deg(f_1(x)).\end{aligned}$$

Hence,

$$\begin{aligned}&((f_0(x) + uf_1(x))^*; 0)\\&= (f_0(x) + ux^jf_1(x); 0)\\&= \left(1 + (1 + x^j)ua'(x)f_1(x) + (1 + x^j)b'(x)\frac{x^\alpha - 1}{f_0(x)}\right) * (f_0(x) + uf_1(x); 0) \in C.\end{aligned}$$

This implies that $((f_0(x) + uf_1(x))^*; 0) \in C$. Next, we will prove that C is a reversible code. Let $\boldsymbol{v}(x) \in C$. Then, there exist $\lambda_1(x), \lambda_2(x) \in R[x]$ such that

$$\boldsymbol{v}(x) = \lambda_1(x) * (f_0(x) + uf_1(x); 0) + \lambda_2(x) * (l(x); g(x) + up(x)).$$

By Proposition 4.4, we get

$$\boldsymbol{v}^r(x) = (\lambda_1(x)(f_0(x) + uf_1(x)); 0)^r + (\lambda_2(x)l(x); \lambda_2(x)(g(x) + up(x)))^r.$$

Suppose that $\lambda_1(x) = b_0 + b_1x + \cdots + b_tx^t \in R[x]$ for some non-negative integer t. We consider

$$\begin{aligned}(\lambda_1(x)(f_0(x) + uf_1(x)); 0)^r &= (\sum_{i=0}^{t} b_ix^i(f_0(x) + uf_1(x)); 0)^r \\ &= \sum_{i=0}^{t} b_i * (x^i(f_0(x) + uf_1(x)); 0)^r.\end{aligned}$$

By Proposition 4.5, there exists $m_i \in \mathbb{Z}^+ \cup \{0\}$ such that

$$(x^i(f_0(x) + uf_1(x)); 0)^r = x^{(m_i+1)\alpha-1-\kappa} * ((f_0(x) + uf_1(x))^*; 0),$$

where $\kappa = \deg(x^i(f_0(x) + uf_1(x)))$ and $0 \le i \le t$. This implies that

$$(\lambda_1(x)(f_0(x) + uf_1(x)); 0)^r \in C.$$

Suppose that $\lambda_2(x) = c_0 + c_1x + \cdots + c_sx^s \in R[x]$ for some non-negative integer s. We consider

$$\begin{aligned}(\lambda_2(x)l(x); \lambda_2(x)(g(x) + up(x)))^r &= (\sum_{j=0}^{s} c_jx^jl(x); \sum_{j=0}^{s} c_jx^j(g(x) + up(x)))^r \\ &= \sum_{j=0}^{s} c_j * (x^jl(x); x^j(g(x) + up(x)))^r.\end{aligned}$$

By Corollary 4.1, there exists $n_j \in \mathbb{Z}^+ \cup \{0\}$ such that

$$(x^jl(x); x^j(g(x) + up(x)))^r = x^{((2n_j\gamma+1)\alpha-1-\deg(x^jl(x))} * (l^*(x); (g(x) + up(x))^*),$$

where $0 \le j \le s$. Then, $(\lambda_2(x)l(x); \lambda_2(x)(g(x)+up(x)))^r \in C$. This implies that $\boldsymbol{v}^r(x) \in C$. Therefore, C is a reversible code. □

Lemma 4.3. *Let $C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle$ be a double cyclic code of length (α, β) over R, where the polynomials $f_0(x), f_1(x), g(x), p(x), a(x), l_1(x)$, $l_2(x)$ satisfy Theorem 3.3 (Type 2) and Proposition 3.2. If C is a reversible code, then*

(1) $f_0(x), f_1(x), g(x)$ and $a(x)$ are self-reciprocal;
(2) $a(x)|(x^ip^*(x) + p(x))$, where $i = \deg(g(x)) - \deg(p(x))$.

Proof. The proof is similar to the proof of Lemma 4.1. □

Lemma 4.4. *Let γ be a non-negative integer and*

$$C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle$$

be a double cyclic code of length $(\alpha, \beta = 2\gamma\alpha)$ over R, where the polynomials $f_0(x), f_1(x)$, $g(x), p(x), a(x), l_1(x), l_2(x)$ satisfy Theorem 3.3 (Type 2) and Proposition 3.2.

Suppose that $\deg(g(x)+up(x)) = (2\gamma-1)\alpha + \deg(l_1(x))$ *and* $i = \deg(g(x)) - \deg(p(x))$. *If* C *is a reversible code, then there exist* $\omega_1(x), \omega_2(x) \in R[x]$ *such that* $f_0(x)+uf_1(x)$ *divides*

$$l_1^*(x) + \left(1 + u\omega_1(x) + \frac{x^\beta - 1}{g(x)}\omega_2(x)\right) l_1(x) + \left(\frac{g(x)}{a(x)}\omega_1(x) + \kappa(x)\omega_2(x) + \kappa'(x)\right) l_2(x)$$

in R_α, *where* $\kappa'(x) \in \mathbb{F}_2[x]$ *such that* $a(x)\kappa'(x) = x^i p^*(x) + p(x)$ *and* $\kappa(x) = \frac{x^\beta-1}{g(x)a(x)}p(x)$.

Proof. Suppose that C is a reversible code. Since $(l_1(x); g(x)+up(x)) \in C$,

$$(l_1(x); g(x)+up(x))^r = (x^{\alpha-1-\deg(l_1(x))}l_1^*(x); x^{\beta-1-\deg(g(x)+up(x))}(g(x)+up(x))^*) \in C.$$

Thus,

$$(l_1^*(x); (g(x)+up(x))^*) = x^{(2\gamma-1)\alpha+\deg(l_1(x))+1} * (l_1(x); g(x)+up(x))^r \in C.$$

Then, there exist $\lambda_1(x), \lambda_2(x), \lambda_3(x) \in R[x]$ such that

$$\begin{aligned}(l_1^*(x); (g(x)+up(x))^*) = {} & \lambda_1(x) * (f_0(x)+uf_1(x); 0) + \lambda_2(x) * (l_1(x); g(x)+up(x)) \\ & + \lambda_3(x) * (l_2(x); ua(x)).\end{aligned}$$

By Lemma 4.3, we get $g(x)$ is self-reciprocal and there exists $\kappa'(x) \in \mathbb{F}_2[x]$ such that $a(x)\kappa'(x) = x^i p^*(x) + p(x)$. Then,

$$[\lambda_1(x)(f_0(x)+uf_1(x)) + \lambda_2(x)l_1(x) + \lambda_3(x)l_2(x)]_{(x^\alpha-1)} = l_1^*(x) \text{ and} \tag{4.5}$$

$$[\lambda_2(x)(g(x)+up(x)) + u\lambda_3(x)a(x)]_{(x^\beta-1)} = g(x) + up(x) + ua(x)\kappa'(x). \tag{4.6}$$

From (4.6), we get $(\lambda_2(x)-1)ug(x) \equiv 0 \pmod{x^\beta-1}$. Then, there exist $\omega_1(x), \omega_2(x) \in R[x]$ such that $\lambda_2(x) = 1 + u\omega_1(x) + \frac{x^\beta-1}{g(x)}\omega_2(x)$. By substituting $\lambda_2(x)$ in (4.6), we have

$$ug(x)\omega_1(x) + \frac{x^\beta-1}{g(x)}p(x)\omega_2(x) + u\lambda_3(x)a(x) + ua(x)\kappa'(x) \equiv 0 \pmod{x^\beta - 1}.$$

Since $a(x)|g(x)$ and $a(x)|\frac{x^\beta-1}{g(x)}p(x)$, we obtain that

$$\left(\frac{g(x)}{a(x)}\omega_1(x) + \frac{x^\beta-1}{g(x)a(x)}p(x)\omega_2(x) + \lambda_3(x) + \kappa'(x)\right) ua(x) \equiv 0 \pmod{x^\beta-1}.$$

Hence, $\lambda_3(x) = \frac{g(x)}{a(x)}\omega_1(x) + \frac{x^\beta-1}{g(x)a(x)}p(x)\omega_2(x) + \kappa'(x) + uh_1(x) + \frac{x^\beta-1}{a(x)}h_2(x)$ for some $h_1(x), h_2(x) \in R[x]$. By substituting $\lambda_2(x)$ and $\lambda_3(x)$ in (4.5), we have

$$\begin{aligned}q(x)(x^\alpha-1) + l_1^*(x) = {} & \lambda_1(x)(f_0(x)+uf_1(x)) + \left(1 + u\omega_1(x) + \frac{x^\beta-1}{g(x)}\omega_2(x)\right) l_1(x) \\ & + \left(\frac{g(x)}{a(x)}\omega_1(x) + \frac{x^\beta-1}{g(x)a(x)}p(x)\omega_2(x) + \kappa'(x)\right) l_2(x) \\ & + uh_1(x)l_2(x) + \frac{x^\beta-1}{a(x)}h_2(x)l_2(x),\end{aligned}$$

where $q(x) \in R[x]$. By Proposition 3.2, we obtain that

$$(f_0(x)+uf_1(x))|uh_1(x)l_2(x) \text{ and } (f_0(x)+uf_1(x))|\left(\frac{x^\beta-1}{a(x)}h_2(x)l_2(x)\right) \text{ in } R_\alpha.$$

This implies that $f_0(x)+uf_1(x)$ divides

$$l_1^*(x) + \left(1 + u\omega_1(x) + \frac{x^\beta-1}{g(x)}\omega_2(x)\right) l_1(x) + \left(\frac{g(x)}{a(x)}\omega_1(x) + \kappa(x)\omega_2(x) + \kappa'(x)\right) l_2(x)$$

in R_α, where $\kappa(x) = \frac{x^\beta - 1}{g(x)a(x)}p(x)$. □

Lemma 4.5. *Let γ be a non-negative integer and*

$$C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle$$

be a double cyclic code of length $(\alpha, \beta = 2\gamma\alpha)$ over R, where the polynomials $f_0(x), f_1(x), g(x), p(x), a(x), l_1(x), l_2(x)$ satisfy Theorem 3.3 (Type 2) and Proposition 3.2.

Suppose that $\deg(ua(x)) = (2\gamma - 1)\alpha + \deg(l_2(x))$. If C is a reversible code, then there exist $\eta_1(x), \eta_2(x) \in R[x]$ such that $f_0(x) + uf_1(x)$ divides

$$l_2^*(x) + \left(1 + \frac{g(x)}{a(x)}\eta_1(x) + \kappa(x)\eta_2(x)\right) l_2(x) + \left(u\eta_1(x) + \frac{x^\beta - 1}{g(x)}\eta_2(x)\right) l_1(x)$$

in R_α, where $\kappa(x) = \frac{x^\beta - 1}{g(x)a(x)}p(x)$.

Proof. Suppose that C is a reversible code. Since $(l_2(x); ua(x)) \in C$,

$$(l_2(x); ua(x))^r = (x^{\alpha - 1 - \deg(l(x))} l_2^*(x); ux^{\beta - 1 - \deg(ua(x))} a^*(x)) \in C.$$

Thus,

$$(l_2^*(x); ua^*(x)) = x^{(2\gamma - 1)\alpha + \deg(l_2(x)) + 1} * (x^{\alpha - 1 - \deg(l_2(x))} l_2^*(x); ux^{\beta - 1 - \deg(ua(x))} a^*(x)) \in C.$$

Then, there exist $\lambda_1(x), \lambda_2(x), \lambda_3(x) \in R[x]$ such that

$$\begin{aligned}(l_2^*(x); ua^*(x)) = {} & \lambda_1(x) * (f_0(x) + uf_1(x); 0) + \lambda_2(x) * (l_1(x); g(x) + up(x)) \\ & + \lambda_3(x) * (l_2(x); ua(x)).\end{aligned}$$

By Lemma 4.3, we get $a(x)$ is self-resiprocal. Then,

(4.7) $$[\lambda_1(x)(f_0(x) + uf_1(x)) + \lambda_2(x)l_1(x) + \lambda_3(x)l_2(x)]_{(x^\alpha - 1)} = l_2^*(x) \text{ and}$$

(4.8) $$[\lambda_2(x)(g(x) + up(x)) + u\lambda_3(x)a(x)]_{(x^\beta - 1)} = ua(x).$$

From (4.8), we get $u\lambda_2(x)g(x) \equiv 0 \pmod{x^\beta - 1}$. Hence, there exist $\eta_1(x), \eta_2(x) \in R[x]$ such that $\lambda_2(x) = u\eta_1(x) + \frac{x^\beta - 1}{g(x)}\eta_2(x)$. By substituting $\lambda_2(x)$ in (4.8), we have

$$ug(x)\eta_1(x) + u\frac{x^\beta - 1}{g(x)}p(x)\eta_2(x) + u\lambda_3(x)a(x) + ua(x) \equiv 0 \pmod{x^\beta - 1}.$$

Since $a(x)|g(x)$ and $a(x)|\frac{x^\beta - 1}{g(x)}p(x)$, we obtain that

$$\left(\frac{g(x)}{a(x)}\eta_1(x) + \frac{x^\beta - 1}{g(x)a(x)}p(x)\eta_2(x) + \lambda_3(x) + 1\right) ua(x) \equiv 0 \pmod{x^\beta - 1}.$$

Hence, $\lambda_3(x) = 1 + \frac{g(x)}{a(x)}\eta_1(x) + \frac{x^\beta - 1}{g(x)a(x)}p(x)\eta_2(x) + uh_1(x) + \frac{x^\beta - 1}{a(x)}h_2(x)$ for some $h_1(x), h_2(x) \in R[x]$. By substituting $\lambda_2(x)$ and $\lambda_3(x)$ in (4.7), we have

$$\begin{aligned}q(x)(x^\alpha - 1) + l_2^*(x) = {} & \lambda_1(x)(f_0(x) + uf_1(x)) + \left(u\eta_1(x) + \frac{x^\beta - 1}{g(x)}\eta_2(x)\right) l_1(x) \\ & + \left(1 + \frac{g(x)}{a(x)}\eta_1(x) + \frac{x^\beta - 1}{g(x)a(x)}p(x)\eta_2(x)\right) l_2(x) \\ & + uh_1(x)l_2(x) + \frac{x^\beta - 1}{a(x)}h_2(x)l_2(x), \text{ where } q(x) \in R[x].\end{aligned}$$

By Proposition 3.2, we obtain that

$$(f_0(x) + uf_1(x))|uh_1(x)l_2(x) \text{ and } (f_0(x) + uf_1(x))|\left(\frac{x^\beta - 1}{a(x)}h_2(x)l_2(x)\right) \text{ in } R_\alpha.$$

This implies that $f_0(x)+uf_1(x)$ divides

$$l_2^*(x)+\left(1+\frac{g(x)}{a(x)}\eta_1(x)+\kappa(x)\eta_2(x)\right)l_2(x)+\left(u\eta_1(x)+\frac{x^\beta-1}{g(x)}\eta_2(x)\right)l_1(x)$$

in R_α, where $\kappa(x)=\frac{x^\beta-1}{g(x)a(x)}p(x)$. □

By Lemmas 4.3, 4.4 and 4.5, we obtain the following Theorem.

Theorem 4.6. *Let γ be a non-negative integer and*

$$C=\langle (f_0(x)+uf_1(x);0),(l_1(x);g(x)+up(x)),(l_2(x);ua(x))\rangle$$

be a double cyclic code of length $(\alpha,\beta=2\gamma\alpha)$ over R, where the polynomials $f_0(x),f_1(x)$, $g(x),p(x),a(x),l_1(x),l_2(x)$ satisfy Theorem 3.3 (Type 2) and Proposition 3.2.

Suppose that $\deg(g(x)+up(x))=(2\gamma-1)\alpha+\deg(l_1(x))$, $\deg(ua(x))=(2\gamma-1)\alpha+\deg(l_2(x))$ and $\kappa(x)=\frac{x^\beta-1}{g(x)a(x)}p(x)$. Then, C is a reversible code if and only if

(1) $f_0(x),f_1(x),g(x)$ and $a(x)$ are self-reciprocal;
(2) $a(x)|(x^ip^*(x)+p(x))$, where $i=\deg(g(x))-\deg(p(x))$;
(3) there exist $\omega_1(x),\omega_2(x)\in R[x]$ such that $f_0(x)+uf_1(x)$ divides $l_1^*(x)+\left(1+u\omega_1(x)+\frac{x^\beta-1}{g(x)}\omega_2(x)\right)l_1(x)+\left(\frac{g(x)}{a(x)}\omega_1(x)+\kappa(x)\omega_2(x)+\kappa'(x)\right)l_2(x)$ in R_α, where $\kappa'(x)\in\mathbb{F}_2[x]$ such that $a(x)\kappa'(x)=x^ip^*(x)+p(x)$;
(4) there exist $\eta_1(x),\eta_2(x)\in R[x]$ such that $f_0(x)+uf_1(x)$ divides $l_2^*(x)+\left(1+\frac{g(x)}{a(x)}\eta_2(x)+\kappa(x)\eta_2(x)\right)l_2(x)+\left(u\eta_1(x)+\frac{x^\beta-1}{g(x)}\eta_2(x)\right)l_1(x)$ in R_α.

Proof. Suppose that C is a reversible code. By Lemmas 4.3, 4.4 and 4.5, we obtain the conditions 1 – 4.

On the other hand, suppose that conditions 1 – 4 are true. We will show that $((f_0(x)+uf_1(x))^*;0)$, $(l_1^*(x);(g(x)+up(x))^*)$ and $(l_2^*(x);ua^*(x))$ are in C. Since $\gcd(f_0(x),\frac{x^\alpha-1}{f_0(x)})=1$, there exist $a'(x),b'(x)\in\mathbb{F}_2[x]$ such that $a'(x)f_0(x)+b'(x)\frac{x^\alpha-1}{f_0(x)}=1$. Then, we obtain that

$$\left[\left(1+(1+x^j)ua'(x)f_1(x)+(1+x^j)b'(x)\frac{x^\alpha-1}{f_0(x)}\right)(f_0(x)+uf_1(x))\right]_{(x^\alpha-1)}$$
$$=f_0(x)+ux^jf_1(x),\ \text{where } j=\deg(f_0(x))-\deg(f_1(x)).$$

Hence, $((f_0(x)+uf_1(x))^*;0)\in C$.

Next, we will show that $(l_1^*(x);(g(x)+up(x))^*)\in C$. Let $i=\deg(g(x))-\deg(p(x))$ and $a(x)\kappa'(x)=x^ip(x)+p(x)$. By condition 3, there exist polynomials $\omega_1(x),\omega_2(x)$ in $R[x]$ such that

$$l_1^*(x)=q_1(x)(x^\alpha-1)+\lambda_1(x)(f_0(x)+uf_1(x))+\left(1+u\omega_1(x)+\frac{x^\beta-1}{g(x)}\omega_2(x)\right)l_1(x)$$
$$+\left(\frac{g(x)}{a(x)}\omega_1(x)+\kappa(x)\omega_2(x)+\kappa'(x)\right)l_2(x),\ \text{where } q_1(x),\lambda_1(x)\in R[x].$$

Consider,

$$\begin{aligned}(l_1^*(x); (g(x)+up(x))^*) &= (l_1^*(x); g(x) + ux^i p^*(x))\\ &= (l_1^*(x); g(x) + up(x) + ua(x)\kappa'(x))\\ &= \lambda_1(x) * (f_0(x) + uf_1(x); 0)\\ &\quad + \left(1 + u\omega_1(x) + \frac{x^\beta - 1}{g(x)}\omega_2(x)\right) * (l_1(x); g(x) + up(x))\\ &\quad + \left(\frac{g(x)}{a(x)}\omega_1(x) + \kappa(x)\omega_2(x) + \kappa'(x)\right) * (l_2(x); ua(x)).\end{aligned}$$

Hence, $(l_1^*(x); (g(x)+up(x))^*) \in C$.

Finally, we will show that $(l_2^*(x); ua^*(x)) \in C$. By condition 4, there exist polynomials $\eta_1(x), \eta_2(x)$ in $R[x]$ such that

$$\begin{aligned}l_2^*(x) &= q_2(x)(x^\alpha - 1) + \lambda_2(x)(f_0(x) + uf_1(x))\\ &\quad + \left(1 + \frac{g(x)}{a(x)}\eta_2(x) + \kappa(x)\eta_2(x)\right) l_2(x) + \left(u\eta_1(x) + \frac{x^\beta - 1}{g(x)}\eta_2(x)\right) l_1(x)\end{aligned}$$

for some $q_2(x), \lambda_2(x) \in R[x]$. Consider,

$$\begin{aligned}(l_2^*(x); ua(x)) &= \lambda_2(x) * (f_0(x) + uf_1(x); 0)\\ &\quad + \left(u\eta_1(x) + \frac{x^\beta - 1}{g(x)}\eta_2(x)\right) * (l_1(x); g(x) + up(x))\\ &\quad + \left(1 + \frac{g(x)}{a(x)}\eta_2(x) + \kappa(x)\eta_2(x)\right) * (l_2(x); ua(x)).\end{aligned}$$

Hence, $(l_2^*(x); ua^*(x)) \in C$. Similar to the proof of Theory 4.5, we obtain that $\boldsymbol{v}^r(x) \in C$ for all $\boldsymbol{v}(x) \in C$. Therefore, C is a reversible code. □

4.2. **Reversible-complement codes.**

Theorem 4.7. [8] *Let C be a double cyclic code of length (α, β) over R. Then, C is a reversible-complement code if and only if*

(1) C is a reversible code and
(2) $(\underbrace{u, u, ..., u}_{\alpha}; \underbrace{u, u, ..., u}_{\beta}) \in C$.

We now discuss the reversible-complement for double cyclic codes of length (α, β) over R using the results obtained in the above subsections. The proofs for the theorems following are straightforward.

Theorem 4.8. *Let γ be a non-negative integer and*

$$C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x)) \rangle$$

be a double cyclic code of length $(\alpha, \beta = 2\gamma\alpha)$ over R, where the polynomials $f_0(x), f_1(x), g(x), p(x), l(x)$ satisfy Theorem 3.3 (Type 1) and Proposition 3.3.

Suppose that $\deg(g(x)+up(x)) = (2\gamma-1)\alpha + \deg(l(x))$. Then, C is a reversible-complement code if and only if

(1) C is a reversible code and
(2) $u\mathbb{I}(x) = (u + ux + \cdots + ux^{\alpha-1}; u + ux + \cdots + ux^{\beta-1}) \in C$.

Theorem 4.9. *Let γ be a non-negative integer and*

$$C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle$$

be a double cyclic code of length $(\alpha, \beta = 2\gamma\alpha)$ over R, where the polynomials $f_0(x), f_1(x), g(x), p(x), a(x), l_1(x), l_2(x)$ satisfy Theorem 3.3 (Type 2) and Proposition 3.2.

Suppose that $\deg(g(x) + up(x)) = (2\gamma - 1)\alpha + \deg(l_1(x))$ and $\deg(ua(x)) = (2\gamma - 1)\alpha + \deg(l_2(x))$. Then, C is a reversible-complement code if and only if

(1) C is a reversible code and
(2) $u\mathbb{I}(x) = (u + ux + \cdots + ux^{\alpha-1}; u + ux + \cdots + ux^{\beta-1}) \in C$.

4.2.1. *Examples.* Now, we will construct some concrete examples to illustrate the above results. Recall that the Hamming distance between two codewords of the same length is defined as the number of coordinates in which two codewords differ and the minimum Hamming distance of a code is the smallest Hamming distance between any two distinct codewords in the code. Let C be a code of length (α, β) over R and $S_{\mathcal{D}_4} = \{A, T, G, C\}$. Recall that $\phi(0) = A, \phi(1) = G, \phi(u) = T$ and $\phi(1 + u) = C$. Define $\Phi : C \to S_{\mathcal{D}_4}^{\alpha+\beta}$ by

$$\Phi(a_0, a_1, ..., a_{\alpha-1}; b_0, b_1, ..., b_{\beta-1}) = (\phi(a_0), \phi(a_1), ..., \phi(a_{\alpha-1}), \phi(b_0), \phi(b_1), ..., \phi(b_{\beta-1})),$$

where $(a_0, a_1, ..., a_{\alpha-1}; b_0, b_1, ..., b_{\beta-1}) \in C$. For any double cyclic code C of length (α, β) over R, we let $\hat{C}$ be a set which is

$$\hat{C} = \{(b; a) \in R^{\beta,\alpha} : (a; b) \in C\} \subseteq R^{\beta,\alpha}.$$

Then, we can generate a DNA code $\mathcal{D}$ from $C \cup \hat{C}$, where C is a double cyclic DNA code of length (α, β) over R.

By using Theorems 4.8 and 4.9, we can construct a double cyclic DNA code C of length (α, β) over R and generate DNA codes as follows:

Example 4.1. *Let $x^3 - 1 = (x + 1)(x^2 + x + 1) = m_1(x)m_2(x)$ and $x^6 - 1 = m_1(x)^2 m_2(x)^2$ over $\mathbb{F}_2$.*

(1) Let $C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x)) \rangle$ be a double cyclic code of length $(3, 6)$ over R, where $f_0(x) + uf_1(x) = (1 + u)m_1(x)m_2(x), l(x) = 1$, $g(x) = m_1(x)m_2(x)$ and $p(x) = 0$. It is easy to check that C is a reversible code. Consider,

$$u\mathbb{I}(x) = u(x^2 + x + 1) * (l(x); g(x) + up(x)) \in C.$$

By Theorem 4.8, we obtain that C is a reversible-complement code. Then, a DNA code $\mathcal{D}$ of length 9 with minimum Hamming distance 3 from $C \cup \hat{C}$ has 64 codewords, which are given in Table 1.

TABLE 1. DNA code $\mathcal{D}$ of length 9 obtained from $C \cup \hat{C}$ in Example 4.1 (1)

TGTTGTTGT	TTGTTGTTG	CGGCGGCGG	GAGGAGGAG	ATCATCATC
GCAGCAGCA	TGCTGCTGC	AAAAAAAAA	TGGTGGTGG	GCTGCTGCT
GTCGTCGTC	GGCGGCGGC	ACTACTACT	AACAACAAC	ACGACGACG
CTTCTTCTT	ATTATTATT	CGTCGTCGT	AGTAGTAGT	CGACGACGA
CCACCACCA	ACAACAACA	AATAATAAT	GTTGTTGTT	CAACAACAA
GACGACGAC	GTAGTAGTA	CATCATCAT	GGTGGTGGT	TACTACTAC
GAAGAAGAA	TAGTAGTAG	GTGGTGGTG	GATGATGAT	TAATAATAA
ATAATAATA	CGCCGCCGC	TGATGATGA	TCTTCTTCT	TCCTCCTCC
GCGGCGGCG	GGGGGGGGG	CCGCCGCCG	CAGCAGCAG	CCTCCTCCT
ACCACCACC	TTCTTCTTC	CTACTACTA	CACCACCAC	TCGTCGTCG
GCCGCCGCC	TCATCATCA	AAGAAGAAG	AGCAGCAGC	TTTTTTTTT
CTGCTGCTG	CCCCCCCCC	TTATTATTA	AGGAGGAGG	ATGATGATG
GGAGGAGGA	CTCCTCCTC	AGAAGAAGA	TATTATTAT	

(2) *Let $C = \langle (f_0(x) + uf_1(x); 0), (l(x); g(x) + up(x)) \rangle$ be a double cyclic code of length $(3, 6)$ over R, where $f_0(x) = m_1(x)m_2(x), f_1(x) = m_1(x), l(x) = 1 + u$, $g(x) = m_1(x)m_2(x)^2$ and $p(x) = 0$. It is easy to check that C is a reversible code. Consider,*

$$u\mathbb{I}(x) = x * (f_0(x) + uf_1(x); 0) + u * (l(x); g(x) + up(x)) \in C.$$

By Theorem 4.8, we obtain that C is a reversible-complement code. Then, a DNA code $\mathcal{D}$ of length 9 with minimum Hamming distance 2 from $C \cup \hat{C}$ has 126 codewords, which are given in Table 2.

TABLE 2. DNA code $\mathcal{D}$ of length 9 obtained from $C \cup \hat{C}$ in Example 4.1 (2)

$TGAGGGGG$	$TTAAAAAAA$	$ATATTTTTT$	$ACGTTTTTT$	$AAAAAAGCT$
$AAAAAATTA$	$AAAAAAGTC$	$TGCAAAAAA$	$CTTGGGGGG$	$CCCCCCACT$
$CCCCCCATC$	$CCCCCCTTG$	$TAGGGGGGG$	$TCACCCCCC$	$CTCTTTTTT$
$GGGGGGCGG$	$CCCCCCGCC$	$TTTTTTGAC$	$CCCCCCCGC$	$TTGCCCCCC$
$AAAAAATCG$	$CCGCCCCCC$	$CCCCCCGAA$	$AAAAAAGAG$	$CGCCCCCCC$
$GTTCCCCCC$	$TTTTTTGCA$	$CCCCCCTGT$	$CCCCCCAAG$	$ACTCCCCCC$
$GCTAAAAAA$	$TTTTTTGGT$	$AGACCCCCC$	$CCTTTTTTT$	$ATGGGGGGG$
$GGGGGGGCG$	$AAAAAAATT$	$TTTTTTCTC$	$TGGTTTTTT$	$GGGGGGGGC$
$GGCGGGGGG$	$AAAAAAGGA$	$AAAAAACTG$	$ACAGGGGGG$	$GGGCCCCCC$
$CCCCCCAGA$	$TCCTTTTTT$	$GTCAAAAAA$	$TTTTTTCAG$	$CCCCCCGGG$
$CCCCCCCAT$	$GGTTTTTTT$	$CCCGGGGGG$	$AGTGGGGGG$	$TCGAAAAAA$
$CACAAAAAA$	$GCCCCCCCC$	$AACGGGGGG$	$GGGGGGAAC$	$TTTTTTATA$
$CCCCCCCTA$	$CGGGGGGGG$	$AAAAAAACC$	$GGGGGGATG$	$GGAAAAAAA$
$AAAAAAAAA$	$GGGGGGTCT$	$GGGGGGACA$	$CCCCCCTAC$	$CCCCCCGTT$
$TAATTTTTT$	$GCGGGGGGG$	$CTACCCCCC$	$TTTTTTTCC$	$TCTGGGGGG$
$TTTTTTCGA$	$TTTTTTTAA$	$CCCCCCTCA$	$GGGGGGAGT$	$CGATTTTTT$
$CCCCCCCCG$	$TACCCCCCC$	$ACCAAAAAA$	$TTTTTTACG$	$GACTTTTTT$
$CGTAAAAAA$	$GGGGGGGTA$	$CATCCCCCC$	$ATTAAAAAA$	$GGGGGGTAG$
$GGGGGGCTT$	$AAAAAATGC$	$AATTTTTTT$	$GCATTTTTT$	$TTTTTTAGC$
$TGTCCCCCC$	$GATGGGGGG$	$TTTTTTGTG$	$GAGAAAAAA$	$GGGGGGTGA$
$AAAAAACAC$	$AAAAAAAGG$	$TTCGGGGGG$	$ATCCCCCCC$	$TATAAAAAA$
$TTTTTTCCT$	$AGCTTTTTT$	$CTGAAAAAA$	$CCAAAAAAA$	$TTTTTTAAT$
$AGGAAAAAA$	$AAGCCCCCC$	$GGGGGGCCC$	$CAGTTTTTT$	$GGGGGGCAA$
$AAAAAACCA$	$CAAGGGGGG$	$GAACCCCCC$	$TTTTTTTGG$	$GTAGGGGGG$
$TTTTTTTTT$	$GTGTTTTTT$	$AAAAAACGT$	$AAAAAATAT$	$GGGGGGTTC$
$GGGGGGGAT$				

The example 4.1 (1) shows that the double cyclic DNA code C can produce reversible-complement codes of lengths 3 and 6 over R. These codes are $C_1 = \langle 1 \rangle$ and $C_2 = \langle m_1(x)m_2(x) \rangle$, respectively. Similarly, in the example 4.1 (2), the double cyclic DNA code C can produce reversible and reversible-complement codes of lengths 3 and 6 over R. These include the reversible code $C_3 = \langle um_1(x) \rangle$ of length 3 over R and the reversible-complement codes $C_4 = \langle 1 + u \rangle$ and $C_5 = \langle m_1(x)m_2(x) \rangle$ of lengths 3 and 6 over R, respectively.

On the other hand, the codes C_1, C_2, C_3, C_4 and C_5 can construct a double cyclic DNA code of length $(3, 6)$ over R by using Theorems 3.3, 4.8 and 4.9 (may not be the same as the code C in examples 4.1 (1) and (2)). Similarly, we can obtain the same results for double cyclic DNA codes using double cyclic code type 2.

Example 4.2. *Let $x^3 - 1 = (x + 1)(x^2 + x + 1) = m_1(x)m_2(x)$ and $x^6 - 1 = m_1(x)^2m_2(x)^2$ over $\mathbb{F}_2$.*

(1) *Let $C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle$ be a double cyclic code of length $(3, 6)$ over R, where $f_0(x) = m_1(x)m_2(x), f_1(x) = m_2(x), l_1(x) = m_2(x), l_2(x) = 0$, $g(x) = m_1(x)m_2(x)^2$, $p(x) = m_2(x)$ and $a(x) = m_1(x)m_2(x)$. It is easy to check that C is a reversible code. Consider,*

$$u\mathbb{I}(x) = u * (l_1(x); g(x) + up(x)) \in C.$$

By Theorem 4.9, we obtain that C is a reversible-complement code. Then, a DNA code $\mathcal{D}$ of length 9 with minimum Hamming distance 2 from $C \cup \hat{C}$ has 60 codewords, which are given in Table 3.

TABLE 3. DNA code $\mathcal{D}$ of length 9 obtained from $C \cup \hat{C}$ in Example 4.2 (1)

$CCCGGGGGG$	$GGGCCCCCC$	$CCCGCGCGC$	$TTTATAATA$	$CCCCCGGGC$
$CCCGGCCCG$	$CCGGGCCCC$	$TAATAAAAA$	$AAAATAATA$	$GGGGGGCCC$
$TATTATAAA$	$TATTATTTT$	$TTTTAATAA$	$CGGGCCCCC$	$TAATAATTT$
$TTTTTATTA$	$AAATTATTA$	$CCCCGGGCC$	$TTATTAAAA$	$GGGCGCGCG$
$AATAATTTT$	$TTTTTTAAA$	$GCGCGCGGG$	$TTTTTTTTT$	$AAAATTATT$
$GGGGCGCGC$	$GGGCCGGGC$	$CGCGCGCCC$	$AATAATAAA$	$GCCCGGCCC$
$GGGCCCGGG$	$ATTATTTTT$	$CCCGGGCCC$	$GGGGGCCCG$	$ATTATTAAA$
$ATAATAAAA$	$ATAATATTT$	$AAATATTAT$	$TTTAAAAAA$	$AAATTTTTT$
$CCCGCCCGG$	$AAAAAAAAA$	$TTTTATTAT$	$CCCCGCGCG$	$CGGGCCGGG$
$GCGCGCCCC$	$AAAAAATTT$	$GGCCCGGGG$	$TTATTATTT$	$GGCCCGCCC$
$CGCGCGGGG$	$GGGGCCCGG$	$TTTAATAAT$	$GGGCGGGCC$	$GCCCGGGGG$
$AAATAATAA$	$CCGGGCGGG$	$CCCCCCGGG$	$TTTATTATT$	$AAAAATAAT$

(2) *Let* $C = \langle (f_0(x) + uf_1(x); 0), (l_1(x); g(x) + up(x)), (l_2(x); ua(x)) \rangle$ *be a double cyclic code of length* $(3, 6)$ *over* R*, where* $f_0(x)+f_1(x) = (1+u)m_2(x), l_1(x) = um_1(x), l_2(x) = 0$*,* $g(x) = m_1(x)^2 m_2(x)$*,* $p(x) = m_1(x)$ *and* $a(x) = m_1(x)m_2(x)$*. It is easy to check that* C *is a reversible code. Consider,*

$$u\mathbb{I}(x) = u * (f_0(x) + uf_1(x); 0) + (x^2 + x + 1) * (l_2(x); ua(x)) \in C.$$

By Theorem 4.9, we obtain that C *is a reversible-complement code. Then, a DNA code* $\mathcal{D}$ *of length* 9 *with minimum Hamming distance* 2 *from* $C \cup \hat{C}$ *has* 254 *codewords, which are given in Table 4.*

5. CONCLUSIONS

This work has studied the algebraic structure and properties of double cyclic codes of length (α, β) over the finite commutative chain ring $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$. The values of α and β are positive odd and even integers, respectively. In terms of the algebraic structure of these codes, we have obtained two types of codes called double cyclic codes type 1 and type 2. In addition, we have constructed theorems for generating DNA codes using the structure of non-separable codes of double cyclic codes types 1 and 2. A double cyclic code of length (α, β) over $\mathbb{F}_2 + u\mathbb{F}_2$ that is suitable for DNA codes is called a double cyclic DNA code. The double cyclic DNA codes from our results can produce reversible and reversible-complement codes of lengths α and β, which are studied in [7, 9]. On the other hand, we can construct a double cyclic DNA code from some codes in [7, 9] by using Theorems 3.3, 4.8 and 4.9. Even though a double cyclic DNA code of length (α, β) over R can give a DNA code $\mathcal{D}$ of odd length, if the β is not zero, the code $\mathcal{D}$ cannot be created from a reversible-complement code of odd length over R. To demonstrate our results, we provided illustrative examples of DNA codes generated by our research. These examples show the potential applications of the codes. We hope our study will pave the way for further improvements in coding theory and bioinformatics, opening up new avenues for cutting-edge research and applications.

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TABLE 4. DNA code $\mathcal{D}$ of length 9 obtained from $C \cup \hat{C}$ in Example 4.2 (2)

AAAATAATA	GCTCGTGGC	GGTCCTTTA	CGAGCACCG	TCGTGCCGG
ATTAGCACG	GCCTGCTCG	GGCGGACCA	GCACGAAAT	CGCGAGCAC
GCGGTGCTC	CCCTAATAA	GGACCAGGC	ACCAGGCGG	GTGCTCGCG
GCCTGGTCC	CCGGCACGA	ACGAGCCGG	GCACGAGGC	CACGAGGCG
CTCGTGCGC	ACCAGGGCC	AATGGACCA	TCCTGGTAA	ACGAGCGCC
ATTTCCTGG	TATCTGGTC	ATACTCGTG	CGGTGCTCG	ACGAGCTAA
CGGTCCTGG	GACCAGATA	TTTTTTCCC	GCGGAGCAC	GGGATAATA
TTTAAAAAA	TAATCGTGC	TATTATGGG	GTGCTCATA	GGACCACCG
GCCACGAGC	TCGTGCGCC	TGGTCCTAA	AGCACGCGG	AGCACGTAA
ATTACGAGC	TTTTATTAT	CGCCAGGAC	CTCGTGTAT	GAGCACATA
CACGAGTAT	TTATTAGGG	CCCAAAAAA	GGCGGTCCT	CGCCTCGTG
TGGTCCATT	TTTATTATT	TGCTCGCGG	ATTATTCCC	TTATTAAAA
GCCAGGACC	GGACCAAAT	CTGGTCTAT	GACCAGGCG	TGGTCCCGG
CGGTGGTCC	CGAGCAAAT	TTTTTTTTT	TCCTGGGCC	TCCTGGCGG
GCGCTCGTG	AATAATGGG	GCGGTCCTG	ATAGTGCTC	TTAGCTCGT
TAAAGGACC	TAATCCTGG	AGGACCGCC	TGCTCGGCC	CCGGGTCCT
ATTAGGACC	CGAGCAGGC	AAAATTATT	TATGTGCTC	GCGCACGAG
GGTCCTCCG	CCAGGAGGC	CCTGGTGGC	AAATAATAA	AAAAAAGGG
TAAACGAGC	TAATAAGGG	TTTTTTAAA	TATGAGCAC	TCGTGCATT
TATTATCCC	AAAAAAAAA	TAAACCAGG	AATGCACGA	GGCGCTCGT
TTACGAGCA	ATAGTCCTG	GTCCTGATA	GGCGCACGA	TATCTCGTG
TAATGCTCG	TAATGGTCC	AGGACCTAA	GACCAGCGC	CCCTTATTA
TATGTCCTG	AATCCTGGT	GTCCTGGCG	GCTCGTTTA	CGGTCGTGC
CTGGTCCGC	TATCACGAG	GGACCATTA	GGTCCTGGC	GCCTCCTGG
CGCCTGGTC	ATAATACCC	AAATTTTTT	GTGCTCTAT	TGCTCGTAA
GGGTATTAT	TTATTATTT	TGCTCGATT	CGCGTCCTG	GCACGATTA
CCGCCTGGT	CGCGTGCTC	CACGAGATA	GAGCACGCG	AGGACCCGG
TTAGGTCCT	CCCATTATT	CTGGTCGCG	TATTATAAA	GGGATTATT
GGCCGTGCT	AATAATCCC	CAGGACATA	TCCTGGATT	CCGCCAGGA
GCGCAGGAC	GCCTCGTGC	CAGGACGCG	TTAGCACGA	AGCACGATT
ATTATTAAA	GGGAAAAAA	CCCTATTAT	CGTGCTCCG	AGCACGGCC
CCGGCTCGT	ATTTGCTCG	TAATAACCC	GGCCCTGGT	CGGACCAGG
CCTGGTAAT	GTCCTGCGC	ATTATTGGG	CCAGGATTA	AATGGTCCT
TTACCTGGT	CCTGGTCCG	CACGAGCGC	ATAATAAAA	AATAATAAA
CAGGACTAT	AATCGAGCA	TATGACCAG	ATTTGGTCC	GGCCCAGGA
CGTGCTAAT	AATCCAGGA	ACGAGCATT	GTGCTCCGC	ACCAGGTAA
GTCCTGTAT	CGGAGCACG	TATCAGGAC	CGGACGAGC	AAATATTAT
ATACAGGAC	CCCTTTTTT	CGAGCATTA	TAATAAAAA	ATAATAGGG
GCTCGTCCG	GGTCCTAAT	TTTTAATAA	TTACGTGCT	CGCGACCAG
CCGCGTGCT	TATTATTTT	ATACACGAG	ATACTGGTC	TTTTTATTA
TAAAGCACG	ACCAGGATT	TTTAATAAT	GGCCGAGCA	GGGTAATAA
TTACCAGGA	GGGTTTTTT	AATCGTGCT	CAGGACCGC	AGGACCATT
CCGCGAGCA	CTCGTGATA	CCAGGACCG	ATTATTTTT	AATGCTCGT
GCGGACCAG	GAGCACTAT	AAAAAATTT	GGGAATAAT	GCCACCAGG
GACCAGTAT	GGGTTATTA	TAATAATTT	CGTGCTGGC	CCCATAATA
CCCAATAAT	TTATTACCC	CTGGTCATA	AAAAATAAT	ATTACCAGG
TCGTGCTAA	CCAGGAAAT	AATAATTTT	GCCAGCACG	ATAGACCAG
CTCGTGGCG	CGTGCTTTA	ATAGAGCAC	TTTATAATA	AAATTATTA
TTTTTTGGG	AAAAAACCC	GCTCGTAAT	TGGTCCGCC	ATAATATTT
CGCCACGAG	TTAGGACCA	GCGCTGGTC	GCACGACCG	CCTGGTTTA
GAGCACCGC	CCGGGACCA	CGGAGGACC	ATTTCGTGC	

REFERENCES

[1] Abualrub, T.; Ghrayeb, A.; Nian Zeng, X. Construction of Cyclic Codes over $GF(4)$ for DNA Computing. *J. Franklin Inst.* **2006**, 343, 448 – 457.

[2] Abualrub, T.; Siap, I. Cyclic Codes over the Rings $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$. *Des. Codes Cryptogr.* **2007**, 42, 273 – 287.

[3] Adleman, L.M. Molecular Computation of Solutions to Combinatorial Problems. *Science* **1994**, 266, 1021–1024.

[4] Borges, J.; Fernández-Córdoba, C.; Ten-Valls, R. $\mathbb{Z}_2$-Double Cyclic Codes. *Des. Codes Cryptogr.* **2017**, 86, 463–479.

[5] Dinh, H.Q.; Pathak, S.; Upadhyay, A.K.; Yamaka, W. New DNA Codes from Cyclic Codes over Mixed Alphabets. *Math.* **2020**, 8, 1977.

[6] Gaborit, P.; King, O.D. Linear Constructions for DNA Codes. *Theor. Comput. Sci.* **2005**, 334, 99–113.

[7] Guenda, K.; Gulliver, T.A. Construction of Cyclic Codes over $\mathbb{F}_2 + u\mathbb{F}_2$ for DNA Computing. *Applicable Algebra Eng. Commun. Comput.* **2013**, 24, 445–459.

[8] Kanlaya, A.; Klin-eam, C. Constructing Double Cyclic Codes over $\mathbb{F}_2 + u\mathbb{F}_2$ for DNA Codes. *J. Comput. Biol.* **2023**, 30, 1112–1130.

[9] Liang, J.; Wang, L. On Cyclic DNA Codes over $\mathbb{F}_2 + u\mathbb{F}_2$. *J. Appl. Math. Compt.* **2015**, 51, 81–91.

[10] McDonald, B.R. Finite Rings with Identity; M. Dekker: New York, 1974.

[11] Siap, I.; Abualrub, T.; Ghrayeb, A. Cyclic DNA Codes over the Ring $\mathbb{F}_2[u]/\langle u^2 - 1\rangle$ Based on the Deletion Distance. *J. Franklin Inst.* **2009**, 346, 731–740.

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