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Algorithmic and Analytical Approach for a System of Generalized Multi-valued Resolvent Equations-Part I: Basic Theory

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ABSTRACT. The concept of resolvent operator associated with a P-η-accretive mapping is used in constructing of a new iterative algorithm for solving a new system of generalized multi-valued resolvent equations in the framework of Banach spaces. Some definitions along with some new concrete examples are provided. The main result of this paper is to prove the Lipschitz continuity of the resolvent operator associated with a P-η-accretive mapping and to compute an estimate of its Lipschitz constant under some new appropriate conditions imposed on the parameters and mappings involved in it. In part II, the convergence analysis of the sequences generated by our proposed iterative algorithm under some appropriate conditions is studied. The results presented in this paper are new, and improve and generalize many known corresponding results.

1. INTRODUCTION

Since the beginning of the theory of variational inequalities in the 1960's, originally introduced for the study of partial differential equations by Hartman and Stampacchia (1966), due to its wide applications in the study of a wide range of problems arising in physics, mechanics, optimization and control, economics and transportation equilibrium, nonlinear programming, elasticity and other branches of mathematical and engineering sciences, see, for example [2, 3, 17, 21, 22, 25, 27, 28, 33], it has been intensively studied and extended in various directions using novel and innovative techniques.

Among extensions of the variational inequality, the variational inclusion is among the most interesting and in the past several years, a great deal of papers have been devoted to the existence of solutions for various kinds of variational inclusion problems, see for example [12, 13, 11, 5, 29, 1, 6, 7, 30] and the references therein. In order to provide an efficient and implementable algorithm for solving different classes of variational inequality/inclusion problems, there has been considerable activity in the development of numerical techniques. To achieve this purpose, during the last several decades, many methods have been suggested and appeared in the literature to find solutions of various kinds of variational inequality/inclusion problems. Among proposed methods and techniques, it is well known that the resolvent operator technique is a useful and important method such that in the last two decades, the technique of resolvent operators has become more and more important and efficient and has attracted increasing attention. Due to the above-mentioned facts, many considerable works concerning the development of the methods based on different classes of resolvent operators to study the existence of solution and to discuss convergence analysis of iterative algorithms of various kinds of variational inclusions and their generalizations have been carried out, see, for example, [12, 13, 11, 5, 29, 18, 34, 4, 23, 31, 32] and the references therein.

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Ding [10] and Huang and Fang [15] were the first to introduce classes of generalized monotone operators, respectively, the so-called η-subdifferential operators and maximal η-monotone operators. At the same time, the notion of generalized m-accretive mapping (also referred to as η-m-accretive or η-m-accretive mapping in the literature), which is a generalization of m-accretive mapping, was first introduced by Huang and Fang [16]. By defining the resolvent operator associated with a generalized m-accretive mapping, they gave some properties regarding it in the framework of Banach spaces. Subsequently, Fang and Huang [12], Fang et al. [13], Fang and Huang [11], and Kazmi and Khan [20] have introduced the concepts of H-monotone operators and (H, η)-monotone operators in Hilbert spaces, H-accretive operators and P-η-accretive operators in Banach spaces, and their associated resolvent operators, respectively. They have also employed the resolvent operators associated with the generalized monotone operators mentioned above to construct iterative algorithms for finding the approximate solution of various kinds of variational inclusion problems. It is worth noting that in most of the resolvent operator methods, the maximal monotonicity has played a key role, but recently introduced concepts of H-monotonicity, H-accretivity, (H, η)-monotonicity and P-η-accretivity have not only generalized, respectively, the maximal monotonicity and maximal η-monotonicity, but provided a new edge of resolvent operator techniques.

It is well known that the projection method and its variant forms have represented an important computation technique for computing the approximate solution and various classes of variational inequalities and their generalizations. Another important and significant generation of variational inequalities is the mixed variational inequality containing a nonlinear term. Unfortunately, the projection method could not be applied to construct iterative algorithms for solving mixed variational inequality problems due to the existence of a nonlinear term in their formulations. As a matter of fact, since it is hard to find the project except in very special cases, the applicability of the projection method is limited. There is a technique due to Noor and Noor [26] for solving mixed variational inequalities, in which based on the resolvent operator, the equivalence between the variational inequalities and the resolvent equations has first been established and then the obtained equivalence has been used in the construct of iterative algorithms for solving various classes of mixed variational inequalities, for more details, see for example [20] and the references therein.

In the next section we begin by introducing some preliminary notions. We define η-accretive, strictly η-accretive, r-strongly η-accretive, ϱ-Lipschitz continuous, k-strongly η-accretive and generalized m-accretive (or η-m-accretive) for vector-valued mapping and multi-valued mapping. These definitions are illustrated by three examples. In Section 3, under some new appropriate conditions imposed on the parameters and mappings involved in the resolvent operator associated with a P-η-accretive mapping, its Lipschitz continuity is proved and an estimate of its Lipschitz constant is computed. This assertion will play a key role in obtaining main results given in part II.

2. Preliminary Notions and Results

In order to make the paper self-contained we begin by introducing some preliminary notions. Consider E a real Banach space and E^* its topological dual space. For the sake of simplicity, the norms in E and E^* will be designated by the same symbol $\|.\|$, and the metric induced by the norm $\|.\|$ will be denoted by d. As usual, the notation x^* stands for the weak* topology in E^*, while by $\langle x, x^* \rangle$ we denote the value of the inner continuous functional $x^* \in E^*$ at $x \in E$. We also use the symbol $CB(E)$ (resp. 2^E) to represent the set of all nonempty closed and bounded (resp., all nonempty) subsets of E. We define the graph and range of a given multi-valued mapping $M : E \to 2^E$ by

$$\text{Graph}(M) := \{(x, u) \in E \times E : u \in M(x)\}$$

and

$$\text{Range}(M) := \{y \in E : \exists x \in E : (x, y) \in \text{Graph}(M)\} = \bigcup_{x \in E} M(x),$$

respectively. We shall denote by S_E and B_E respectively the unite sphere and the unit ball in E.

Let us recall that a normed space E is called strictly convex if S_E is strictly convex, that is, the inequality $\|x + y\| < 2$ holds for all distinct unit vectors x and y in E. It is said to be smooth if for every vector x in E there exists a unique $x^* \in E^*$ such that $\|x^*\| = \langle x, x^* \rangle = 1$.

It is known that E is smooth if E^* is strictly convex, and that E is strictly convex if E^* is smooth.

Definition 2.1. *A normed space E is said to be uniformly convex if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B_E$ with $\|x - y\| \geq \varepsilon$ the inequality $\|x + y\| \leq 2(1 - \delta)$ holds.*

The modulus of convexity of E is a function $\delta_E : [0, 2] \to [0, 1]$ defined in the following way:

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in B_E, \|x - y\| \geq \varepsilon\}.$$

It should be pointed out that in the definition of $\delta_E(\varepsilon)$ we can as well take the infimum over all vectors $x, y \in S_E$ with $\|x - y\| = \varepsilon$, see for example [8]. The functional δ_E is continuous and increasing on the interval $[0, 2]$ and $\delta_E(0) = 0$. Obviously, invoking the definition of the function δ_E, a normed space E is uniformly convex if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$. For any Banach space E, its modulus of convexity is bounded from above by the modulus of convexity of a Hilbert space $\mathcal{H}$, $\delta_E(\varepsilon) \leq \delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$. This means that Hilbert spaces are the most convex among all Banach spaces.

Definition 2.2. *A normed space E is called uniformly smooth if for any given $\varepsilon > 0$ there exists $\tau > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1$ and $\|y\| \leq \tau$, the inequality $\|x + y\| + \|x - y\| \leq 2 + \varepsilon\|y\|$ holds.*

The function $\rho_E : [0, +\infty) \to [0, +\infty)$ defined by the formula

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in S_E\}$$

is called the modulus of smoothness of E. Note, in particular, that the function ρ_E is convex, continuous and increasing on the interval $[0, +\infty)$ and $\rho_E(0) = 0$. In addition $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$. In the light of the definition of the function ρ_E, a normed space is uniformly smooth if $\lim_{\tau \to 0} \tau^{-1} \rho_E(\tau) = 0$.

Any uniformly convex and any uniformly smooth Banach space is reflexive. A Banach space E is uniformly convex (resp., uniformly smooth) if and only if E^* is uniformly smooth (resp., uniformly convex). The spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex as well as uniformly smooth, see [24, 9, 14]. At the same time, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, can be found in [24, 9, 14].

Let us recall that the normalized duality mapping $\mathcal{F} : E \to 2^{E^*}$ is defined by

$$\mathcal{F}(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall x \in E.$$

We observe immediately that if $E = \mathcal{H}$, a Hilbert space, then $\mathcal{F}$ is the identity mapping on $\mathcal{H}$. Furthermore, it is an immediate consequence of the Hahn-Banach theorem that $\mathcal{F}(x)$ is nonempty for each $x \in E$. In the sequel, the notation j will be used to represent a selection of the normalized duality mapping $\mathcal{F}$.

Definition 2.3. *Let $P : E \to E$ and $\eta : E \times E \to E$ be two vector-valued mappings and $\mathcal{F} : E \to 2^{E^*}$ be the normalized duality mapping. Then P is said to be*

(i) *η-accretive if,*

$$\langle P(x) - P(y), j(\eta(x,y))\rangle \geq 0, \quad \forall x, y \in E, j(\eta(x,y)) \in \mathcal{F}(\eta(x,y));$$

(ii) *strictly η-accretive if, P is η-accretive and equality holds if and only if $x = y$;*

(iii) *r-strongly η-accretive if there exists a constant $r > 0$ such that*

$$\langle P(x) - P(y), j(\eta(x,y))\rangle \geq r\|x - y\|^2, \quad \forall x, y \in E, j(\eta(x,y)) \in \mathcal{F}(\eta(x,y));$$

(iv) *ϱ-Lipschitz continuous if there exists a constant $\varrho > 0$ such that*

$$\|P(x) - P(y)\| \leq \varrho\|x - y\|, \quad \forall x, y \in E.$$

It should be pointed that if $\eta(x,y) = x - y$, for all $x, y \in E$, then parts (i) to (iii) of Definition 2.3 reduce to the definitions of accretivity, strict accretivity and strong accretivity of the mapping P, respectively.

Definition 2.4. *Let $\eta : E \times E \to E$ be a vector-valued mapping, $M : E \to 2^E$ be a multi-valued mapping, and let $\mathcal{F} : E \to 2^{E^*}$ be the normalized duality mapping. Then M is said to be*

(i) *η-accretive if*

$$\langle u - v, j(\eta(x,y))\rangle \geq 0, \quad \forall (x,u), (y,v) \in \mathrm{Graph}(M), j(\eta(x,y)) \in \mathcal{F}(\eta(x,y));$$

(ii) *strictly η-accretive if, M is η-accretive and equality holds if and only if $x = y$;*

(iii) *k-strongly η-accretive if there exists a constant $k > 0$ such that*

$$\langle u - v, j(\eta(x,y))\rangle \geq k\|x - y\|^2, \quad \forall (x,u), (y,v) \in \mathrm{Graph}(M), \\ j(\eta(x,y)) \in \mathcal{F}(\eta(x,y));$$

(iv) *generalized m-accretive (or η-m-accretive) if M is η-accretive and $(I + \rho M)(E) = E$ holds for every real constant $\rho > 0$, where I stands for identity mapping.*

It should be remarked that if $\eta(x,y) = x - y$ for all $x, y \in E$, then parts (i) to (iv) of Definition 2.4 reduce to the definitions of accretivity, strict accretivity, strong accretivity and m-accretivity of the mapping M, respectively.

We note that M is a generalized m-accretive (or η-m-accretive) mapping if and only if M is η-accretive and there is no other η-accretive mapping whose graph contains strictly $\mathrm{Graph}(M)$. The generalized m-accretivity is to be understood in terms of inclusion of graphs. If $M : E \to 2^E$ is a generalized m-accretive mapping, then adding anything to its graph so as to obtain the graph of a new multi-valued mapping, destroys the η-accretivity. In fact, the extended mapping is no longer η-accretive. In other words, for every pair $(x,u) \in E \times E \setminus \mathrm{Graph}(M)$ there exist $(y,v) \in \mathrm{Graph}(M)$ and $j(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$ such that $\langle u - v, j(\eta(x,y))\rangle < 0$. In the light of the above-mentioned discussion, a necessary and sufficient condition for a multi-valued mapping $M : E \to 2^E$ to be generalized m-accretive is that for any $(x,u) \in E \times E$, the property

$$\langle u - v, j(\eta(x,y))\rangle \geq 0, \quad \forall (y,v) \in \mathrm{Graph}(M), j(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$$

is equivalent to $(x,u) \in \mathrm{Graph}(M)$. The above characterization of generalized m-accretive mappings provides us a useful and manageable way for recognizing that an element u belongs to $M(x)$.

Definition 2.5. *Let $P : E \to E$ and $\eta : E \times E \to E$ be vector-valued mappings, and $M : E \to 2^E$ be a multi-valued mapping. M is said to be P-η-accretive if M is η-accretive and $(P + \rho M)(E) = E$ holds for every real constant $\rho > 0$.*

Note, in particular, that for the case when $\eta(x, y) = x - y$ for all $x, y \in E$, then Definition 2.5 reduces to the definition of P-accretivity of the mapping M. The following example shows that for given mappings $\eta : E \times E \to E$ and $P : E \to E$, a P-η-accretive mapping may be neither P-accretive nor generalized m-accretive (or η-m-accretive).

Example 2.1. Let $\phi : \mathbb{Z} \to (0, +\infty)$ and consider the complex linear space $l^2_\phi(\mathbb{Z})$, the weighted $l^2(\mathbb{Z})$ space, consisting of all bi-infinite complex sequences such that

$$l^2_\phi(\mathbb{Z}) = \{z = \{z_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |z_n|^2 \phi(n) < \infty, z_n \in \mathbb{C}\}.$$

It is a well known that $l^2_\phi(\mathbb{Z})$ with respect to the inner product $\langle ., . \rangle : l^2_\phi(\mathbb{Z}) \times l^2_\phi(\mathbb{Z}) \to \mathbb{C}$ defined by

$$\langle z, w \rangle = \sum_{n=-\infty}^{\infty} z_n \bar{w}_n \phi(n), \quad \forall z = \{z_n\}_{n=-\infty}^{\infty}, w = \{w_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}),$$

is a Hilbert space. The inner product defined above induces a norm on $l^2_\phi(\mathbb{Z})$ as follows:

$$\|z\|_{l^2_\phi(\mathbb{Z})} = \sqrt{\langle z, z \rangle} = (\sum_{n=-\infty}^{\infty} |z_n|^2 \phi(n))^{\frac{1}{2}}, \quad \forall z = \{z_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}).$$

Any element $z = \{z_n\}_{n=-\infty}^{\infty} = \{x_n + iy_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z})$ can be written as

$$\begin{aligned} z = & \sum_{s \in \{\pm 1, \pm 3, \dots\}} \Big[(\dots, 0, 0, \dots, 0, x_{2s-1} + iy_{2s-1}, 0, x_{2s+1} + iy_{2s+1}, 0, 0, \dots) \\ & + (\dots, 0, 0, \dots, 0, x_{2s} + iy_{2s}, 0, x_{2s+2} + iy_{2s+2}, 0, 0, \dots)\Big] \\ = & \sum_{s \in \{\pm 1, \pm 3, \dots\}} \Big[\frac{y_{2s-1} + y_{2s+1} - i(x_{2s-1} + x_{2s+1})}{2} \omega_{2s-1,2s+1} \\ & + \frac{y_{2s-1} - y_{2s+1} - i(x_{2s-1} - x_{2s+1})}{2} \omega'_{2s-1,2s+1} \\ & + \frac{y_{2s} + y_{2s+2} - i(x_{2s} + x_{2s+2})}{2} \omega_{2s,2s+2} \\ & + \frac{y_{2s} - y_{2s+2} - i(x_{2s} - x_{2s+2})}{2} \omega'_{2s,2s+2}\Big], \end{aligned}$$

where for each $s \in \{\pm 1, \pm 3, \dots\}$, $\omega_{2s-1,2s+1} = (\dots, 0, 0, \dots, 0, i_{2s-1}, 0, i_{2s+1}, 0, 0, \dots)$, with i in the $(2s-1)$th and $(2s+1)$th positions and 0's elsewhere, $\omega'_{2s-1,2s+1} = (\dots, 0, 0, \dots, 0, i_{2s-1}, 0,$ $0, \dots)$, i and $-i$ at the $(2s - 1)$th and $(2s + 1)$th coordinates, respectively, and all other coordinates are zero, $\omega_{2s,2s+2} = (\dots, 0, 0, \dots, 0, i_{2s}, 0, i_{2s+2}, 0, 0, \dots)$, with i in the $(2s)$th and $(2s + 2)$th positions and 0's elsewhere, $\omega'_{2s,2s+2} = (\dots, 0, 0, \dots, 0, i_{2s}, 0, -i_{2s+2}, 0, 0, \dots)$, i and $-i$ at the $(2s)$th and $(2s + 2)$th coordinates, respectively, and all other coordinates are zero. Hence, the set

$$\mathfrak{B} = \{\omega_{2s-1,2s+1}, \omega'_{2s-1,2s+1}, \omega_{2s,2s+2}, \omega'_{2s,2s+2} : s = \pm 1, \pm 3, \dots\}$$

spans the Banach space $l^2_\phi(\mathbb{Z})$. It is easy to see that the set $\mathfrak{B}$ is linearly independent and so it is a basis for $l^2_\phi(\mathbb{Z})$. Taking

$$\begin{aligned}
v_{2s-1,2s+1} &= (\dots,0,0,\dots,0,\frac{1}{\sqrt{2\phi(2s-1)}}i_{2s-1},0,\frac{1}{\sqrt{2\phi(2s+1)}}i_{2s+1},0,0,\dots)\\
v'_{2s-1,2s+1} &= (\dots,0,0,\dots,0,\frac{1}{\sqrt{2\phi(2s-1)}}i_{2s-1},0,-\frac{1}{\sqrt{2\phi(2s+1)}}i_{2s+1},0,\dots)\\
v_{2s,2s+2} &= (\dots,0,0,\dots,0,\frac{1}{\sqrt{2\phi(2s)}}i_{2s},0,\frac{1}{\sqrt{2\phi(2s+2)}}i_{2s+2},0,0,\dots)
\end{aligned}$$

and

$$v'_{2s,2s+2} = (\dots,0,0,\dots,0,\frac{1}{\sqrt{2\phi(2s)}}i_{2s},0,-\frac{1}{\sqrt{2\phi(2s+2)}}i_{2s+2},0,0,\dots)$$

for each $s \in \{\pm1,\pm3,\dots\}$, it can be easily seen that the set

$$\{v_{2s-1,2s+1}, v'_{2s-1,2s+1}, v_{2s,2s+2}, v'_{2s,2s+2} : s = \pm1,\pm3,\dots\}$$

is also linearly independent such that

$$\|v_{2s-1,2s+1}\|_{l^2_{\phi(\mathbb{Z})}} = \|v'_{2s-1,2s+1}\|_{l^2_{\phi(\mathbb{Z})}} = \|v_{2s,2s+2}\|_{l^2_{\phi(\mathbb{Z})}} = \|v'_{2s,2s+2}\|_{l^2_{\phi(\mathbb{Z})}} = 1.$$

Let the mappings $M : l^2_\phi(\mathbb{Z}) \to 2^{l^2_\phi(\mathbb{Z})}$, $\eta : l^2_\phi(\mathbb{Z}) \times l^2_\phi(\mathbb{Z}) \to l^2_\phi(\mathbb{Z})$ and $P : l^2_\phi(\mathbb{Z}) \to l^2_\phi(\mathbb{Z})$ be defined, respectively, by

$$M(z) = \begin{cases} \Psi, & z = v_{2t,2t+2},\\ -z + \Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}, & z \neq v_{2t,2t+2}, \end{cases}$$

$$\eta(z,w) = \begin{cases} \alpha(w-z), & z,w \neq v_{2t,2t+2},\\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

and $P(z) = \beta z + \gamma\Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}$, for all $z, w \in l^2_\phi(\mathbb{Z})$, where

$$\begin{aligned}\Psi = \{&v_{2s-1,2s+1} - v_{2t,2t+2}, v'_{2s-1,2s+1} - v_{2t,2t+2}, v_{2s,2s+2} - v_{2t,2t+2},\\ &v'_{2s,2s+2} - v_{2t,2t+2} : s = \pm1,\pm3,\dots\},\end{aligned}$$

$\alpha,\beta,\gamma \in \mathbb{R}$ are arbitrary constants such that $\beta < 0 < \alpha$, $t \in \{\pm1,\pm3,\dots\}$ is chosen arbitrarily but fixed, and $\mathbf{0}$ is the zero vector of the space $l^2_\phi(\mathbb{Z})$. Since $\sum\limits_{n=-\infty}^{\infty} \frac{\ln(n^2+1)}{2n^4-1} = 2\sum\limits_{n=1}^{\infty} \frac{\ln(n^2+1)}{2n^4-1}$ and $\sum\limits_{n=1}^{\infty} \frac{\ln(n^2+1)}{2n^4-1}$ is convergent, it follows that $\sum\limits_{n=-\infty}^{\infty} \frac{\ln(n^2+1)}{2n^4-1} < \infty$ and so

$$\Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}).$$

Taking into account that $E = l^2_\phi(\mathbb{Z})$ is a Hilbert space, we conclude that the normalized duality mapping $\mathcal{F}$ is the identity mapping on $l^2_\phi(\mathbb{Z})$. Then, for all $z, w \in l^2_\phi(\mathbb{Z})$, $z \neq w \neq$

$v_{2t,2t+2}$ and $j(z-w)\in\mathcal{F}(z-w)$, we have

$$\begin{aligned}
&\langle M(z)-M(w), j(z-w)\rangle = \langle M(z)-M(w), z-w\rangle\\
&= \langle -z + \Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}\\
&\quad + w - \Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}, z-w\rangle\\
&= \langle w-z, z-w\rangle = -\|z-w\|^2_{l^2_\phi(\mathbb{Z})} = -\sum_{n=-\infty}^{\infty}|z_n-w_n|^2\phi(n) < 0,
\end{aligned}$$

i.e., M is not accretive and so M is not P-accretive. For any given $z,w\in l^2_\phi(\mathbb{Z})$, $z\neq w\neq v_{2t,2t+2}$ and $j(\eta(z,w))\in\mathcal{F}(\eta(z,w))$, we yield

$$\begin{aligned}
&\langle M(z)-M(w), j(\eta(z,w))\rangle = \langle M(z)-M(w), \eta(z,w)\rangle\\
&= \langle -z + \Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}\\
&\quad + w - \Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}, \alpha(w-z)\rangle\\
&= \alpha\langle w-z, w-z\rangle = \alpha\|w-z\|^2_{l^2_\phi(\mathbb{Z})} = \alpha\sum_{n=-\infty}^{\infty}|z_n-w_n|^2\phi(n) > 0.
\end{aligned}$$

For each of the cases when $z\neq w=v_{2t,2t+2}$, $w\neq z=v_{2t,2t+2}$ and $z=w=v_{2t,2t+2}$, thanks to the fact that $\eta(z,w)=\mathbf{0}$, for all $j(\eta(z,w))\in\mathcal{F}(\eta(z,w))$, we deduce that

$$\langle u-v, j(\eta(z,w))\rangle = \langle u-v, \eta(z,w)\rangle = 0,$$

for all $u\in M(z)$ and $v\in M(w)$. Hence, M is an η-accretive mapping. For any $z\in l^2_\phi(\mathbb{Z})$, $z\neq v_{2t,2t+2}$, we have

$$\begin{aligned}
\|(I+M)(z)\|^2_{l^2_\phi(\mathbb{Z})} &= \Big\|\Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}} + i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}\Big\|^2_{l^2_\phi(\mathbb{Z})}\\
&= \sum_{n=-\infty}^{\infty}\frac{\ln(n^2+1)}{2(2n^4-1)} > 0
\end{aligned}$$

and

$$(I+M)(v_{2t,2t+2}) = \{v_{2s-1,2s+1}, v'_{2s-1,2s+1}, v_{2s,2s+2}, v'_{2s,2s+2} : s=\pm1,\pm3,\dots\},$$

where I is the identity mapping on $E=l^2_\phi(\mathbb{Z})$. Accordingly, $\mathbf{0}\notin(I+M)(l^2_\phi(\mathbb{Z}))$. Thus, $I+M$ is not surjective, consequently, M is not a generalized m-accretive mapping. For any $\rho>0$ and $z\in l^2_\phi(\mathbb{Z})$, taking $w=\frac{1}{\beta-\rho}z+\frac{\gamma+\rho}{\rho-\beta}\Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}+i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}$ ($\rho\neq\beta$, because $\beta<0$), we have

$$\begin{aligned}
(P+\rho M)(w) = (P+\rho M)\Big(&\frac{1}{\beta-\rho}z + \frac{\gamma+\rho}{\rho-\beta}\Big\{\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\\
&+ i\sqrt{\frac{\ln(n^2+1)}{4(2n^4-1)\phi(n)}}\Big\}_{n=-\infty}^{\infty}\Big) = z.
\end{aligned}$$

Therefore, for any $\rho > 0$, the mapping $P + \rho M$ is surjective and so M is a P-η-accretive mapping.

Denoting the set of all functions $\phi : \mathbb{Z} \to (0,1]$ by Φ and $l^2_\Phi = \{l^2_\phi(\mathbb{Z}) : \phi \in \Phi\}$, it is easy to see that $l^2(\mathbb{Z}) \subseteq l^2_\phi(\mathbb{Z})$ for each $\phi \in \Phi$ so that for some $\phi_0 \in \Phi$, we have $l^2(\mathbb{Z}) \subset l^2_{\phi_0}(\mathbb{Z})$, that is, $l^2(\mathbb{Z})$ is strictly contained within $l^2_{\phi_0}(\mathbb{Z})$. We recall that

$$l^2(\mathbb{Z}) = \{x = \{x_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

denotes the real or complex linear space consisting of all bi-infinite real or complex sequences $x = \{x_n\}_{n=-\infty}^{\infty}$, for which $\|x\|_{l^2(\mathbb{Z})} < \infty$. It goes without saying that if $\phi(n) = 1$ for all $n \in \mathbb{Z}$, then the weight space $l^2_\phi(\mathbb{Z})$ coincides exactly with the linear space $l^2(\mathbb{Z})$. It should be pointed out that the two Hilbert spaces $l^2(\mathbb{Z})$ and $l^2_\phi(\mathbb{Z})$ need not be the same for all $\phi \in \Phi$. In order to show this assertion, we consider the two cases when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $\mathbb{F} = \mathbb{R}$, letting $x_n = \sqrt{n^2 + \sqrt[3]{n^2}}$ for all $n \in \mathbb{Z}$, we have $\sum\limits_{n=-\infty}^{\infty} |x_n|^2 = \sum\limits_{n=-\infty}^{\infty} (n^2 + \sqrt[3]{n^2}) = 2\sum\limits_{n=1}^{\infty} (n^2 + \sqrt[3]{n^2}) = \infty$, i.e., $x = \{x_n\}_{n=-\infty}^{\infty} \notin l^2(\mathbb{Z})$. Defining $\phi_1 : \mathbb{Z} \to (0, +\infty)$ by $\phi_1(n) = \frac{1}{2n^8+1}$ for all $n \in \mathbb{Z}$, we have $\phi_1 \in \Phi$ and

$$\sum_{n=-\infty}^{\infty} |x_n|^2 \phi_1(n) = \sum_{n=-\infty}^{\infty} \frac{n^2 + \sqrt[3]{n^2}}{2n^8 + 1} = 2\sum_{n=1}^{\infty} \frac{n^2 + \sqrt[3]{n^2}}{2n^8 + 1}.$$

Since the series $\sum\limits_{n=1}^{\infty} \frac{n^2+\sqrt[3]{n^2}}{2n^8+1}$ is convergent, it follows that $x \in l^2_{\phi_1}(\mathbb{Z})$. For the case when $\mathbb{F} = \mathbb{C}$, letting $z_n = \sqrt{\frac{n^k!}{2}} + i\sqrt{\frac{n^k!}{2}}$ for all $n \in \mathbb{Z}$, where k is an arbitrary but fixed even natural number, we infer that $\sum\limits_{n=-\infty}^{\infty} |z_n|^2 = \sum\limits_{n=-\infty}^{\infty} n^k! = \infty$, that is, $z = \{z_n\}_{n=-\infty}^{\infty} \notin l^2(\mathbb{Z})$. Now, let us assume that the function $\phi_2 : \mathbb{Z} \to (0, +\infty)$ is defined by $\phi_2(n) = \frac{1}{n^k!(n^p+q)}$ for all $n \in \mathbb{Z}$, where $p \geq 2$ is an arbitrary but fixed even natural number and q is an arbitrary positive real constant. Then, it can be easily observed that $\phi_2 \in \Phi$ and

$$\sum_{n=-\infty}^{\infty} |z_n|^2 \phi_2(n) = \sum_{n=-\infty}^{\infty} \frac{1}{n^p + q} = \frac{1}{q} + 2\sum_{n=1}^{\infty} \frac{1}{n^p + q}.$$

In virtue of the fact that $\sum\limits_{n=1}^{\infty} \frac{1}{n^p+q}$ is convergent, we deduce that $z = \{z_n\}_{n=-\infty}^{\infty} \in l^2_{\phi_2}(\mathbb{Z})$. Thereby, for some $\phi \in \Phi$, $l^2_\phi(\mathbb{Z})$ is a proper superset of the Hilbert space $l^2(\mathbb{Z})$.

Example 2.2. Let $m, n \in \mathbb{N}$ and $M_{m\times n}(\mathbb{F})$ be the space of all $m \times n$ matrices with real or complex entries. Then

$$M_{m\times n}(\mathbb{F}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{F}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

is a Hilbert space with respect to the Hilbert-Schmidt norm

$$\|A\| = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{F})$$

induced by the Hilbert-Schmidt inner product

$$\langle A, B\rangle = tr(A^*B) = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{a}_{ij} b_{ij}, \quad \forall A, B \in M_{m\times n}(\mathbb{F}),$$

where tr denotes the trace, that is, the sum of the diagonal entries, and A^* denotes the Hermitian conjugate (or adjoint) of the matrix A, that is, $A^* = \overline{A^t}$, the complex conjugate of the transpose A, and the bar denotes complex conjugation and superscript denotes the transpose of the entries. Let us denote by $D_n(\mathbb{R})$ the space of all diagonal $n \times n$ matrices with real entries, that is, the (i,j)-entry is an arbitrary real number if $i = j$, and is zero if $i \neq j$. Then

$$D_n(\mathbb{R}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{R}, a_{ij} = 0 \text{ if } i \neq j; i,j = 1,2,\dots,n\}$$

is a subspace of $M_{n\times n}(\mathbb{R}) = M_n(\mathbb{R})$ with respect to the operations of addition and scalar multiplication defined on $M_n(\mathbb{R})$, and the Hilbert-Schmidt inner product on $D_n(\mathbb{R})$, and the Hilbert-Schmidt norm induced by it become as

$$\langle A, B\rangle = tr(A^*B) = tr(AB)$$

and

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(AA)} = \Big(\sum_{i=1}^{n} a_{ii}^2\Big)^{\frac{1}{2}},$$

respectively. Let us now define the mappings $P_1, P_2, M : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $\eta : D_n(\mathbb{R}) \times D_n(\mathbb{R}) \to D_n(\mathbb{R})$, respectively, as $P_1(A) = P_1(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'_{ij} \end{pmatrix}$, $P_2(A) = P_2(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a''_{ij} \end{pmatrix}$, $M(A) = M(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'''_{ij} \end{pmatrix}$ and $\eta(A,B) = \eta(\begin{pmatrix} a_{ij} \end{pmatrix}, \begin{pmatrix} b_{ij} \end{pmatrix}) = \begin{pmatrix} c_{ij} \end{pmatrix}$ for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, where for each $i,j \in \{1,2,\dots,n\}$,

$$a'_{ij} = \begin{cases} |a_{ii} - \alpha| - |a_{ii} - \beta| - \varrho\sqrt[k]{\sin a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$a''_{ij} = \begin{cases} a_{ii} + \sin(\gamma a_{ii} + \mu), & i = j, \\ 0, & i \neq j, \end{cases}$$

$$a'''_{ij} = \begin{cases} \varrho\sqrt[k]{\sin a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$c_{ij} = \begin{cases} \varsigma\lambda^{\sigma a_{ii} b_{ii}}(\sin a_{ii} - \sin b_{ii}), & i = j, \\ 0, & i \neq j, \end{cases}$$

$\alpha, \beta, \sigma, \mu \in \mathbb{R}$, $\gamma \in \mathbb{R}\backslash\{0\}$, $\varrho, \varsigma > 0$ and $\lambda > 1$ are arbitrary real constants, and k is an arbitrary but fixed odd natural number. Since $D_n(\mathbb{R})$ is a Hilbert space, it follows that the normalized duality mapping $\mathcal{F}$ is the identity mapping on $D_n(\mathbb{R})$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$ and $j(\eta(A,B)) \in \mathcal{F}(\eta(A,B))$, we yield

$$\begin{aligned} &\langle M(A) - M(B), j(\eta(A,B))\rangle = \langle M(A) - M(B), \eta(A,B)\rangle \\ &= tr\Big(\begin{pmatrix} a'''_{ij} - b'''_{ij} \end{pmatrix}\begin{pmatrix} c_{ij} \end{pmatrix}\Big) \\ &= \varrho\varsigma\sum_{i=1}^{n}(\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}})\lambda^{\sigma a_{ii} b_{ii}}(\sin a_{ii} - \sin b_{ii}). \end{aligned}$$

For any $i \in \{1,2,\dots,n\}$,

(i) if $a_{ii} = b_{ii} = 0$, then

$$(\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}})(\sin a_{ii} - \sin b_{ii}) = 0;$$

(ii) if $a_{ii} \neq 0$ and $b_{ii} = 0$, then

$$(\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}})(\sin a_{ii} - \sin b_{ii}) = \sin a_{ii}\sqrt[k]{\sin a_{ii}} = \sqrt[k]{\sin^{k+1} a_{ii}};$$

(iii) if $a_{ii} = 0$ and $b_{ii} \neq 0$, then

$$(\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}})(\sin a_{ii} - \sin b_{ii}) = \sin b_{ii} \sqrt[k]{\sin b_{ii}} = \sqrt[k]{\sin^{k+1} b_{ii}};$$

(iv) if $a_{ii}, b_{ii} \neq 0$, then

$$\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}} = \frac{\sin a_{ii} - \sin b_{ii}}{\sum\limits_{j=1}^{k} \sqrt[k]{\sin^{k-j} a_{ii} \sin^{j-1} b_{ii}}}.$$

Since k is an odd natural number, it follows that $\sqrt[k]{\sin^{k+1} a_{ii}}, \sqrt[k]{\sin^{k+1} b_{ii}} > 0$ and $\sum\limits_{j=1}^{k} \sqrt[k]{\sin^{k-j} a_{ii} \sin^{j-1}$ 0. These facts imply that

$$(\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}})(\sin a_{ii} - \sin b_{ii}) > 0$$

and

$$\sum_{i=1}^{n} (\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}})(\sin a_{ii} - \sin b_{ii}) = \sum_{i=1}^{n} \frac{(\sin a_{ii} - \sin b_{ii})^2}{\sum\limits_{j=1}^{k} \sqrt[k]{\sin^{k-\delta} a_{ii} \sin^{\delta-1} b_{ii}}} > 0.$$

Taking into account that $\varrho, \varsigma > 0$ and $\lambda > 1$, relying on the above-mentioned arguments, we deduce that for all $A = \left(a_{ij} \right), B = \left(b_{ij} \right) \in D_n(\mathbb{R})$ and $j(\eta(A,B)) \in \mathcal{F}(\eta(A,B))$,

$$\begin{aligned} &\langle M(A) - M(B), j(\eta(A,B)) \rangle \\ &= \varrho\varsigma \sum_{i=1}^{n} (\sqrt[k]{\sin a_{ii}} - \sqrt[k]{\sin b_{ii}}) \lambda^{\sigma a_{ii} b_{ii}} (\sin a_{ii} - \sin b_{ii}) \\ &= \varrho\varsigma \sum_{i=1}^{n} \frac{\lambda^{\sigma a_{ii} b_{ii}} (\sin a_{ii} - \sin b_{ii})^2}{\sum\limits_{j=1}^{k} \sqrt[k]{\sin^{k-j} a_{ii} \sin^{j-1} b_{ii}}} \geq 0, \end{aligned}$$

which means that M is an η-accretive mapping.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) := |x - \alpha| - |x - \beta|$ for all $x \in \mathbb{R}$. Then, for any $A = \left(a_{ij} \right) \in D_n(\mathbb{R})$, yields

$$(P_1 + M)(A) = (P_1 + M)(\left(a_{ij} \right)) = \left(a'_{ij} + a'''_{ij} \right) = \left(\hat{a}_{ij} \right),$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\hat{a}_{ij} = \begin{cases} |a_{ii} - \alpha| - |a_{ii} - \beta|, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} f(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Considering the fact that $f(\mathbb{R}) = [-|\alpha - \beta|, |\alpha - \beta|]$, it follows that $(P_1 + M)(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, which ensures that the mapping $P_1 + M$ is not surjective, and so M is not a P_1-η-accretive mapping. Now, suppose that the real constant $\rho > 0$ is chosen arbitrarily but fixed and let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + \sin(\gamma x + \mu) + \rho\varrho \sqrt[k]{\sin x}$, for all $x \in \mathbb{R}$. Then, for any $A = \left(a_{ij} \right) \in D_n(\mathbb{R})$, we obtain

$$(P_2 + \rho M)(A) = (P_2 + \rho M)(\left(a_{ij} \right)) = \left(a''_{ij} + \rho a'''_{ij} \right) = \left(\widetilde{a}_{ij} \right),$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\widetilde{a}_{ij} = \begin{cases} a_{ii} + \sin(\gamma a_{ii} + \mu) + \rho\varrho \sqrt[k]{\sin a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} g(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Thanks to the fact that $g(\mathbb{R}) = \mathbb{R}$, we conclude that $(P_2 + \rho M)(D_n(\mathbb{R})) = D_n(\mathbb{R})$, that is, $P_2 + \rho M$ is a surjective mapping. Taking into account the arbitrariness in the choice of $\rho > 0$, it follows that M is a P_2-η-accretive mapping.

The following example illustrates that for given mappings $P : E \to E$ and $\eta : E \times E \to E$, a generalized m-accretive mapping need not be P-η-accretive.

Example 2.3. Assume that $H_2(\mathbb{C})$ is the set of all Hermitian matrices with complex entries. Let us recall that a square matrix A is said to be Hermitian (or self-adjoint) if it is equal to its own Hermitian conjugate, i.e., $A^* = \overline{A^t} = A$. In the light of the definition of a Hermitian 2×2 matrix, the condition $A^* = A$ implies that the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is Hermitian iff $a, d \in \mathbb{R}$ and $b = \bar{c}$. Consequently,

$$H_2(\mathbb{C}) = \Big\{ \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} | x, y, z, w \in \mathbb{R} \Big\}.$$

Then, $H_2(\mathbb{C})$ is a subspace of $M_2(\mathbb{C})$, the space of all 2×2 matrices with complex entries, with respect to the operations of addition and scalar multiplication defined on $M_2(\mathbb{C})$, when $M_2(\mathbb{C})$ is considered as a real vector space. By considering the scalar product on $H_2(\mathbb{C})$ as $\langle A, B \rangle := \frac{1}{2} tr(AB)$, for all $A, B \in H_2(\mathbb{C})$, it is not hard to see that $\langle ., . \rangle$ is an inner product, that is, $(H_2(\mathbb{C}), \langle ., . \rangle)$ is an inner product space. The inner product defined above induces a norm on $H_2(\mathbb{C})$ as follows:

$$\begin{aligned} \|A\| = \sqrt{\frac{1}{2} tr(AA)} &= \Big\{ \frac{1}{2} tr\Big(\begin{pmatrix} x^2 + y^2 + z^2 & (z + w)(x - iy) \\ (z + w)(x + iy) & x^2 + y^2 + w^2 \end{pmatrix} \Big) \Big\}^{\frac{1}{2}} \\ &= \sqrt{x^2 + y^2 + \frac{1}{2}(z^2 + w^2)}, \quad \forall A \in H_2(\mathbb{C}). \end{aligned}$$

Taking into account that every finite dimensional normed space is a Banach space, it follows that $(H_2(\mathbb{C}), \|.\|)$ is a Hilbert space. Suppose that the mappings $P, M : H_2(\mathbb{C}) \to H_2(\mathbb{C})$ and $\eta : H_2(\mathbb{C}) \times H_2(\mathbb{C}) \to H_2(\mathbb{C})$ are defined, respectively, by

$$P(A) = P\Big(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \Big) = \begin{pmatrix} \alpha z^{2k} & x^2 - iy^2 \\ x^2 + iy^2 & \sigma + \delta \sin^l w \end{pmatrix},$$

$$M(A) = M\Big(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \Big) = \begin{pmatrix} \alpha z^k & x - iy \\ x + iy & \frac{\beta}{\gamma + \theta \sin^q w} \end{pmatrix}$$

and

$$\begin{aligned} \eta(A, B) &= \eta\Big(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}, \begin{pmatrix} \hat{z} & \hat{x} - i\hat{y} \\ \hat{x} + i\hat{y} & \hat{w} \end{pmatrix} \Big) \\ &= \begin{pmatrix} \varrho e^{s(w + \hat{w})}(z^p - \hat{z}^p) & x - \hat{x} - i(y - \hat{y}) \\ x - \hat{x} + i(y - \hat{y}) & \xi \cos^m z \cos^m \hat{z}(\sin^q \hat{w} - \sin^q w) \end{pmatrix}, \end{aligned}$$

for all $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}, B = \begin{pmatrix} \hat{z} & \hat{x} - i\hat{y} \\ \hat{x} + i\hat{y} & \hat{w} \end{pmatrix} \in H_2(\mathbb{C})$, where $s, \sigma \in \mathbb{R}$ are arbitrary constants, $\alpha, \beta, \gamma, \theta, \varrho, \xi, t$ and δ are positive real constants, k and p are two arbitrary but fixed odd natural numbers, and m, q, l are arbitrary but fixed even natural numbers.

Then, for any $A = \begin{pmatrix} z_1 & x_1 - iy_1 \\ x_1 + iy_1 & w \end{pmatrix}, B = \begin{pmatrix} z_2 & x_2 - iy_2 \\ x_2 + iy_2 & w_2 \end{pmatrix} \in H_2(\mathbb{C})$, we get

$$\begin{aligned}
&\langle M(A)-M(B), j(\eta(A,B))\rangle = \langle M(A)-M(B), \eta(A,B)\rangle \\
&= \frac{1}{2}tr\Big(\left(\begin{array}{cc} \alpha(z_1^k-z_2^k) & x_1-x_2-i(y_1-y_2) \\ x_1-x_2+i(y_1-y_2) & \frac{\beta\theta(\sin^q w_2-\sin^q w_1)}{(\gamma+\theta\sin^q w_1)(\gamma+\theta\sin^q w_2)} \end{array}\right), \\
&\left(\begin{array}{cc} \varrho e^{s(w_1+w_2)}(z_1^p-z_2^p) & x_1-x_2-i(y_1-y_2) \\ x_1-x_2+i(y_1-y_2) & \xi\cos^m z_1\cos^m z_2(\sin^q w_2-\sin^q w_1) \end{array}\right)\Big) \\
&= \frac{1}{2}tr\Big(\left(\begin{array}{cc} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{array}\right)\Big) \\
&= \frac{\alpha\varrho}{2}(z_1-z_2)^2 e^{s(w_1+w_2)}\Big(\sum_{r=1}^{k} z_1^{k-r}z_2^{r-1}\Big)\Big(\sum_{t=1}^{p} z_1^{p-t}z_2^{t-1}\Big) + (x_1-x_2)^2 \\
&\quad + (y_1-y_2)^2 + \frac{\beta\theta\xi\cos^m z_1\cos^m z_2(\sin^q w_2-\sin^q w_1)^2}{2(\gamma+\theta\sin^q w_1)(\gamma+\theta\sin^q w_2)},
\end{aligned}$$

where

$$\begin{aligned}
\Omega_{11} &= \alpha\varrho(z_1-z_2)^2 e^{s(w_1+w_2)}\Big(\sum_{r=1}^{k} z_1^{k-r}z_2^{r-1}\Big)\Big(\sum_{t=1}^{p} z_1^{p-t}z_2^{t-1}\Big) \\
&\quad + (x_1-x_2)^2 + (y_1-y_2)^2, \\
\Omega_{12} &= (x_1-x_2-i(y_1-y_2))(\alpha(z_1^k-z_2^k) + \xi\cos^m z_1\cos^m z_2(\sin^q w_2-\sin^q w_1), \\
\Omega_{21} &= (x_1-x_2+i(y_1-y_2))(\varrho e^{s(w_1+w_2)}(z_1^p-z_2^p) \\
&\quad + \frac{\beta\theta(\sin^q w_2-\sin^q w_1)}{(\gamma+\theta\sin^q w_1)(\gamma+\theta\sin^q w_2)}, \\
\Omega_{22} &= (x_1-x_2)^2 + (y_1-y_2)^2 + \frac{\beta\theta\xi\cos^m z_1\cos^m z_2(\sin^q w_2-\sin^q w_1)^2}{(\gamma+\theta\sin^q w_1)(\gamma+\theta\sin^q w_2)}.
\end{aligned}$$

Owing to the fact that k and p are odd natural numbers, it can be easily observed that $\sum_{r=1}^{k} z_1^{k-r}z_2^{r-1} \geq 0$ and $\sum_{t=1}^{p} z_1^{p-t}z_2^{t-1} \geq 0$. Since $\alpha,\beta,\gamma,\theta,\xi,\varrho > 0$, m and q are even natural numbers, and $A \neq B$, from the latter relation it follows that

$$\langle M(A)-M(B), j(\eta(A,B))\rangle \geq 0, \quad \forall A,B \in H_2(\mathbb{C}), j(\eta(A,B)) \in \mathcal{F}(\eta(A,B)),$$

that is, M is an η-accretive mapping.

Let us now define the functions $f,g,h:\mathbb{R}\to\mathbb{R}$ as $f(\tau)=\alpha(\tau^{2k}+\tau^k)=\alpha\tau^k(\tau^k+1)$, $g(\tau)=\sigma+\delta\sin^l\tau+\frac{\beta}{\gamma+\theta\sin^q\tau}$ and $h(\tau)=\tau^2+\tau$ for all $\tau\in\mathbb{R}$. Then, for any $A = \left(\begin{array}{cc} z & x-iy \\ x+iy & w \end{array}\right) \in H_2(\mathbb{C})$, we derive that

$$\begin{aligned}
(P+M)(A) &= (P+M)\Big(\left(\begin{array}{cc} z & x-iy \\ x+iy & w \end{array}\right)\Big) \\
&= \left(\begin{array}{cc} \alpha z^k(z^k+1) & x^2+x-i(y^2+y) \\ x^2+x+i(y^2+y) & \sigma+\delta\sin^l w+\frac{\beta}{\gamma+\theta\sin^q w} \end{array}\right) \\
&= \left(\begin{array}{cc} f(z) & h(x)-ih(y) \\ h(x)+ih(y) & g(w) \end{array}\right).
\end{aligned}$$

Taking into account consideration the fact that k is an odd natural number, it is easy to see that $f(\mathbb{R}) \neq \mathbb{R}$. Since

$$g(\tau) = \sigma + \delta \sin^l \tau + \frac{\beta}{\gamma + \theta \sin^q \tau} \geq \sigma + \frac{\beta}{\gamma + \theta}$$

and

$$h(\tau) = \tau^2 + \tau = (\tau + \frac{1}{2})^2 - \frac{1}{4} \geq -\frac{1}{4},$$

it follows that $g(\mathbb{R}) = [\frac{\sigma(\gamma+\theta)+\beta}{\gamma+\theta}, +\infty) \neq \mathbb{R}$ and $h(\mathbb{R}) = [-\frac{1}{4}, +\infty) \neq \mathbb{R}$. The above-mentioned arguments imply that $(P+M)(H_2(\mathbb{C})) \neq H_2(\mathbb{C})$, that is, $P+M$ is not surjective, consequently, M is not P-η-accretive. Now, let $\rho > 0$ be an arbitrary positive real constant and suppose that the mappings $\widetilde{f}, \widetilde{g}, \widetilde{h} : \mathbb{R} \to \mathbb{R}$ are defined for all $\tau \in \mathbb{R}$ by $\widetilde{f}(\tau) := \tau + \rho\alpha\tau^k$, $\widetilde{g}(\tau) := \tau + \frac{\rho\beta}{\gamma+\theta \sin^q \tau}$ and $\widetilde{h}(\tau) := (1+\rho)\tau$, respectively. Then, for any $A = \begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix} \in H_2(\mathbb{C})$, we have

$$\begin{aligned}
(I + \rho M)(A) &= (I + \rho M)\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix} \Big) \\
&= \begin{pmatrix} z + \rho\alpha z^k & (1+\rho)x - i(1+\rho)y \\ (1+\rho)x + i(1+\rho)y & w + \frac{\rho\beta}{\gamma+\theta\sin^q w} \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{f}(z) & \widetilde{h}(x) - i\widetilde{h}(y) \\ \widetilde{h}(x) + i\widetilde{h}(y) & \widetilde{g}(w) \end{pmatrix},
\end{aligned}$$

where I is the identity mapping on $H_2(\mathbb{C})$. By virtue of the fact that $\widetilde{f}(\mathbb{R}) = \widetilde{g}(\mathbb{R}) = \widetilde{h}(\mathbb{R}) = \mathbb{R}$, it follows that $(I + \rho M)(H_2(\mathbb{C})) = H_2(\mathbb{C})$, i.e., the mapping $I + \rho M$ is surjective. Since $\rho > 0$ was arbitrary, we conclude that M is a generalized m-accretive mapping.

3. Main Results

In the this section we prove two main theorems showing that if P is a δ-strongly η-accretive mapping, η is a τ-Lipschitz continuous mapping and M is a P-η-accretive mapping this implies that P-η-proximal-point mapping J_ρ^M is $\frac{\tau}{\delta}$-Lipschitz continuous.

All results in [19] have been derived based on Definition 2.1 and Definition 2.3 in [19], it is to be noted that the normalized duality mapping in these definitions is denoted by J. Employing these definitions, Kazmi and Khan [19] presented some properties of P-η-accretive mappings in [19, Theorem 2.1].

In the proof of [19, Theorem 2.1(a)], the authors assumed that there exists $(u_0, x_0) \notin \mathrm{Graph}(M)$ such that

$$\langle u_0 - v, j(\eta(x_0, y)) \rangle \geq 0, \quad \forall (y, v) \in \mathrm{Graph}(M). \tag{3.1}$$

Then in virtue of the fact that M is P-η-accretive, they deduced the existence of $(x_1, u_1) \in \mathrm{Graph}(M)$ such that

$$P(x_1) + \rho u_1 = P(x_0) + \rho u_0. \tag{3.2}$$

Setting $(y, v) = (x, u_1)$ in (3.1) and making use of (3.2), they claimed that there exists a selection $j(\eta(x_0, x_1)) \in \mathcal{F}(\eta(x_0, x_1))$ such that

$$0 \leq \rho\langle u_0 - u_1, j(\eta(x_0, x_1)) \rangle = \langle P(x_1) - P(x_0), j(\eta(x_0, x_1)) \rangle \tag{3.3}$$

and then they deduced that

$$\langle P(x_0) - P(x_1), j(\eta(x_0, x_1)) \rangle \leq 0. \tag{3.4}$$

But, the equality in (3.3) has some flaw. In fact, relying on the fact that P and M are η-accretive, according to their definitions, there are $j_1(\eta(x_0,x_1)) \in \mathcal{F}(\eta(x_0,x_1))$ and $j_2(\eta(x_0,x_1)) \in \mathcal{F}(\eta(x_0,x_1))$ such that

$$\langle P(x_1) - P(x_0), j_1(\eta(x_0,x_1))\rangle \geq 0 \text{ and } \langle u_0 - u_1, j_2(\eta(x_0,x_1))\rangle \geq 0.$$

Since the two selections $j_1(\eta(x_0,x_1))$ and $j_2(\eta(x_0,x_1))$ are not the same necessarily, it follows that (3.2) doesn't imply (3.3), and thereby (3.4) is not also true, necessarily. At the same time, in the proof of [19, Theorem 2.1(b)], for any given $z \in E$ and a constant $\rho > 0$, and letting $x, y \in (P+\rho M)^{-1}(z)$, the authors deduced that $\rho^{-1}(z - P(x)) \in M(x)$ and $\rho^{-1}(z-P(y)) \in M(y)$. Then, using η-accretiveness of M, they asserted that

$$\begin{aligned} 0 &= \rho\langle \rho^{-1}(z-P(x)) - \rho^{-1}(z-P(y)), j(\eta(x,y))\rangle \\ &\quad + \langle P(x) - P(y), j(\eta(x,y))\rangle \\ &\geq \langle P(x) - P(y), j(\eta(x,y))\rangle. \end{aligned} \tag{3.5}$$

Finally, in the light of strict η-accretiveness of P, they deduced that $x = y$. But, thanks to the definition of strict η-accretivity of the mapping P given in Definition 2.1 in [19], P is strictly η-accretive if P is η-accretive, i.e., there exists $j_1(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$ such that

$$\langle P(x) - P(y), j_1(\eta(x,y))\rangle \geq 0, \tag{3.6}$$

and equality holds if and only if $x = y$. On the other hand, taking into account that M is η-accretive, invoking Definition 2.3 in [19], there is $j_2(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$ such that

$$\langle \rho^{-1}(z-P(x)) - \rho^{-1}(z-P(y)), j_2(\eta(x,y))\rangle \geq 0,$$

which implies that

$$\langle P(x) - P(y), j_2(\eta(x,y))\rangle \leq 0. \tag{3.7}$$

However, (3.6) and (3.7) do not guarantee the existence of a selection $j(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$ such that

$$\langle P(x) - P(y), j(\eta(x,y))\rangle = 0,$$

and so strict η-accretivity of P doesn't imply $x = y$ necessarily.

Applying Definitions 2.3 and 2.4 instead of [19, Definitions 2.1 and 2.3]), respectively, we now present the new version of [19, Theorem 2.1] along with its proof as follows.

Theorem 3.1. *Let $\eta : E \times E \to E$ be a vector-valued mapping and $P : E \to E$ be a strictly η-accretive mapping. Suppose further that $\mathcal{F} : E \to 2^{E^*}$ is the normalized duality mapping and $M : E \to 2^E$ is an η-accretive mapping. Then*

- (a) *for any given point $(x,u) \in E \times E$, if $\langle u - v, j(\eta(x,y))\rangle \geq 0$ holds for all $(y,v) \in \mathrm{Graph}(M)$ and $j(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$, then $(x,u)\,\mathrm{Graph}(M)$;*
- (b) *the single-valued mapping $(P+\rho M)^{-1} : \mathrm{Range}(P+\rho M) \to E$ is single-valued for every real constant $\rho > 0$.*

Proof. (a) On the contrary suppose that $(x,u) \notin \mathrm{Graph}(M)$. Since M is P-η-accretive, we have $(P+\rho M)(E) = E$ for every real constant $\rho > 0$. Then, there exists $(x_0,u_0) \in \mathrm{Graph}(M)$ such that

$$P(x_0) + \rho u_0 = P(x) + \rho u. \tag{3.8}$$

Picking $(y,v) = (x_0,v_0)$, from the assumption we conclude that

$$\langle u - u_0, j(\eta(x,x_0))\rangle \geq 0, \quad \forall j(\eta(x,x_0)) \in \mathcal{F}(\eta(x,x_0)). \tag{3.9}$$

Using (3.8) and (3.9) and taking into account that P is η-accretive, it follows that for all $j(\eta(x,x_0)) \in \mathcal{F}(\eta(x,x_0))$,

$$0 \leq \rho\langle u-u_0, j(\eta(x,x_0))\rangle = -\langle P(x)-P(x_0), j(\eta(x,x_0))\rangle \leq 0,$$

which implies that

$$\langle P(x)-P(x_0), j(\eta(x,x_0))\rangle = 0.$$

Considering the fact that P is strictly η-accretive, we deduce that $x = x_0$. Thereby, making use of Eq. (3.8), it follows that $u = u_0$. Clearly, this is in contradiction to our assumption.

(b) Let the real constant $\rho > 0$ be chosen arbitrarily. For any given point $u \in \mathrm{Range}(P+\rho M)$, let $x, y \in (P+\rho M)^{-1}(u)$. Then, we have $u = (P+\rho M)(x) = (P+\rho M)(y)$, which implies that

$$\rho^{-1}(u-P(x)) \in M(x) \text{ and } \rho^{-1}(u-P(y)) \in M(y).$$

Since M is η-accretive, it follows that for all $j(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$,

$$\langle \rho^{-1}(u-P(x)) - \rho^{-1}(u-P(y)), j(\eta(x,y))\rangle \geq 0,$$

from which we deduce that

$$\langle P(x)-P(y), j(\eta(x,y))\rangle \leq 0, \quad \forall j(\eta(x,y)) \in \mathcal{F}(\eta(x,y)). \tag{3.10}$$

On the other hand, taking into account that P is η-accretive, we have

$$\langle P(x)-P(y), j(\eta(x,y))\rangle \geq 0, \quad \forall j(\eta(x,y)) \in \mathcal{F}(\eta(x,y)). \tag{3.11}$$

Now, making use (3.3) and (3.4) and thanks to the strict η-accretiveness of the mapping P it follows that $x = y$. This fact implies that the mapping $P+\rho M$ from $\mathrm{Range}(P+\rho M)$ into E is single-valued. This completes the proof. □

It is worth mentioning that if $P = I$, then the definition of I-η-accretive mappings is that of generalized m-accretive mappings. In fact, the class of P-η-accretive mappings has close relation with that of generalized m-accretive mappings. This fact is illustrated in Theorem 3.1(a).

As an immediate consequence of part (b) of the last result, we obtain the following assertion.

Corollary 3.1. *Suppose that $\eta : E \times E \to E$ is a vector-valued mapping and $P : E \to E$ is a strictly η-accretive mapping. Let $\mathcal{F} : E \to 2^{E^*}$ be the normalized duality mapping and $M : E \to 2^E$ be a P-η-accretive mapping. Then, the mapping $(P+\rho M)^{-1} : E \to E$ is single-valued for every real constant $\rho > 0$.*

According to Theorem 3.1(b), Kazmi and Khan [20] introduced the P-η-proximal-point mapping, denoted as J_ρ^M, associated with a P-η-accretive mapping M. Here, $\rho > 0$ remains a constant, $\eta : E \times E \to E$ is a vector-valued mapping, and $P : E \to E$ represents a strictly η-accretive mapping. This mapping is defined as follows:

$$J_\rho^M(z) = (P+\rho M)^{-1}(z), \quad \forall z \in E. \tag{3.12}$$

Their investigation, presented in [20], concludes section 2 by establishing the Lipschitz continuity of the P-η-proximal-point mapping J_ρ^M and providing an estimation of its Lipschitz constant.

Theorem 3.2. [20, Theorem 2.2] *Let $P : E \to E$ be a δ-strongly η-accretive mapping. Let $\eta : E \times E \to E$ be a τ-Lipschitz continuous mapping and $M : E \to 2^E$ be a P-η-accretive mapping. Then P-η-proximal-point mapping J_ρ^M is $\frac{\tau}{\delta}$-Lipschitz continuous, i.e.,*

$$\|J_\rho^M(x) - J_\rho^M(y)\| \leq \frac{\tau}{\delta}\|x-y\|, \quad \forall x, y \in E.$$

In the rest of the paper, instead of J_ρ^M, we denote the P-η-proximal-point mapping associated with a P-η-accretive mapping $M : E \to 2^E$ by $J_{\rho,\eta}^{M,P}$, where $\rho > 0$ is an arbitrary positive real constant, $P : E \to E$ is strictly η-accretive mapping and $\eta : E \times E \to E$ is a vector-valued mapping, and is defined based on Corollary 3.1 as (3.12).

We now close this section by presenting a revised version of the proof of [19, Theorem 2.2].

Proof. Taking into account that M is P-η-accretive, for any given points $x, y \in E$ with $\|J_{\rho,\eta}^{M,P}(x) - J_{\rho,\eta}^{M,P}(y)\| \neq 0$, we have $J_{\rho,\eta}^{M,P}(x) = (P + \rho M)^{-1}(x)$ and $J_{\rho,\eta}^{M,P}(y) = (P + \rho M)^{-1}(y)$, which implies that

$$\rho^{-1}(x - P(J_{\rho,\eta}^{M,P}(x)) \in M(J_{\rho,\eta}^{M,P}(x)) \text{ and } \rho^{-1}(y - P(J_{\rho,\eta}^{M,P}(y)) \in M(J_{\rho,\eta}^{M,P}(y)).$$

Since M is η-accretive, we conclude that for all $j(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y))) \in \mathcal{F}(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y)))$,

$$\rho^{-1}\langle x - P(J_{\rho,\eta}^{M,P}(x)) - (y - P(J_{\rho,\eta}^{M,P}(y))), j(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y)))\rangle \geq 0.$$

Considering the fact that $\rho^{-1} > 0$, from the preceding inequality, δ-strong η-accretiveness of P, and τ-Lipschitz continuity of η, it follows that for all $j(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y))) \in \mathcal{F}(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y)))$,

$$\begin{aligned}
\tau\|x - y\|\|J_{\rho,\eta}^{M,P}(x) - J_{\rho,\eta}^{M,P}(y)\| &\geq \|x - y\|\|\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y))\| \\
&= \|x - y\|\|j(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y)))\| \\
&\geq \langle x - y, j(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y)))\rangle \\
&\geq \langle P(J_{\rho,\eta}^{M,P}(x)) - P(J_{\rho,\eta}^{M,P}(y)), \\
&\quad j(\eta(J_{\rho,\eta}^{M,P}(x), J_{\rho,\eta}^{M,P}(y)))\rangle \\
&\geq \delta\|J_{\rho,\eta}^{M,P}(x) - J_{\rho,\eta}^{M,P}(y)\|^2.
\end{aligned}$$

In view of the fact that $\|J_{\rho,\eta}^{M,P}(x) - J_{\rho,\eta}^{M,P}(y)\| \neq \emptyset$, the last inequality implies that

$$\|J_{\rho,\eta}^{M,P}(x) - J_{\rho,\eta}^{M,P}(y)\| \leq \frac{\tau}{\delta}\|x - y\|.$$

This completes the proof. □

Conclusions:

In conclusion, this paper has introduced a novel iterative algorithm for solving a new system of generalized multi-valued resolvent equations within the framework of Banach spaces. Through the utilization of the resolvent operator associated with a P-η-accretive mapping, the algorithm demonstrates promising potential in addressing complex problems. The main contribution of this work lies in establishing the Lipschitz continuity of the resolvent operator linked with a P-η-accretive mapping. Additionally, an estimation of its Lipschitz constant has been computed under newly defined conditions, thereby enhancing the understanding of the algorithm's behavior. Furthermore, this paper has provided definitions and concrete examples to elucidate the concepts presented. Notably, the presented results advance and generalize existing knowledge in the field, showcasing the significance of this research endeavor. Moving forward, part II of this study delves into the convergence analysis of the sequences generated by the proposed iterative algorithm. By exploring appropriate conditions, this analysis aims to deepen our understanding of the algorithm's convergence properties and its practical applicability.

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