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New class of n-order fractional differential equations and solvability in the double sequence space $m^2(\Delta_v^u, \phi, p)$

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ABSTRACT. First, we define a new class of fractional differential equations of order $n-1 < \vartheta \le n$, $(n \ge 2)$. Also, we define a new Banach double sequence space $m^2(\Delta_v^u, \phi, p)$ and a Hausdorff MNC on it. By using this MNC, we prove the existence of solution of infinite system of a new class of fractional differential equations of order $\vartheta \in (n-1, n]$, $(n \ge 2)$ in $m^2(\Delta_v^u, \phi, p)$. Finally, we give two examples to verify the usefulness of our main result.

1. INTRODUCTION AND PRELIMINARIES

Fractional differential equations (FDE) arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of economy, biology, physics, chemistry, engineering and many other fields ([16, 18, 19, 20, 33]).

Bromwich in 1965 [8] established the primary work on double sequences. Then, Altay and Basar defined some new spaces of double sequences BS, CS, $BS(t)$, CS_p, CS_{bp} [2]. Also, convergence of double sequences spaces such as Pringsheimś sense, statistically null in Pringsheimś sense, bounded statistically null in Pringsheimś sense and etc extended by several authors.

In 1957, Goldenstein et al. [11] defined the Hausdorff MNC χ and studied by Goldenstein and Markus [12]. Recently, several authors [3, 4, 7, 10, 13, 14, 15, 17, 23, 24, 25, 26, 28, 29, 31] studied the problems of existence of solutions of differential equations, fractional differential equations and integral equations in various spaces by using the techniques of measures of noncompactness.

Motivated by the above papers, we first define a new class of fractional differential equations of order $\vartheta \in (n-1, n]$, $(n \ge 2)$. Also, we define a new Banach double sequence space $m^2(\Delta_v^u, \phi, p)$ and we define a Hausdorff MNC and by using this MNC we discuss the existence of solutions of infinite systems of new class of FDEs of order $\vartheta \in (n-1, n]$, $(n \ge 2)$ in $m^2(\Delta_v^u, \phi, p)$ and we present two examples illustrating the obtained results.

Let Λ be a real Banach space, and $\emptyset \ne \mathfrak{L} \subset \Lambda$. Then
- Conv$\mathfrak{L}$ closed convex hull and $\overline{\mathfrak{L}}$ the closure of $\mathfrak{L}$.
- $D(\nu, \sigma)$ is a closed ball in Λ.
- $\mathfrak{N}_\Lambda$ is the family of relatively compact subsets of Λ.
- $\mathfrak{M}_\Lambda$ is the family of bounded subsets of Λ.

Definition 1.1. [5, 6] *The function $\tilde{\mu} : \mathfrak{M}_\Lambda \to [0, +\infty)$ is a measure of noncompactness (MNC) in Λ if for any $\mathcal{U}, \mathcal{V} \in \mathfrak{M}_\Lambda$ we have:*

(i) $\mathfrak{N}_\Lambda \supseteq \ker \tilde{\mu} = \{\mathcal{U} \in \mathfrak{M}_\Lambda : \tilde{\mu}(\mathcal{U}) = 0\} \ne \emptyset$.

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(*ii*) *If* $\mathcal{U} \subset \mathcal{V}, \Rightarrow \tilde{\mu}(\mathcal{U}) \leq \tilde{\mu}(\mathcal{V})$.

(*iii*) $\tilde{\mu}(\overline{\mathcal{U}}) = \tilde{\mu}(\mathcal{U}) = \tilde{\mu}(Conv\mathcal{U})$.

(*iv*) $\tilde{\mu}(\lambda\mathcal{U} + (1-\lambda)\mathcal{V}) \leq \lambda\tilde{\mu}(\mathcal{U}) + (1-\lambda)\tilde{\mu}(\mathcal{V})$ *for each* $\lambda \in [0,1]$.

(*v*) *If for each* $n \in \mathbb{N}, \overline{\mathcal{U}_n} = \mathcal{U}_n \subseteq \mathfrak{M}_\Lambda, \mathcal{U}_{n+1} \subset \mathcal{U}_n$ *If* $\lim_{n\to\infty} \tilde{\mu}(\mathcal{U}_n) = 0, \Rightarrow \emptyset \neq \mathcal{U}_\infty = \bigcap_{n=1}^{\infty} \mathcal{U}_n$.

Definition 1.2. [5, 6] *Let* (X,d) *be a metric space and* $\mathcal{Q} \in \mathfrak{M}_X$. *The Kuratowski MNC of* $\mathcal{Q}$ *is defined as follow:*

$$\vartheta(\mathcal{Q}) = \inf\Big\{\varepsilon > 0 : \mathcal{Q} \subset \bigcup_{j=1}^{m} S_j, S_j \subset X, diam(S_j) < \varepsilon \; (j = 1, \dots, m);\; m \in \mathbb{N}\Big\},$$

where $diam(S_j) = \sup\{d(\varsigma, \nu) : \varsigma, \nu \in S_j\}$.

The Hausdorff MNC of the bounded set $\mathcal{Q}$, is

$$\chi(\mathcal{Q}) = \inf\Big\{\varepsilon > 0 : \mathcal{Q} \subset \bigcup_{j=1}^{m} D(y_j, v_j), y_j \in X, v_j < \varepsilon \; (j = 1, \dots, m);\; m \in \mathbb{N}\Big\}.$$

Definition 1.3. [22] *Let* $\tilde{\mu}$ *be an arbitrary MNC on Banach space* Λ *and* $\emptyset \neq \mathcal{B} \subseteq \Lambda$. *The operator* $F : \mathcal{B} \to \mathcal{B}$ *is a Meir–Keeler condensing operator if for any bounded subset* $\mathcal{U}$ *of* $\mathcal{B}$ *and for each* $\varepsilon > 0, \exists\, \delta > 0$ *so that*

$$\varepsilon \leq \tilde{\mu}(\mathcal{U}) < \varepsilon + \delta \quad \textit{implies} \quad \tilde{\mu}(F(\mathcal{U})) < \varepsilon$$

Theorem 1.1. [1] *Let* $\emptyset \neq \mathcal{B} = \overline{\mathcal{B}} \subseteq \Lambda$ *is convex, bounded and* $\tilde{\mu}$ *be an arbitrary MNC on* Λ *and* $F : \mathcal{B} \to \mathcal{B}$ *be a continuous Meir–Keeler condensing operator. Then* F *has at least one fixed point.*

Lemma 1.1. [5, 6] *Let* $\Omega \subseteq C(J, \Lambda)$ *be bounded and equicontinuous. Then* $\tilde{\mu}(\Omega(.))$ *is continuous on* J *and*

$$\tilde{\mu}(\Omega) = \sup_{\tau \in J} \tilde{\mu}(\Omega(\tau)), \quad \tilde{\mu}\Big(\int_0^{\tau} \Omega(\varrho) d\varrho\Big) \leq \int_0^{\tau} \tilde{\mu}(\Omega(\varrho)) d\varrho,$$

where $C(J, \Lambda)$ *is Banach space with norm*

$$\|z\|_{C(J,\Lambda)} := \sup\{\|z(\tau)\| : \tau \in J\},\; z \in C(J, \Lambda).$$

2. DOUBLE SEQUENCE SPACE $m^2(\Delta_v^u, \phi, p)$

Let ω be the set of all complex sequences and $\mathcal{C}$ the space whose elements are finite sets of distinct positive integers. for any element σ of $\mathcal{C}$, we denote

$$c(\sigma) = \begin{cases} c_n(\sigma) = 1, & n \in \sigma, \\ c_n(\sigma) = 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{C}_r = \Big\{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq r\Big\},$$

the set of those σ whose support has cardinality at most s, and let

$$\Phi = \Big\{\phi = (\phi_n) \in \omega : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1},\; (n+1)\phi_n \geq n\phi_{n+1}\Big\},$$

([27]). Let $u > 0$ be fixed, $\phi \in \Phi$ and $0 \neq v = (v_k)$ be any fixed sequence of complex numbers. The sequence space $m(\Delta_v^u, \phi, p)$ is defined by ([34])

$$m(\Delta_v^u, \phi, p) = \big\{x = (x_k) \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \big(\frac{1}{\phi_s} \sum_{k \in \sigma} |\Delta_v^u x_k|^p\big) < \infty,\; p \in [0, \infty)\big\},$$

where

$$\begin{aligned} \Delta_v^0 x_k &= v_k x_k, \\ \Delta_v^1 x_k &= v_k x_k - v_{k+1} x_{k+1}, \\ \Delta_v^u x_k &= \Delta_v^{u-1} x_k - \Delta_v^{u-1} x_{k+1}, \end{aligned}$$

such that
$$\Delta_v^u x_k = \sum_{i=0}^{u} (-1)^i \begin{bmatrix} u \\ i \end{bmatrix} v_{k+i} x_{k+i}.$$

Theorem 2.2. [34] *For $\phi \in \Phi$ the sequence space $m(\Delta_v^u, \phi, p)$ is a Banach space by norm*
$$\|x\|_{m(\Delta_v^u,\phi,p)} = \sum_{i=1}^{u} |x_i| + \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \Big(\frac{1}{\phi_s} \sum_{k\in\sigma} |\Delta_v^u x_k|^p\Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Let $\mathbb{B} = \langle b_{nk} \rangle$ be double sequence, then $\overline{\theta} = (\theta, \theta, \dots,)$ is zero single sequence, and $2\overline{\theta} = \langle \theta \rangle$ is zero double sequence, $e = (1, 1, \dots,)$ and $e_k = (0, 0, \dots, 1, 0, 0 \dots,)$, where k-th place is 1. The double sequence $z = \langle z_{jk} \rangle$ of complex (real numbers) is bounded if
$$\|z\|_{(\infty,2)} = \sup_{j,k} |z_{jk}| < \infty.$$

Definition 2.4. [9] *The double sequence of functions $\{f_{kl}\}$ is Pringsheim limit function (pointwise convergent) to f on the set $S \subset \mathbb{R}$ if, for each point $x \in S$ and for each $\varepsilon > 0$, $\exists\, N(x, \varepsilon) > 0$ so that $\forall\, k, l > N$*
$$|f_{kl}(x) - f(x)| < \varepsilon, \ \lim_{k,l\to\infty} f_{kl}(x) = f(x).$$

We define a new double sequence space $m^2(\Delta_v^u, \phi, p)$ as follow:
$$m^2(\Delta_v^u, \phi, p) = \Big\{x = (x_{kl}) \in \omega^2 : \|x\|_{m^2(\Delta_v^u,\phi,p)} = \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}} \Big(\sum_{k\in\sigma_1} \sum_{l\in\sigma_2} |\Delta_v^u x_{k,l}|^p \Big)^{\frac{1}{p}} < \infty \Big\},$$
where ϕ_{se} denote the class of subsets $\sigma = \sigma_1 \times \sigma_2$ in $\mathbb{N} \times \mathbb{N}$ so that the elements of σ_1 and σ_2 are most s and e, respectively $\{\phi_{se}\}$ is a increasing double sequence of the positive real numbers so that
$$s\phi_{s+1,e} \leq (s+1)\phi_{se}, \quad t\phi_{s,e+1} \leq (e+1)\phi_{se}.$$

Theorem 2.3. *The double sequence $m^2(\Delta_v^u, \phi, p)$ (convergent in Pringsheim's sense space) is a Banach space.*

Proof. Let $\xi > 0$ and x_{kl}^i be a Cauchy sequence in $m^2(\Delta_v^u, \phi, p)$, choose $m_0 \in \mathbb{N}$ such that
$$\|x_{kl}^i - x_{kl}^j\|_{m^2(\Delta_v^u,\phi,p)} < \xi,$$
$\forall, k, l \in \mathbb{N}$ and $i, j \geq m_0$. So
$$\sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}} \Big(\sum_{k\in\sigma_1} \sum_{l\in\sigma_2} |\Delta_v^u (x_{k,l}^i - x_{k,l}^j)|^p \Big)^{\frac{1}{p}} < \xi,$$
$\forall\, i, j \geq m_0$. we obtain
$$|\Delta_v^u (x_{k,l}^i - x_{k,l}^j)|^p < \xi,$$
$\forall\, i, j \geq m_0$ and for each fixed $(k, l) \in \mathbb{N} \times \mathbb{N}$. So x_{kl}^i is a Cauchy sequence in $\mathbb{C}$. So there exists $x_{kl} \in \mathbb{C}$, such that $x_{kl}^i \to x_{kl}$, as $i \to \infty$, $\forall\, (k, l) \in \mathbb{N} \times \mathbb{N}$. So, for all $i, j \geq m_0$ and $\sigma_1 \times \sigma_2 \in \phi_{se}$. Thus we have
$$\sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}} \Big(\sum_{k\in\sigma_1} \sum_{l\in\sigma_2} |\Delta_v^u (x_{k,l}^i - x_{k,l})|^p \Big)^{\frac{1}{p}} < \xi,$$
$\forall\, i, j \geq m_0$. This implies that $x_{kl}^i - x_{kl} \in m^2(\Delta_v^u, \phi, p)\ \forall\, i, j \geq m_0$. □

Theorem 2.4. ([27].) *Let $E \subseteq l_p$ $(1 \leq p \leq \infty)$ be bounded. If $P_n : l_p \to l_p$ be the projector $\big(P_n(x) = x^{[n]} = (x_0, x_1, \dots, x_n, 0, 0, \dots)\ \forall\, x \in l_p\big)$. Then,*
$$\mathcal{X}(E) = \lim_{n\to\infty} \Big(\sup_{x\in E} \|(I - P_n)(x)\|_{l_p} \Big).$$
If $E \in \mathfrak{M}_{l_p}$, then
$$\mathcal{X}(E) = \lim_{n\to\infty} \Big(\sup_{x\in E} \sum_{k\geq n} |x_k|^p \Big).$$

Theorem 2.5. *Let* $Q \subseteq m^2(\Delta_v^u, \phi, p)$ *be bounded. If* $P_{kl} : m^2(\Delta_v^u, \phi, p) \to m^2(\Delta_v^u, \phi, p)$ *be the projector* $\big(P_{kl}(x) = (x_{k1}, x_{k2}, \dots, x_{kl}, x_{1l}, x_{2l}, \dots, x_{kl}, , 2\overline{\theta}, 2\overline{\theta}, \dots)\big)$, *where* $x = x_{kl} \in m^2(\Delta_v^u, \phi, p)\ \forall\, k, l \in \mathbb{N}$. *Then, If* $Q \in \mathfrak{M}_{m^2(\Delta_v^u,\phi,p)}$, *we have*

$$\chi(Q) = \lim_{k,l\to\infty} \Big\{ \sup_{x\in Q} \Big\{ \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}} \Big(\sum_{k\in\sigma_1} \sum_{l\in\sigma_2} |\Delta_v^u x_{k,l}|^p \Big)^{\frac{1}{p}} \Big\}\Big\}. \tag{2.1}$$

Proof. Let $Q \subseteq m^2(\Delta_v^u, \phi, p)$ be bounded and P_{kl} a projection function then,

$$Q \subset P_{kl}Q + (1 - P_{kl})Q. \tag{2.2}$$

By (2.2) and properties of χ, we have

$$\begin{aligned} \chi(Q) &\leq \chi(P_{kl}Q) + \chi\big((I - P_{kl})Q\big) = \chi\big((I - P_{kl})Q\big) \\ &\leq \dim\big((I - P_{kl})Q\big) = \sup_{x\in Q} \|(I - P_{kl})x\|_{m^2(\Delta_v^u,\phi,p)}. \end{aligned}$$

Then

$$\chi(Q) \leq \lim_{n,k\to\infty} \sup_{x\in Q} \|(I - P_{kl})x\|_{m^2(\Delta_v^u,\phi,p)}. \tag{2.3}$$

Let $\{L^1, L^2, \dots, L^k\}$ be a $[\chi(Q) + \varepsilon]$-net of Q. Then

$$Q \subset \{L^1, L^2, \dots, L^k\} + [\chi(Q) + \varepsilon]\mathbb{B}\big(m^2(\Delta_v^u, \phi, p)\big),$$

where $\mathbb{B}_{\left(m^2(\Delta_v^u,\phi,p)\right)}(2\overline{\theta}, 1)$ is the unit ball of $m^2(\Delta_v^u, \phi, p)$. Then

$$\sup_{x\in Q} \|(I - P_{kl})x\|_{m^2(\Delta_v^u,\phi,p)} \leq \sup_{1\leq i\leq k} \|(I - P_{kl})L^i\|_{m^2(\Delta_v^u,\phi,p)} + [\chi(Q) + \varepsilon].$$

Hence

$$\lim_{n,k\to\infty} \sup_{x\in Q} \|(I - P_{kl})x\|_{m^2(\Delta_v^u,\phi,p)} \leq \chi(Q) + \varepsilon. \tag{2.4}$$

By combining Eq. (2.3) and (2.4), we get

$$\chi(Q) = \lim_{k,l\to\infty} \Big(\sup_{x\in Q} \|(I - P_{kl})(x)\|_{m^2(\Delta_v^u,\phi,p)} \Big)$$

or

$$\chi(Q) = \lim_{k,l\to\infty} \Big\{ \sup_{x\in Q} \Big\{ \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}} \Big(\sum_{k\in\sigma_1} \sum_{l\in\sigma_2} |\Delta_v^u x_{k,l}|^p \Big)^{\frac{1}{p}} \Big\}\Big\}.$$

□

3. APPLICATION

Now, we define a new class of fractional deferential equation of order $\vartheta \in (n-1, n]$, $(n \geq 2)$ and consider the solutions in $m^2(\Delta_v^u, \phi, p)$ also, we present two examples to effectiveness of the obtained result.

Definition 3.5. ([32]) *The fractional integral of order* ϑ *is defined as*

$$I^\vartheta f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t \frac{f(\rho)}{(t-\rho)^{1-\vartheta}} d\rho,\ \vartheta > 0,$$

Definition 3.6. ([32]) *The Caputo fractional derivative of order* $\vartheta > 0$ *of function* $f : [0, \infty) \to \mathbb{R}$, *is defined by*

$${}^cD^\vartheta f(t) = \frac{1}{\Gamma(n-\vartheta)} \int_0^t \frac{f^{(n)}(\rho)}{(t-\rho)^{\vartheta-n+1}} d\rho,$$

where $n = [\vartheta] + 1$.

Lemma 3.2. *([21]) Let $u \in C([0,\infty)) \cap L^1([0,\infty))$ with the Caputo fractional derivative of order ϑ that belongs to $C([0,\infty)) \cap L^1([0,\infty))$. Then*

$$I^{\vartheta}\, {}^cD^{\vartheta}u(t) = u(t) + c_1 + c_2t + c_3t^2 + \ldots + c_nt^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\vartheta]$.

Lemma 3.3. *Let $f \in l^1([0,\infty))$ be continuous function and $n-1 < \vartheta \leq n$, $(n \geq 2)$. Then the BVP problem of fractional differential equation*

$$\begin{cases} {}^cD^{\vartheta}u(t) = f(t, u(t)),\ 0 \leq t < 1, \\ u(0) = u'(0) = u^{(4)}(0) = u^{(5)}(0) = \ldots = u^{(n)}(0) = 0, \\ u'(1) = u''(\eta),\ u''(1) = \beta \int_0^{\eta} u(\rho)d\rho,\ \beta \in \mathbb{R},\ 0 < \eta < 1 \end{cases} \tag{3.5}$$

has a unique solution

$$\begin{aligned} u(t) \quad = \quad & \int_0^t \frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} f(\rho)d\rho + \frac{3t^2}{6-\beta\eta^3}\Bigg(\beta\int_0^{\eta}\Big(\int_0^s \frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)} f(m)dm\Big)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho \\ & + \frac{\beta\eta^4 - 24}{4(3-6\eta)}\Big(\int_0^{\eta} \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(\rho)d\rho\Big)\Bigg) \\ & + \frac{t^3}{3-6\eta}\Big(\int_0^{\eta} \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(\rho)d\rho\Big). \end{aligned}$$

Proof. By Lemma 3.2, the equation (3.5) is equivalent to the integral form

$$u(t) = I^{\vartheta}f(t) + c_1 + c_2t + c_3t^2 + c_4t^3 + \ldots + c_{n+1}t^n$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, 3, 4, \ldots, n+1$.
By the boundary value conditions for (3.5), we find that

$$c_1 = c_2 = c_5 = c_6 = \ldots = c_{n+1} = 0$$

and

$$u(t) = \int_0^t \frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} f(\rho)d\rho + c_3t^2 + c_4t^3. \tag{3.6}$$

Applying, $u'(1) = u''(\eta)$ we get

$$\int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(\rho)d\rho + 2c_3 + 3c_4 = \int_0^{\eta} \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho + 2c_3 + 6c_4\eta$$

which imply that

$$(3-6\eta)c_4 = \int_0^{\eta} \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(\rho)d\rho.$$

Consequently,

$$c_4 = \frac{1}{3-6\eta}\Big(\int_0^{\eta} \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(\rho)d\rho\Big).$$

By boundary condition $u''(1) = \beta\int_0^{\eta} u(\rho)d\rho$ we have

$$\begin{aligned} \int_0^1 \frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho + 2c_3 + 6c_4 \quad &= \quad \beta\int_0^{\eta}\Big(\int_0^{\rho} \frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)} f(m)dm + c_3\rho^2 + c_4\rho^3\Big)d\rho \\ &= \quad \beta\int_0^{\eta}\Big(\int_0^{\rho} \frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)} f(m)dm\Big)d\rho + \beta c_3\frac{\eta^3}{3} + \beta c_4\frac{\eta^4}{4}. \end{aligned}$$

Then, we have

$$\Bigg(\beta\int_0^{\eta}\Big(\int_0^{\rho} \frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)} f(m)dm\Big)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)} f(\rho)d\rho\Bigg) + (\beta\frac{\eta^4}{4} - 6)c_4 = (2 - \frac{\beta\eta^3}{3})c_3$$

Consequently,

$$c_3 = \frac{3}{6-\beta\eta^3}\Bigg(\beta\int_0^\eta\Big(\int_0^\rho \frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)}f(m)dm\Big)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f(\rho)d\rho + \frac{\beta\eta^4-24}{4}\Big(\frac{1}{3-6\eta}\Big(\int_0^\eta \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f(\rho)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f(\rho)d\rho\Big)\Big)\Bigg).$$

Substituting the value of c_3 and c_4 in (3.6), it yields

$$u(t) = \int_0^t \frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}f(\rho)d\rho + \frac{3t^2}{6-\beta\eta^3}\Bigg(\beta\int_0^\eta\Big(\int_0^\rho \frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)}f(m)dm\Big)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f(\rho)d\rho + \frac{\beta\eta^4-24}{4(3-6\eta)}\Big(\int_0^\eta \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f(\rho)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f(\rho)d\rho\Big)\Bigg) + \frac{t^3}{3-6\eta}\Big(\int_0^\eta \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f(\rho)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f(\rho)d\rho\Big).$$

The proof is completed. □

Consider the following conditions.

(A_1) For $k,l\in\mathbb{N}$, $f_{kl}\in C\big(I\times\mathbb{R}^\infty,\mathbb{R}\big)$, the continuous operator $f: I\times m^2(\Delta_v^u,\phi,p)\to m^2(\Delta_v^u,\phi,p)$ is defined by $(f\omega)(t)=f(t,\omega(t))=\langle f_{kl}(t,\omega(t))\rangle$, where $I=[0,1]$ and the family of functions $\{(f\omega)(t)\}_{t\in I}$ is equicontinuous in $m^2(\Delta_v^u,\phi,p)$.

(A_2) Assume that

$$0<\Bigg[\Big|\frac{1}{\vartheta\Gamma(\vartheta)}\Big|^p+\Big|\frac{3}{6-\beta\eta^3}\Big|^p\Bigg(\Big|\frac{\beta}{\vartheta(\vartheta+1)\Gamma(\vartheta)}\Big|^p+\Big|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}\Big|^p+\Big|\frac{\beta-24}{4(3-6\eta)}\Big|^p\Big(\Big|\frac{1}{(\vartheta-2)\Gamma(\vartheta-2)}\Big|^p + \Big|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}\Big|^p\Big)\Bigg)+\Big|\frac{1}{3-6\eta}\Big|^p\Big(\Big|\frac{1}{(\vartheta-2)\Gamma(\vartheta-2)}\Big|^p+\Big|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}\Big|^p\Big)\Bigg]=A<\infty.$$

(A_3) For each $t\in I$ and $\omega\in m^2(\Delta_v^u,\phi,p)$ the following inequalities holds:

$$|\Delta_v^u f_{kl}(t,\omega(t))|^p\le|\varphi_{kl}(t)|^p|\Delta_v^u\omega_{kl}(t)|^p,$$

where $\varphi_{kl}(t)$ is real functions continuous such that $\langle\varphi_{kl}(t)\rangle$ is uniformly equibounded on I, Put

$$\sup_{t\in I}\sup_{k,l}|\varphi_{kl}(t)|^p=H.$$

Theorem 3.6. *Let $(A_1)-(A_3)$ hold, if $2^{6p}AH<1$, then (3.5) for each $t\in I$ has at least one solution $\omega=\langle\omega_{kl}(t)\rangle\in C\big(I,m^2(\Delta_v^u,\phi,p)\big)$.*

Proof. Let $\omega=\langle\omega_{kl}\rangle$ be a double sequence function which fulfils the equation (3.5) and $\forall\ k,l\in\mathbb{N}$, ω_{kl} be continuous on I. We define the operator $F: C\big(I,m^2(\Delta_v^u,\phi,p)\big)\to C\big(I,m^2(\Delta_v^u,\phi,p)\big)$ by

$$(F\omega)(t) = \int_0^t \frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(\rho,\omega(\rho))d\rho + \frac{3t^2}{6-\beta\eta^3}\Bigg(\beta\int_0^\eta\Big(\int_0^\rho \frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(m,\omega(m))dm\Big)d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho + \frac{\beta\eta^4-24}{4(3-6\eta)}\Big(\int_0^\eta \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho\Big)\Bigg) + \frac{t^3}{3-6\eta}\Big(\int_0^\eta \frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho - \int_0^1 \frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho\Big).$$

By our assumptions, we get

$$
\begin{aligned}
&\|(F\omega)(t)\|^p_{m^2(\Delta_v^u,\phi,p)}\\
= &\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\bigg(\int_0^t\frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(\rho,\omega(\rho))d\rho\\
&+\frac{3t^2}{6-\beta\eta^3}\bigg(\beta\int_0^\eta\Big(\int_0^\rho\frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(m,\omega(m))dm\Big)d\rho-\int_0^1\frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho\\
&+\frac{\beta\eta^4-24}{4(3-6\eta)}\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho-\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho\Big)\bigg)\\
&+\frac{t^3}{3-6\eta}\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho-\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho\Big)|^p\Big)\\
\leq\ &2^{6p}\bigg[\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^t\frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\\
&+|\frac{3t^2}{6-\beta\eta^3}|^p\bigg(\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\beta\int_0^\eta\Big(\int_0^\rho\frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(m,\omega(m))dm\Big)d\rho|^p\\
&+\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^1\frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\\
&+|\frac{\beta\eta^4-24}{4(3-6\eta)}|^p\Big(\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\\
&+\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\bigg)\\
&+|\frac{t^3}{3-6\eta}|^p\bigg(\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{st}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\\
&+\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\bigg)\bigg]\\
\leq\ &2^{6p}\bigg[|\frac{t^\vartheta}{\vartheta\Gamma(\vartheta)}|^p|\varphi_{kl}(t)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)\\
&+|\frac{3t^2}{6-\beta\eta^3}|^p\bigg(|\varphi_{kl}(t)|^p|\frac{\beta\eta^{\vartheta+1}}{\vartheta(\vartheta+1)\Gamma(\vartheta)}|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)\\
&+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(t)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)\\
&+|\frac{\beta\eta^4-24}{4(3-6\eta)}|^p\bigg(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p|\varphi_{kl}(t)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)\\
&+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(t)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)\bigg)\bigg)\\
&+|\frac{t^3}{3-6\eta}|^p\bigg(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p|\varphi_{kl}(t)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)\\
&+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(t)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)\bigg)\bigg]\\
\leq\ &\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(t)|^p\Big)H2^{6p}\bigg[|\frac{1}{\vartheta\Gamma(\vartheta)}|^p+|\frac{3}{6-\beta\eta^3}|^p\bigg(|\frac{\beta}{\vartheta(\vartheta+1)\Gamma(\vartheta)}|^p\\
&+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p+|\frac{\beta-24}{4(3-6\eta)}|^p\Big(|\frac{1}{(\vartheta-2)\Gamma(\vartheta-2)}|^p+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p\Big)\bigg)\\
&+|\frac{1}{3-6\eta}|^p\Big(|\frac{1}{(\vartheta-2)\Gamma(\vartheta-2)}|^p+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p\Big)\bigg].
\end{aligned}
$$

Also, $C\big(I, m^2(\Delta_v^u, \phi, p)\big)$ is equipped the classical norm

$$\|\omega\|_{C\big(I,m^2(\Delta_v^u,\phi,p)\big)} = \sup_{t\in I} \|\omega(t)\|_{m^2(\Delta_v^u,\phi,p)}.$$

So, we have

$$\|(F\omega)\|^p_{C\big(I,m^2(\Delta_v^u,\phi,p)\big)} \leq (HA2^{6p}\|\omega\|_{C\big(I,m^2(\Delta_v^u,\phi,p)\big)})^p.$$

Then

$$r \leq 2^{6p}HAr. \tag{3.7}$$

Let r_0 be the optimal solution of (3.7). Now, we consider the set

$$B_{m^2(\Delta_v^u,\phi,p)} = B_{m^2(\Delta_v^u,\phi,p)}(\omega, r_0) = \{\omega \in C\big(I, m^2(\Delta_v^u, \phi, p)\big) : \|\omega\|_{C\big(I,m^2(\Delta_v^u,\phi,p)\big)} \leq r_0\}.$$

Clearly, $B_{m^2(\Delta_v^u,\phi,p)} = \overline{B}_{m^2(\Delta_v^u,\phi,p)}$ is convex and bounded, and F is bounded on B. Let $\varepsilon > 0$ by (A_1), $\exists$, $\delta > 0$ so that if $v \in B_{m^2(\Delta_v^u,\phi,p)}$ and $\|\omega - v\|_{C\big(I,m^2(\Delta_v^u,\phi,p)\big)} \leq \delta$, then $\|f\omega - fv\|^p_{C\big(I,m^2(\Delta_v^u,\phi,p)\big)} \leq \frac{\varepsilon^p}{2^{6p}A}$. Thus, for each $t \in I$, We get

$$\|(F\omega)(t) - (Fv)(t)\|^p_{m^2(\Delta_v^u,\phi,p)}$$

$$\begin{aligned}
\leq\ & 2^{6p}\Bigg[|\frac{t^\vartheta}{\vartheta\Gamma(\vartheta)}|^p \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big) \\
& +|\frac{3t^2}{6-\beta\eta^3}|^p\Bigg(|\frac{\beta\eta^{\vartheta+1}}{\vartheta(\vartheta+1)\Gamma(\vartheta)}|^p \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big) \\
& +|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big) \\
& +|\frac{\beta\eta^4-24}{4(3-6\eta)}|^p\Bigg(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big) \\
& +|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big)\Bigg)\Bigg) \\
& +|\frac{t^3}{3-6\eta}|^p\Bigg(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big) \\
& +|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big)\Bigg)\Bigg] \\
\leq\ & \sup_{(s,e)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\phi_{se}} \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \big|\Delta_v^u(f_{kl}(\rho,\omega(\rho)) - f_{kl}(\rho, v(\rho)))\big|^p\Big)2^{6p}A.
\end{aligned}$$

Then

$$\|(F\omega) - (Fv)\|_{C\big(I,m^2(\Delta_v^u,\phi,p)\big)} \leq \varepsilon.$$

So, F is continuous. Let $t_0 \in I$, $\varepsilon > 0$, $t > t_0$ so that if $|t - t_0| < \varepsilon$, then we can write

$$
\begin{aligned}
&\|(F\omega)(t)-(F\omega)(t_0)\|^p_{m^2(\Delta_v^u,\phi,p)}\\
\leq\ & 2^{6p}\Bigg[\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^t\frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(\rho,\omega(\rho))d\rho\\
&-\int_0^{t_0}\frac{(t_0-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\\
&+|\frac{3(t^2-t_0^2)}{6-\beta\eta^3}|^p\Bigg(\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\beta\int_0^\eta\Big(\int_0^\rho\frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(m,\omega(m))dm\Big)d\rho|^p\\
&+\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^1\frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\\
&+|\frac{\beta\eta^4-24}{4(3-6\eta)}|^p\Big(\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\\
&+\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{st}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\Bigg)\\
&+|\frac{t^3-t_0^3}{3-6\eta}|^p\Big(\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\\
&+\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\Big)\Bigg]\\
\leq\ & 2^{6p}\Bigg[|\frac{t^\vartheta-t_0^\vartheta}{\vartheta\Gamma(\vartheta)}|^p|\varphi_{kl}(\rho)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big)\\
&+|\frac{3(t-t_0)(t+t_0)}{6-\beta\eta^3}|^p\Bigg(|\varphi_{kl}(\rho)|^p|\frac{\beta\eta^{\vartheta+1}}{\vartheta(\vartheta+1)\Gamma(\vartheta)}|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u u_{kl}(\rho)|^p\Big)\\
&+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{st}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u u_{kl}(\rho)|^p\Big)\\
&+|\frac{\beta\eta^4-24}{4(3-6\eta)}|^p\Big(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p|\varphi_{kl}(\rho)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u u_{kl}(\rho)|^p\Big)\\
&+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(\rho)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u u_{kl}(\rho)|^p\Big)\Big)\Bigg)\\
&+|\frac{(t-t_0)(t^2+tt_0+t_0^2)}{3-6\eta}|^p\Big(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p|\varphi_{kl}(\rho)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u u_{kl}(\rho)|^p\Big)\\
&+|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(\rho)|^p\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u u_{kl}(\rho)|^p\Big)\Big)\Bigg].
\end{aligned}
$$

Since $|t-t_0|<\varepsilon$ and $n-1<\vartheta\leq n$ so $|t^\vartheta-t_0^\vartheta|^p<\varepsilon$. Then $(F\omega)$ is continuous for each $t\in I$. Finally, we show that F is a Meir–Keeler condensing operator w.r.t. the Hausdorff MNC χ on the space $C\big(I,m^2(\Delta_v^u,\phi,p)\big)$. By Theorem 2.5 and Lemma 1.1 we concluded that

$$
(\mathcal{X}_{C\left(I,m^2(\Delta_v^u,\phi,p)\right)}(B_{m^2(\Delta_v^u,\phi,p)}))^p=\sup_{t\in I}(\mathcal{X}_{m^2(\Delta_v^u,\phi,p)}(B_{m^2(\Delta_v^u,\phi,p)}(t)))^p. \tag{3.8}
$$

Again by using (2.1), Lemma 1.1 and our assumption, we have

$$
\begin{aligned}
&(\mathcal{X}_{m^2(\Delta_v^u,\phi,p)}[(FB_{m^2(\Delta_v^u,\phi,p)})(t)])^p \\
&= \lim_{k,l\to\infty}\Big\{\sup_{u\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}} \\
&\quad \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\Big(\int_0^t\frac{(t-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(\rho,\omega(\rho))d\rho \\
&\quad +\frac{3t^2}{6-\beta\eta^3}\Big(\beta\int_0^\eta\Big(\int_0^s\frac{(\rho-m)^{\vartheta-1}}{\Gamma(\vartheta)}f_{kl}(m,\omega(m))dm\Big)d\rho-\int_0^1\frac{(1-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho \\
&\quad +\frac{\beta\eta^4-24}{4(3-6\eta)}\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho-\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho\Big)\Big) \\
&\quad +\frac{t^3}{3-6\eta}\Big(\Big(\int_0^\eta\frac{(\eta-\rho)^{\vartheta-3}}{\Gamma(\vartheta-2)}f_{kl}(\rho,\omega(\rho))d\rho-\int_0^1\frac{(1-\rho)^{\vartheta-2}}{\Gamma(\vartheta-1)}f_{kl}(\rho,\omega(\rho))d\rho|^p\Big)\Big)\Big\}\Big\} \\
&\leq 2^{6p}\Big[|\frac{t^\vartheta}{\vartheta\Gamma(\vartheta)}|^p|\varphi_{kl}(\rho)|^p\lim_{k,l\to\infty}\Big\{\sup_{u\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big)\Big\}\Big\} \\
&\quad +|\frac{3t^2}{6-\beta\eta^3}|^p\Big(|\varphi_{kl}(\rho)|^p|\frac{\beta\eta^{\vartheta+1}}{\vartheta(\vartheta+1)\Gamma(\vartheta)}|^p\lim_{k,l\to\infty}\Big\{\sup_{\omega\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}} \\
&\quad \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u u_{kl}(\rho)|^p\Big)\Big\}\Big\} \\
&\quad +|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(\rho)|^p\lim_{k,l\to\infty}\Big\{\sup_{\omega\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big)\Big\}\Big\} \\
&\quad +|\frac{\beta\eta^4-24}{4(3-6\eta)}|^p\Big(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p|\varphi_{kl}(\rho)|^p\lim_{k,l\to\infty}\Big\{\sup_{\omega\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}} \\
&\quad \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big)\Big\}\Big\} \\
&\quad +|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(\rho)|^p\lim_{k,l\to\infty}\Big\{\sup_{\omega\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big)\Big\}\Big\}\Big) \\
&\quad +|\frac{t^3}{3-6\eta}|^p\Big(|\frac{\eta^{\vartheta-2}}{(\vartheta-2)\Gamma(\vartheta-2)}|^p|\varphi_{kl}(\rho)|^p\lim_{k,l\to\infty}\Big\{\sup_{\omega\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}} \\
&\quad \frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big\}\Big) \\
&\quad +|\frac{1}{(\vartheta-1)\Gamma(\vartheta-1)}|^p|\varphi_{kl}(s)|^p\lim_{k,l\to\infty}\Big\{\sup_{\omega\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big)\Big\}\Big)\Big] \\
&\leq \lim_{k,l\to\infty}\Big\{\sup_{\omega\in m^2(\Delta_v^u,\phi,p)}\Big\{\sup_{(s,e)\geq(1,1)}\sup_{\sigma_1\times\sigma_2\in\phi_{se}}\frac{1}{\phi_{se}}\Big(\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}|\Delta_v^u\omega_{kl}(\rho)|^p\Big)\Big\}\Big\}H2^{6p}A.
\end{aligned}
$$

Therefore, we deduce

$$\sup_{t\in I}(\mathcal{X}_{m^2(\Delta_v^u,\phi,p)}[(FB_{m^2(\Delta_v^u,\phi,p)})(t)])^p\leq H2^{6p}A(\mathcal{X}_{C\left(I,m^2(\Delta_v^u,\phi,p)\right)}(B_{m^2(\Delta_v^u,\phi,p)}))^p.$$

By (3.8), we have

$$(\mathcal{X}_{C\left(I,m^2(\Delta_v^u,\phi,p)\right)}(FB_{m^2(\Delta_v^u,\phi,p)}))^p\leq H2^{6p}A(\mathcal{X}_{C\left(I,m^2(\Delta_v^u,\phi,p)\right)}(B_{m^2(\Delta_v^u,\phi,p)}))^p.$$

Hence

$$(\mathcal{X}_{C\left(I,m^2(\Delta_v^u,\phi,p)\right)}(B_{m^2(\Delta_v^u,\phi,p)}))<\frac{\varepsilon}{(H2^{6p}A)^{\frac{1}{p}}}.$$

Take $\delta=\varepsilon(\frac{1}{(H2^{6p}A)^{\frac{1}{p}}}-1)$, we get F is a Meir-–Keeler condensing operator on $B_{m^2(\Delta_v^u,\phi,p)}\subset m^2(\Delta_v^u,\phi,p)$. Thus, Theorem 1.1 grantees that F has a fixed point in $B_{m^2(\Delta_v^u,\phi,p)}$, so (3.5) has at least one solution in $C\big(I,m^2(\Delta_v^u,\phi,p)\big)$. □

Example 3.1. *Consider the equations*

$$\begin{cases} {}^cD^{\frac{11}{2}}u(t)=f(t,u(t)),\ 0\le t<1,\\ u(0)=u'(0)=u^{(4)}(0)=u^{(5)}(0)=0,\\ u'(1)=u''(\frac{1}{3}),\ u''(1)=0.001\int_0^{\frac{1}{3}}u(\rho)d\rho. \end{cases} \tag{3.9}$$

The Eq. (3.9) is a special case of the Eq. (3.5) when $f(t,\omega(t))=\langle f_{kl}(t,\omega(t))\rangle$,

$$f_{kl}(t,\omega(t))=\sum_{k=a,l=b}\left(\big(\ln(\frac{k(k+2)}{(k+1)^2})\big)\frac{\sin(\omega_{kl}(t))\cos(t^4+2t^2)}{2^6e^t}\right),$$

Clearly, $\langle f_{kl}(t,\omega(t))\rangle$ *are continuous and the family of functions* $\{(f\omega)(t)\}_{t\in I}$ *is equicontinuous. Let* $\varepsilon>0$ *and* $\langle\omega_{kl}(t)\rangle\in m^2(\Delta_v^u,\phi,p)$. *Thus, by taking* $v=\langle v_{kl}(t)\rangle\in m^2(\Delta_v^u,\phi,p)$ *such that* $\|\omega(t)-v(t)\|_{m^2(\Delta_v^u,\phi,p)}\le\delta=\frac{\varepsilon 2^6}{|\ln\frac{1}{2}|}$, *then*

$$\|f(t,\omega(t))-f(t,u(t))\|_{m^2(\Delta_v^u,\phi,p)}\le\frac{|\ln\frac{1}{2}|}{2^6}\|\omega(t)-v(t)\|_{m^2(\Delta_v^u,\phi,p)}=\varepsilon.$$

Then the conditions (A_1) *and* (A_2) *hold. To this end, take*

$$\varphi_{kl}(t)=\sum_{k=a,l=b}\big(\ln(\frac{k(k+2)}{(k+1)^2})\big)\frac{\cos(t^4+2t^2)}{2^6}$$

is continuous such that $\langle\varphi_{kl}(t)\rangle$ *is equibounded on* I, *and we have* $H=|\ln(\frac{1}{2})|^p\frac{1}{2^{6p}}$, *then the hypothesis* (A_3) *of Th 3.6 holds. Indeed, for any* $t\in I$ *if* $\langle\omega_{kl}(t)\rangle\in m^2(\Delta_v^u,\phi,p)$, *then* $\langle\triangle f_{kl}(t,\omega(t))\rangle\in m^2(\Delta_v^u,\phi,p)$ *and* $f(t,\omega(t))=\langle f_{kl}(t,\omega(t))\rangle\in m^2(\Delta_v^u,\phi,p)$, *we have*

$$\begin{aligned} |\Delta_v^u f_{kl}(t,\omega(t))|^p &= |\Delta_v^u\sum_{k=a,l=b}\left(\big(\ln(\frac{k(k+2)}{(k+1)^2})\big)\frac{\sin(\omega_{kl}(t))\cos(t^4+2t^2)}{2^6e^t}\right)|^p\\ &\le |\sum_{k=a,l=b}\left(\big(\ln(\frac{k(k+2)}{(k+1)^2})\big)\frac{\cos(t^4+2t^2)}{2^6}\right)|^p|\Delta_v^u\omega_{kl}(t))|^p \end{aligned}$$

Moreover, $2^{6p}AH<1$. *Now, Th 3.6 grantees that infinite system (3.9) has at least one solution in* $C(I,m^2(\Delta_v^u,\phi,p))$.

Example 3.2. *Consider the fractional differential equations*

$$\begin{cases} {}^cD^{\frac{9}{2}}u(t)=f(t,u(t)),\ 0\le t<1,\\ u(0)=u'(0)=u^{(4)}(0)=0,\\ u'(1)=u''(\frac{2}{7}),\ u''(1)=0.006\int_0^{\frac{2}{7}}u(\rho)d\rho. \end{cases} \tag{3.10}$$

Observe that the Eq. (3.10) is a particular case of the Eq. (3.5) when $f(t,\omega(t))=\langle f_{kl}(t,\omega(t))\rangle$,

$$f_{kl}(t,\omega(t))=\sum_{k=a,l=b}\left((\frac{1}{k(k+1)(k+2)})\frac{\arctan(\omega_{kl}(t))\sin(5t^6+4t^4)}{3^8e^{3t}}\right),$$

Clearly, $\langle f_{kl}(t,\omega(t))\rangle$ *are continuous and the family of functions* $\{(f\omega)(t)\}_{t\in I}$ *is equicontinuous. Let* $\varepsilon>0$ *and* $\langle\omega_{kl}(t)\rangle\in m^2(\Delta_v^u,\phi,p)$. *Thus, by taking* $v=\langle v_{kl}(t)\rangle\in m^2(\Delta_v^u,\phi,p)$ *such that* $\|\omega(t)-v(t)\|_{m^2(\Delta_v^u,\phi,p)}\le\delta=\frac{\varepsilon 3^8}{\frac{1}{4}}$, *then*

$$\|f(t,\omega(t))-f(t,u(t))\|_{m^2(\Delta_v^u,\phi,p)}\le\frac{\frac{1}{4}}{3^8}\|\omega(t)-v(t)\|_{m^2(\Delta_v^u,\phi,p)}=\varepsilon.$$

Then the conditions (A_1) *and* (A_2) *hold. To this end, take*

$$\varphi_{kl}(t)=\sum_{k=a,l=b}(\frac{1}{k(k+1)(k+2)})\frac{\sin(5t^6+4t^4)}{3^8},$$

is continuous such that $\langle\varphi_{kl}(t)\rangle$ *is equibounded on* I*, and we have* $H = (\frac{1}{4})^p\frac{1}{3^{8p}}$*, then the hypothesis* (A_3) *of Th 3.6 holds. Indeed, for any* $t \in I$ *if* $\langle\omega_{kl}(t)\rangle \in m^2(\Delta_v^u, \phi, p)$*, then* $\langle\triangle f_{kl}(t,\omega(t))\rangle \in m^2(\Delta_v^u, \phi, p)$ *and* $f(t,\omega(t)) = \langle f_{kl}(t,\omega(t))\rangle \in m^2(\Delta_v^u, \phi, p)$*, we have*

$$\begin{aligned} |\Delta_v^u f_{kl}(t,\omega(t))|^p &= |\Delta_v^u \sum_{k=a,l=b}\left((\frac{1}{k(k+1)(k+2)})\frac{\arctan(\omega_{kl}(t))\sin(5t^6+4t^4)}{3^8e^{3t}}\right)|^p \\ &\leq |\sum_{k=a,l=b}\left((\frac{1}{k(k+1)(k+2)})\frac{\sin(5t^6+4t^4)}{3^8}\right)|^p|\Delta_v^u\omega_{kl}(t))|^p \end{aligned}$$

Moreover, $3^{8p}AH < 1$*. Now, Th 3.6 grantees that infinite system (3.10) has at least one solution in* $C(I, m^2(\Delta_v^u, \phi, p))$*.*

4. Conclusion

First, we defined a new fractional differential equation of order $\vartheta \in (n-1, n]$, $(n \geq 2)$. Vakeel, A.K. [34] constructed the sequence space $m(\Delta_v^u, \phi, p)$. In this work, we have defined the Banach double sequence space $m^2(\Delta_v^u, \phi, p)$ and constructed the Hausdorff MNC on this space. By using this Hausdorff MNC, we studied the existence of solutions of fractional differential equation in the double sequence space $m^2(\Delta_v^u, \phi, p)$. Further, we constructed two examples to support the usefulness of the obtained main results.

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