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Hybrid CG-Like Algorithm for Nonlinear Equations and Image Restoration

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ABSTRACT. This paper introduces a hybrid spectral-conjugate gradient (SCG) method to solve nonlinear monotone operator equations efficiently. The proposed method incorporates a hybrid parameter that encompasses the Polak–Ribière–Polyak (PRP), Liu-Storey (LS), Fletcher-Reeves (FR), and conjugate descent (CD) methods as particular instances. Additionally, we derive the spectral parameter to ensure that the search direction adheres to the sufficient descent condition. The search direction is also designed to be bounded, and under specific conditions, we demonstrate that the sequence produced by our hybrid SCG algorithm converges toward a solution. Furthermore, to underscore the effectiveness of our proposed method, we conducted extensive numerical experiments comparing its performance against that of existing algorithms. These experiments were based on a selection of benchmark nonlinear monotone operator equations, highlighting our proposed algorithm's superior efficiency and potential in practical applications.

1. INTRODUCTION

Let $\mathcal{A} \subseteq \mathbf{R}^n$ be a nonempty, closed and convex and $H : \mathcal{A} \to \mathbf{R}^n$ be monotone and Lipschitz continuous operator. This work considers the problem of searching a point $y \in \mathcal{A}$ such that

$$H(y) = 0. \tag{1.1}$$

Many real-world applications such as the economic equilibrium problems [16], the chemical equilibrium systems [27] and compressive sensing [34] can be modelled in the form of (1.1), this has led to an increasing interest of researchers in studying methods for solving (1.1). Given a good starting point, numerical methods such as Newton's and Quasi-Newton's methods and their variants are quite interesting. However, due to the need for computation of Jacobian of the underlying operator or approximation of it, these methods are not suitable for handling large-scale nonlinear problems see [13, 32, 29, 14] for an overview of these methods.

Following the famous projection method of Solodov and Svaiter [30], the Conjugate Gradient (CG) method for solving problems (1.1) is proposed as one of the first-order optimization method, which is known for its simplicity and low storage requirements. Several CG methods have been introduced, studied, and extended over the years for solving (1.1), as discussed in [1, 6, 5, 19, 21, 20] and references therein.

Generally, a CG method for solving (1.1) generates an iterative sequence $\{y_k\}$, defined by, for $k = 0, 1, 2, \dots$

$$y_{k+1} = y_k + \alpha_k d_k, \tag{1.2}$$

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where $\alpha_k > 0$ is a line search and d_k is a direction given by

$$d_k = \begin{cases} -H_k & \text{if } k = 0, \\ -H_k + \beta_k d_{k-1} & \text{if } k \geq 0, \end{cases} \tag{1.3}$$

where H_k is the function evaluation of H at y_k and β_k is a scalar called the CG parameter. Different choices of the parameter β_k correspond to different CG methods. Several Hybrid methods have been introduced recently as a combination of CG methods to take advantage of some vital features of each of the combined methods, for example Mtagulwa in [26] proposed a hybrid method consisting of Polak–Ribière–Polyak (PRP) and Fletcher-Reeves (FR). Under some suitable conditions, it was shown that their method not only possesses the good computational effect as the PRP method but also holds all the nice convergence properties of the FR method. Another hybrid method given as a convex combination of Liu-Storey (LS) and FR methods was considered in [18]; the proposed method utilised the strong convergence property of the FR method and the good numerical performance of the LS method. The search direction was shown to satisfy the descent condition under some suitable conditions.

Moreover, inspired by the work of Andrei [10], the fact that the LS method usually performs better in practice than Dia-Liao (DY) method, and on the other hand, the DY method has stronger convergence property than the LS method. Liu [24] established a more efficient and robust hybrid CG method by utilising the advantages of LS and DY methods for solving unconstrained optimization problems with suitable conditions. The parameter β_k in the proposed method is computed as a convex combination of the β_k parameters of LS and DY methods. Another kind of hybrid combination is the three-term method known for its good theoretic properties which has been introduced and studied in the literature. Notably, Amini et al. [9] proposed a modified HS method for solving unconstrained optimization with a new direction that satisfies descent condition independent of the choice of line search. It has been shown that the proposed method inherits the good theoretical properties of the three terms combinations and in particular, the numerical efficiency of Hestens-Stiefel (HS).

In the same vein, a three-term spectral conjugate gradient (CG) method based on the conjugate descent (CD) CG parameter was proposed in [4], where two directions were obtained by adding a term to the CD direction. Under some assumptions, it has been shown that sufficient descent property is satisfied. Additionally, Narushima *et al.* [28] introduced a generalised three-term CG methods type. Like [9], it has been proved that the sufficient descent condition is satisfied independent of the choice of β_k parameter and line search. Motivated and inspired by the strong convergence properties of FR and conjugate descent (CD) methods together with the good numerical performance of PRP and LS method, this article to the best of our knowledge proposes for the first a generalised CG-method for (1.1). The proposed method uses the PRP, LS, FR, and CG methods in particular cases. Additionally, using some benchmark test problems, the numerical performance of the proposed method in comparison with some existing CG methods in the literature is presented.

The paper's outline is given as follows: in section 2 we present some preliminaries required for the formulation of the proposed method, and we present the convergence analysis of the hybrid method in Section 4. In the last section, we report the proposed method's numerical experiments compared with some existing methods in the literature.

Notation. Unless otherwise stated, the symbol $\|\cdot\|$ stands for Euclidean norm on $\mathbf{R}^n$. $H(y_k)$ is abbreviated to H_k. Furthermore, $P_{\mathcal{A}}[\cdot]$ is the projection mapping from $\mathbf{R}^n$ onto

$\mathcal{A}$ given by $P_{\mathcal{A}}[y] = \arg\min\{\|y - z\| \ : y \in \mathbf{R}^n, \ z \in \mathcal{A}\}$, for a nonempty closed and convex set $\mathcal{A} \in \mathbf{R}^n$.

2. Hybrid PRP-LS-FR-CD

This section will propose a new hybrid three-term spectral-conjugate gradient algorithm for finding approximate solutions to problem (1.1). The search direction comprises two parameters, namely, the hybrid conjugate gradient (CG) parameter denoted by β_k^{PLFC} and the parameter denoted by θ_k^{PLFC} derived to ensure the search direction satisfies the sufficient descent condition. The parameter β_k^{PLFC} combines four well-known CG parameters. It is defined as

$$\beta_k^{PLFC} := \frac{\omega_1 H_k^T z_{k-1} + \omega_2 \|H_k\|^2}{\omega_3 \|H_{k-1}\| - \omega_4 d_{k-1}^T H_{k-1}}, \tag{2.4}$$

where $z_{k-1} = H_k - H_{k-1}$. Note that,

- if $\omega_1 = \omega_3 = 1$ and $\omega_2 = \omega_4 = 0$, then $\beta_k^{PLFC} = \beta_k^{PRP}$.
- if $\omega_1 = c_4 = 1$ and $\omega_2 = \omega_3 = 0$, then $\beta_k^{PLFC} = \beta_k^{LS}$.
- if $\omega_2 = \omega_3 = 1$ and $\omega_1 = \omega_4 = 0$, then $\beta_k^{PLFC} = \beta_k^{FR}$.
- if $\omega_2 = \omega_4 = 1$ and $\omega_1 = \omega_3 = 0$, then $\beta_k^{PLFC} = \beta_k^{CD}$.

Next, we propose the new hybrid three-term direction as

$$d_0 = -H_0, \ d_k := d_k^{PLFC} := -H_k + \beta_k^{PLFC} s_{k-1} - \theta_k^{PLFC} H_k, \ k \geq 1, \tag{2.5}$$

where $s_k = \alpha_k d_k$, β_k^{PLFC} is given by (2.4) and θ_k^{PLFC} is defined in such a way that

$$H_k^T d_k = -\|H_k\|^2, \tag{2.6}$$

holds.

It is easy to see that for $k = 0$, (2.6) is satisfied. As for $k \geq 1$, multiplying both sides of (2.5) by H_k^T, we have

$$H_k^T d_k = -\left(1 - \beta_k^{PLFC} \frac{H_k^T s_{k-1}}{\|H_k\|^2} + \theta_k^{PLFC}\right) \|H_k\|^2.$$

For (2.6) to be satisfied, we require

$$\theta_k^{PLFC} = \beta_k^{PLFC} \frac{H_k^T s_{k-1}}{\|H_k\|^2}. \tag{2.7}$$

In what follows, we denote the solution set of (1.1) by $Sol(\mathcal{A}, H)$ and assume that:

Assumption 2.1. *$Sol(\mathcal{A}, H)$ is nonempty.*

Assumption 2.2. *The operator H is monotone. That is for any $y_1, y_2 \in \mathbf{R}^n$,*

$$(H(y_1) - H(y_2))^T (y_1 - y_2) \geq 0. \tag{2.8}$$

Assumption 2.3. *The operator H is L-Lipschitz continuous on $\mathbb{R}^n$. That is for any $y_1, y_2 \in \mathbf{R}^n$, $L > 0$,*

$$\|H(y_1) - H(y_2)\| \leq L\|y_1 - y_2\|. \tag{2.9}$$

Next, we present the algorithm for the method we proposed above.

Algorithm 1: HYBRIDSCG

Input. Choose an initial guess $y_0 \in \mathcal{A}, \omega_1,\ \omega_2,\ \omega_3,\ \omega_4,\ t > 0, 0 < \theta < 2, 0 < \mu < 1,$ $\sigma > 0$, $tol > 0$ and $k := 0$.

Step 1. If $\|H_k\| \leq tol$, terminate. Else move to **Step 2**.

Step 2. Compute d_k using (2.5).

Step 3. Compute

$$\gamma_k = y_k + \alpha_k d_k, \tag{2.10}$$

$\alpha_k = t\mu^i$, for $i = 0, 1, \cdots$, where i is the least nonnegative integer satisfying

$$-H(\gamma_k)^T d_k \geq \sigma\alpha_k\|d_k\|^2. \tag{2.11}$$

Step 4. If $\gamma_k \in \mathcal{A}$ and $\|H(\gamma_k)\| \leq tol$, stop. Else, compute

$$y_{k+1} := P_{\mathcal{A}}[y_k - \theta\varphi_k H(\gamma_k)], \tag{2.12}$$

where

$$\varphi_k := \frac{H(\gamma_k)^T(y_k - \gamma_k)}{\|H(\gamma_k)\|^2}. \tag{2.13}$$

Step 5. Let $k = k + 1$ and repeat from **Step 1**.

3. Theoretical results

In this section, we will need the following results in order to establish the sequence of iterates generated by Algorithm 1 converges to a solution of (1.1).

Lemma 3.1. *The search direction defined by* (2.5) *satisfies the sufficient descent property* (2.6).

Proof. If $k = 0$,

$$H_0^T d_0 = -\|H_0\|^2.$$

If $k \geq 1$, utilizing (2.4), (2.5) and (2.7), we get

$$\begin{aligned} H_k^T d_k &= -\|H_k\|^2 + \beta_k^{PLFC} H_k^T s_{k-1} - \theta_k^{PLFC}\|H_k\|^2 \\ &= -\|H_k\|^2 + \beta_k^{PLFC} H_k^T s_{k-1} - \beta_k^{PLFC}\frac{H_k^T s_{k-1}}{\|H_k\|^2}\|H_k\|^2 \\ &= -\|H_k\|^2. \end{aligned}$$

Therefore, (2.6) is satisfied. □

Lemma 3.2. *Suppose Assumption 2.1, Assumption 2.2 and Assumption 2.3 are satisfied. If $\{d_k\}$, $\{\gamma_k\}$ and $\{y_k\}$ are sequences defined by* (2.5)*, (2.10) and (2.12), respectively, then*
(i) for all k, there is $\alpha_k = t\mu^i$ satisfying (2.11) for some $i \in \mathbb{N} \cup \{0\}$ and $\forall k \geq 0$.
(ii) α_k obtained via (2.11) *satisfy*

$$\alpha_k > \frac{\mu\|H_k\|^2}{(L + \sigma)\|d_k\|^2}. \tag{3.14}$$

Proof. (i) Suppose on the contrary there exists $k_0 \geq 0$ such that (2.11) does not hold for any non-negative integer i, i.e.,

$$-H(y_{k_0} + t\mu^i d_{k_0})^T d_{k_0} < \sigma t\mu^i\|d_{k_0}\|^2.$$

By assumption 2.3 and allowing $i \to \infty$, we get

$$-H(y_{k_0})^T d_{k_0} \leq 0. \tag{3.15}$$

On the other hand, from (2.6),

$$-H(y_{k_0})^T d_{k_0} = \|H(y_{k_0})\|^2 > 0,$$

which contradicts (3.15). Hence, the step size is well defined. (ii) If $\alpha_k \neq t$, then $\alpha_k^{'} = \frac{\alpha_k}{\mu}$ does not satisfy (2.11), that is

$$-H(y_k + \alpha_k^{'} d_k)^T d_k < \sigma \alpha_k^{'} \|d_k\|^2.$$

Using (2.6) and assumption 2.3, then

$$\begin{aligned} \|H_k\|^2 &= -H_k^T d_k \\ &= (H(y_k + \alpha_k^{'} d_k) - H_k)^T d_k - H(y_k + \alpha_k^{'} d_k)^T d_k \\ &< L\alpha_k^{'} \|d_k\|^2 + \sigma \alpha_k^{'} \|d_k\|^2 \\ &= (L + \sigma)\alpha_k \mu^{-1} \|d_k\|^2. \end{aligned}$$

Hence,

$$\alpha_k > \frac{\mu \|H_k\|^2}{(L+\sigma)\|d_k\|^2}.$$

□

Lemma 3.3. [2] *If Aassumption 2.1, 2.2 and 2.3 are satisfied, then the sequences $\{\gamma_k\}$ and $\{y_k\}$ defined by (2.10) and (2.12) in Algorithm 1 are bounded. In addition,*

$$\lim_{k\to\infty} \alpha_k \|d_k\| = 0. \tag{3.16}$$

Lemma 3.4. *Let $\{y_k\}$ be the sequence generated by Algorithm 2 under assumption 2.1, 2.2 and 2.3, it holds that*

$$\|y_{k+1} - \bar{y}\|^2 \leq \|y_k - \bar{y}\|^2. \tag{3.17}$$

Proof. The proof follows from [2, Lemma 4]. □

Remark 3.1. *Since $\{y_k\}$ is bounded from Lemma 3.3 and F is continuous from Assumption 2.3, $\{H_k\}$ is also bounded. That is, there exist $c_1,\ c_2 > 0$ such that for all k*

$$\|y_k\| \leq c_1,\ \|H_k\| \leq c_2. \tag{3.18}$$

Theorem 3.4. *Suppose Assumption 2.1, 2.2 and 2.3 are satisfied. If $\{y_k\}$ is a sequence defined by* (2.12)*, then,*

$$\liminf_{k\to\infty} \|H_k\| = 0. \tag{3.19}$$

Furthermore, the sequence $\{y_k\}$ converges to a solution of problem (1.1).

Proof. Suppose $\liminf\limits_{k\to\infty} \|H_k\| \neq 0$, then, there exists $c_3 > 0$ such that for all $k \geq 0$

$$\|H_k\| \geq c_3. \tag{3.20}$$

Next, we will show that d_k defined by (2.5) is bounded.
For $k = 0$,

$$\|d_0\| = \|H_0\| \leq c_4.$$

Now for $k \geq 1$, using (2.4),

$$
\begin{aligned}
|\beta_k^{PLFC}| &= \left| \frac{\omega_1 H_k^T z_{k-1} + \omega_2 \|H_k\|^2}{\omega_3 \|H_{k-1}\| - \omega_4 d_{k-1}^T H_{k-1}} \right| \\
&\leq \frac{\omega_1 \|H_k\| \|z_{k-1}\| + \omega_2 \|H_k\|^2}{-\omega_4 d_{k-1}^T H_{k-1}} \\
&= \frac{\omega_1 \|H_k\| \|z_{k-1}\| + \omega_2 \|H_k\|^2}{\omega_4 \|H_{k-1}\|^2}. \qquad (3.21)
\end{aligned}
$$

Also using (2.7) and (3.21),

$$
\begin{aligned}
|\theta_k^{PLFC}| &= \left| \beta_k^{PLFC} \frac{H_k^T s_{k-1}}{\|H_k\|^2} \right| \\
&\leq \left(\frac{\omega_1 \|H_k\| \|z_{k-1}\| + \omega_2 \|H_k\|^2}{\omega_4 \|H_{k-1}\|^2} \right) \frac{\|H_k\| \|s_{k-1}\|}{\|H_k\|^2} \\
&= \left(\frac{\omega_1 \|H_k\| \|z_{k-1}\| + \omega_2 \|H_k\|^2}{\omega_4 \|H_{k-1}\|^2 \|H_k\|} \right) \|s_{k-1}\| \\
&= \left(\frac{\omega_1 \|z_{k-1}\| + \omega_2 \|H_k\|}{\omega_4 \|H_{k-1}\|^2} \right) \|s_{k-1}\|. \qquad (3.22)
\end{aligned}
$$

So, using (2.5), (3.18), (3.20) (3.21), (3.22) and assumption 2.2,

$$
\begin{aligned}
\|d_k\| &= \left\| -H_k + \beta_k^{PLFC} s_{k-1} - \theta_k^{PLFC} H_k \right\| \\
&\leq \|H_k\| + |\beta_k^{PLFC}| \|s_{k-1}\| + |\theta_k^{PLFC}| \|H_k\| \\
&\leq \|H_k\| + \left(\frac{\omega_1 \|H_k\| \|z_{k-1}\| + \omega_2 \|H_k\|^2}{\omega_4 \|H_{k-1}\|^2} \right) \|s_{k-1}\| + \left(\frac{\omega_1 \|H_k\| \|z_{k-1}\| + \omega_2 \|H_k\|^2}{\omega_4 \|H_{k-1}\|^2} \right) \|s_{k-1}\| \\
&\leq \|H_k\| + 2 \left(\frac{\omega_1 \|H_k\| L(\|y_k\| + \|y_{k-1}\|) + \omega_2 \|H_k\|^2}{\omega_4 \|H_{k-1}\|^2} \right) \alpha_{k-1} \|d_{k-1}\| \\
&\leq c_2 + 2 \frac{(\omega_1 c_2 L(2c_1) + \omega_2 c_2^2)}{\omega_4 c_3^2} \alpha_{k-1} \|d_{k-1}\|.
\end{aligned}
$$

By equality (3.16), we have for any $c_5 > 0$, there exists $k_0 \in \mathbb{N}$ for which $\alpha_{k-1}\|d_{k-1}\| < c_5$, $\forall k > k_0$. So, if we chose $c_5 = \omega_4 c_3^2$ and $c_7 = \max\{\|d_0\|, \|d_1\|, \cdots, \|d_{k_0}\|, c_6\}$, where $c_6 = c_2(1 + 4L\omega_1 c_1 + 2\omega_2 c_2)$. Letting $M = \max\{c_4, c_7\}$, we have

$$
\|d_k\| \leq M, \ \forall k \in \mathbb{N}. \qquad (3.23)
$$

Now, multiplying both side of (3.14) by $\|d_k\|$, we have

$$
\alpha_k \|d_k\| > \frac{\mu \|H_k\|^2}{(L + \sigma)\|d_k\|} \geq \frac{\mu c_3^2}{(L + \sigma) M} > 0.
$$

This contradicts (3.16) and hence $\liminf_{k \to \infty} \|H_k\| = 0$.

Since H is continuous and (3.19) hold, then the sequence $\{y_k\}$ has some accumulation point say $\bar{y}$ for which $H(\bar{y}) = 0$, that is, $\bar{y}$ is a solution of (1.1). From (3.17), it holds that $\{\|y_k - \bar{y}\|\}$ converges, and since $\bar{y}$ is an accumulation point of $\{y_k\}$, Then we must have that $\{y_k\}$ converges to $\bar{y}$. $\square$

TABLE 1. List of test problems with their references

S/N	**Problem & Reference**
1	Modified exponential function 2 [22]
2	Logarithmic function [22]
3	Nonsmooth function [35]
4	Strictly convex function I [22]
5	Strictly convex function II [22]
6	Tridiagonal exponential function [11]
7	Nonsmooth function [33]
8	Problem 4 in [15]
9	Pursuit-evasion problem [7]

4. NUMERICAL EXPERIMENTS

In this section, the numerical strength of the proposed hybrid algorithm called HYBRIDSCG is tested based on some standard metrics. All numerical simulations are implemented in Matlab R2020b on an HP laptop with 8 GB RAM and 2.40 GHz processor. The standard metrics are: the number of iterations (NOI), the number of function evaluations (NFE), and the CPU time (TIME). To show the strength of the proposed algorithm based on the above metrics, we compare it with the algorithm called DFPB1 proposed by Ahookhosh et al. in [8] and the algorithm called MFRM proposed by Abubakar et al. [3]. We use the following for the experiments:

- Problems: Nine test problems.
- Parameters: $t = 1,\ \mu = 0.8,\ \sigma = 10^{-4},\ \theta = 1.2,\ \omega_1 = \omega_2 = \omega_3 = \omega_4 = 1.$ As for DFPB1 and MFRM, all parameters come from [8] and [3], respectively.
- Termination criterion: Iterations terminate when $\|H_k\| \leq 10^{-5}$ and/or the number of iterations exceed 1000 without reaching a solution.
- The symbol $-$ is used to indicate that an algorithm failed due to number of iterations exceeding 1000.

The list of the nine test problems are given in Table 1 below. The results of the experiments on the nine test problems can be found in the Appendix. In adition, we employ the performance profiles of Dolan and Morè [17] to plot the graph of comparison for each metric. Figure 1, 2 and 3 represent the performance profiles of HYBRIDSCG, DFPB1 and MFRM based on NOI, NFE and TIME, respectively. By Figure 1, HYBRIDSCG is the best solver with 60% success. It is also the best solver based on Figure 2 with more than 70% success. Likewise in Figure 3, it is the most successful with over based 60% success. The percentage of success for the algorithms by Ahookhosh et al. in [8] and Abubakar et al. [3] based on each plot is summarized as follows

- Figure 1: DFPB1 ($< 10\%$), MFRM (almost 50%).
- Figure 2: DFPB1 ($< 10\%$), MFRM (around 30%).
- Figure 3: DFPB1 ($< 10\%$), MFRM (around 35%).

5. APPLICATIOM IN IMAGE RESTORATION

In this subsection, we aim at evaluating the efficiency of the HYBRIDSCG in image restoration. Image restoration problem can be mathematically formulated as:

$$b = Ay + v, \tag{5.24}$$

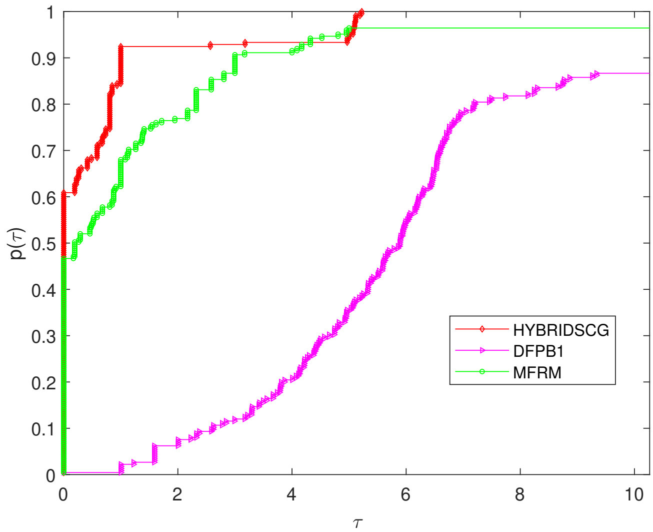

FIGURE 1. Performance profiles for the number of iterations (NOI)

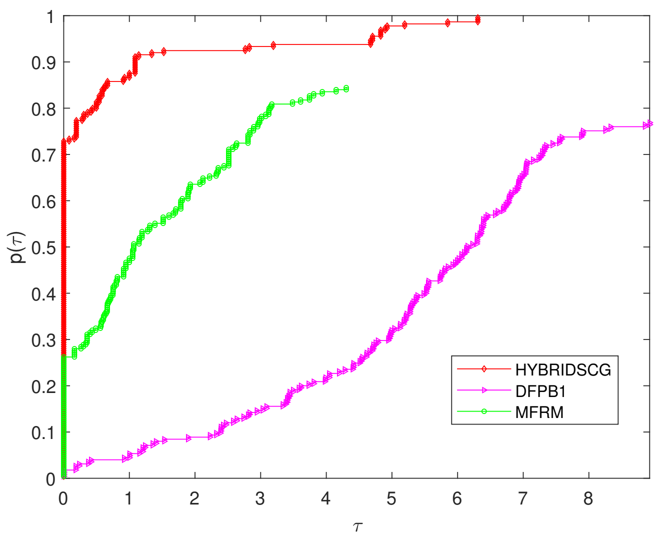

FIGURE 2. Performance profiles for the number of function evaluations (NFE)

where $b \in \mathbb{R}^k$ is the observed data, $A \in \mathbb{R}^{k\times n}$ and $\upsilon \in \mathbb{R}^k$ is an error term. See [18] for more details.

The HYBRIDSCG method is compared with some existing methods such as the modified Fletcher-Reeves conjugate gradient method for monotone nonlinear equations with applications (MFRM) [3] and the conjugate gradient method for solving convex constrained monotone equations with applications in compressive sensing (CGD) by Xiao et al. [31].

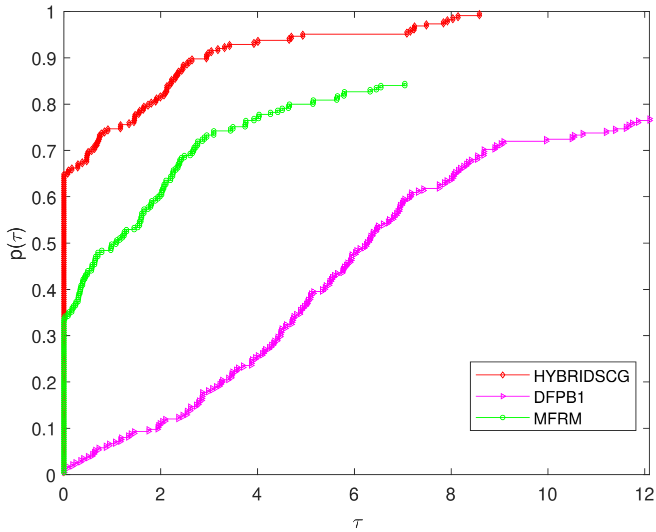

FIGURE 3. Performance profiles for the CPU time (in seconds)

In the numerical implementation, three (3) colored images of different sizes are degraded using a Gaussian noise operator and a Gaussian blur with a standard deviation of 10^{-2}, then we apply the three methods to restore the degraded images. Experimental results for HYBRIDSCG, MFRM, and CGD are given in Table 2. The comparison is based on the signal-to-noise ratio (SNR), peak signal-to-noise ratio (PSNR) [12], and the structural similarity index (SSIM) [23]. HYBRIDSCG is implemented using the following specified parameters: $t = 1$, $\mu = 0.01$, $\sigma = 10^{-4}$, $\theta = 1$, and $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1$ with a merit function defined as follows:

$$f(y) = \frac{1}{2}\|Ay - b\|_2^2 + \lambda\|y\|_1. \tag{5.25}$$

Using the same initial point, all algorithms were implemented. The regularization parameter λ is selected based on the approach presented by Liu and Li [25].
$y_0 = A^T b$ is used in starting the experiment and

$$\frac{|f_k - f_{k-1}|}{|f_{k-1}|} < 10^{-4},$$

as the stopping criterion, where f_k is the function value at y_k.

In Table 2, we report the results of the image restoration process by the proposed and compared methods. The results obtained by HYBRIDSCG are much better than the results obtained by the compared methods. Notably, HYBRIDSCG has a larger value of SNR, PSNR and SSIM, indicating it performs better than the compared methods in restoring the degraded images.

6. CONCLUSIONS

This article proposes a new hybrid algorithm for solving nonlinear monotone operator equations with convex constraints. To the best of our knowledge, this is the first algorithm

TABLE 2. Efficiency comparison for HYBRIDSCG, MFRM and CGD based on SNR, PSNR and SSIM

	HYBRIDSGC			MFRM			CGD		
Images	SNR	PSNR	SSIM	SNR	PSNR	SSIM	SNR	PSNR	SSIM
Lenna	17.16	22.49	0.9200	16.66	21.99	0.9110	16.83	22.17	0.9137
Tiffany	21.51	23.34	0.9240	21.03	22.87	0.913	21.02	22.86	0.9140
Peppers	15.89	21.81	0.9040	15.31	21.23	0.890	15.59	21.51	0.896

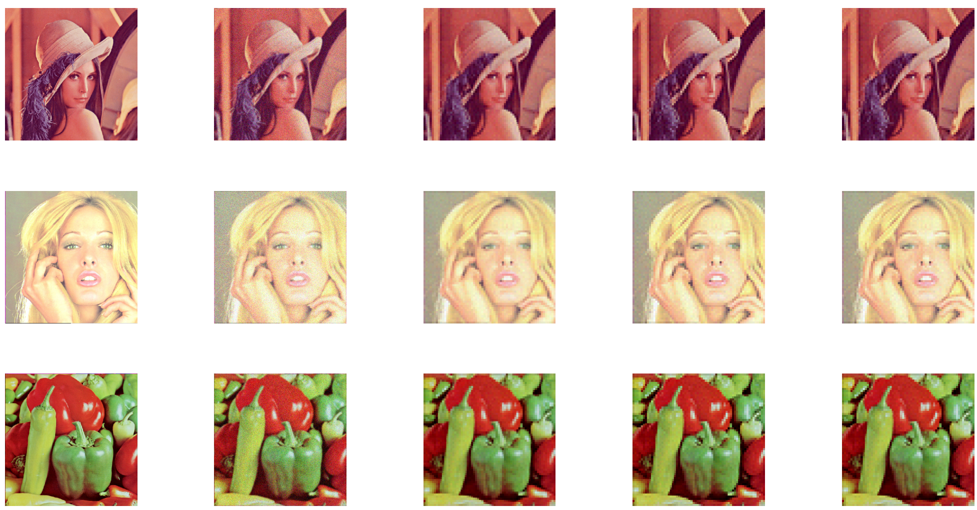

FIGURE 4. Restoration of the test images. From left to right: original image, degraded image, restored image by HYBRIDSGC, restored image by MFRM, and restored image by CGD.

of its kind, and it is based on the SCG method. The PRP, LS, FR, and CD methods are special cases of this new hybrid method. One of the notable features of the new method is that its search direction is both descent and bounded, independent of the line search. Under certain favourable assumptions, the sequence generated by this method converges globally. The new method's efficiency was demonstrated through numerical experiments on several benchmark test problems. The results indicate that the new algorithm outperforms the existing methods it was compared with, showcasing its superior efficiency. The proposed algorithm's performance was evaluated based on multiple criteria, including convergence rate and computational cost. These comprehensive tests validate the robustness and practical applicability of the new method in solving complex nonlinear problems with convex constraints.

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APPENDIX

Throughout, "DIM" denotes the dimension, "INP" denotes initial point, "NOI" denotes number of iterations, "NFE" denotes number of function evaluations, "TIME" denotes the CPU running time and "Norm" denotes the norm of a function at the approximate solution. The associated initial points used for these experiments are $x_1 = (0.1, 0.1, \cdots, 0.1)^T, x_2 = (0.2, 0.2, \cdots, 0.2)^T$,
$x_3 = (0.5, 0.5, \cdots, 0.5)^T, x_4 = (1.5, 1.5, \cdots 1.5)^T, x_5 = (2, 2, \cdots, 2)^T$. The symbol $-$ is used to indicate that an algorithm failed due to number of iterations exceeding 1000.

TABLE 3. Results for the three algorithms on Problem 1.

		HYBRIDSCG				**DFPB1**				**MFRM**			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	1	7	0.00208	0.00E+00	59	184	0.017397	9.97E-06	23	98	0.013288	9.01E-06
	x_2	1	7	0.001195	0.00E+00	52	162	0.013355	9.95E-06	7	35	0.005672	8.82E-06
	x_3	1	7	0.001198	0.00E+00	143	436	0.0285	9.71E-06	8	40	0.004216	9.74E-06
	x_4	1	9	0.001595	0.00E+00	465	1405	0.073153	9.91E-06	5	31	0.003143	0
	x_5	1	10	0.002063	0.00E+00	4	23	0.004625	0	31	134	0.009485	7.65E-06
5000	x_1	1	7	0.00214	0.00E+00	88	270	0.056742	9.80E-06	8	38	0.00848	5.63E-06
	x_2	1	7	0.00243	0.00E+00	89	274	0.058434	9.86E-06	8	40	0.008403	2.59E-06
	x_3	1	7	0.002771	0.00E+00	302	916	0.21207	9.96E-06	8	40	0.008365	6.41E-06
	x_4	1	9	0.004404	0.00E+00	4	23	0.006772	0	5	31	0.006826	0
	x_5	1	10	0.004374	0.00E+00	3	20	0.005763	0	31	134	0.02749	8.10E-06
10000	x_1	1	7	0.004689	0.00E+00	48	149	0.068357	9.92E-06	5	26	0.011089	3.70E-06
	x_2	1	7	0.005319	0.00E+00	132	404	0.16097	9.97E-06	8	40	0.016022	3.64E-06
	x_3	1	7	0.005584	0.00E+00	302	916	0.33523	9.96E-06	8	40	0.016474	5.44E-06
	x_4	1	9	0.005512	0.00E+00	5	26	0.012497	0	5	31	0.013106	0
	x_5	1	10	0.009738	0.00E+00	3	20	0.011495	0	28	122	0.048459	7.18E-06
50000	x_1	1	7	0.013687	0.00E+00	123	376	0.55323	9.55E-06	5	26	0.039405	3.58E-06
	x_2	1	7	0.013792	0.00E+00	280	849	1.2166	9.74E-06	8	40	0.063794	8.10E-06
	x_3	1	7	0.013815	0.00E+00	435	1315	1.9343	9.93E-06	8	40	0.057732	4.54E-06
	x_4	1	9	0.017657	0.00E+00	3	20	0.044262	0	5	31	0.048293	0
	x_5	1	10	0.025617	0.00E+00	4	23	0.046637	0	20	90	0.12298	6.44E-06
100000	x_1	1	7	0.025428	0.00E+00	180	548	1.753	9.60E-06	5	26	0.075056	4.59E-06
	x_2	1	7	0.028532	0.00E+00	403	1220	3.5145	9.96E-06	9	43	0.11747	1.59E-06
	x_3	1	7	0.024189	0.00E+00	432	1307	3.8515	9.96E-06	8	40	0.11518	4.96E-06
	x_4	1	9	0.0323	0.00E+00	2	17	0.067078	0	32	138	0.35109	7.09E-06
	x_5	1	10	0.033243	0.00E+00	-	-	-	-	17	78	0.2203	9.31E-06

TABLE 4. Results for the three algorithms on Problem 2.

		HYBRIDSCG				DFPB1				MFRM			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	5	14	0.002712	5.40E-06	91	278	0.023238	9.30E-06	3	8	0.002453	5.17E-07
	x_2	5	14	0.002174	6.67E-06	80	246	0.026381	9.62E-06	3	8	0.002031	6.04E-06
	x_3	6	16	0.002883	1.15E-06	36	111	0.0081	8.33E-06	4	11	0.002095	4.37E-07
	x_4	6	16	0.002547	1.26E-06	151	459	0.034901	9.83E-06	5	14	0.001851	1.10E-06
	x_5	7	18	0.003591	3.90E-06	217	658	0.062891	9.84E-06	6	17	0.002024	1.74E-08
5000	x_1	5	15	0.006807	8.82E-06	83	255	0.086791	9.92E-06	3	8	0.003307	1.75E-07
	x_2	6	17	0.007096	1.12E-06	64	196	0.064129	9.90E-06	3	8	0.003025	3.13E-06
	x_3	6	16	0.007799	2.51E-06	149	453	0.15207	9.63E-06	4	11	0.00427	1.42E-07
	x_4	6	16	0.007371	3.40E-06	329	995	0.33947	9.99E-06	5	14	0.006139	4.05E-07
	x_5	7	18	0.009422	8.69E-06	472	1425	0.46621	9.88E-06	6	17	0.005815	2.36E-09
10000	x_1	6	17	0.016245	1.29E-06	59	182	0.11499	9.72E-06	3	8	0.005392	1.21E-07
	x_2	6	17	0.013211	1.59E-06	66	202	0.12517	8.76E-06	3	8	0.006238	2.79E-06
	x_3	6	16	0.01863	3.54E-06	220	667	0.42825	9.77E-06	4	11	0.007939	9.73E-08
	x_4	6	16	0.013821	4.91E-06	483	1458	0.93875	9.87E-06	5	14	0.009098	2.93E-07
	x_5	7	19	0.015851	8.89E-06	692	2086	1.3808	9.84E-06	6	17	0.010531	1.24E-09
50000	x_1	6	17	0.044504	2.89E-06	98	299	0.72483	9.63E-06	3	8	0.022176	6.32E-08
	x_2	6	17	0.048016	3.57E-06	216	655	1.5813	9.96E-06	3	8	0.020866	3.37E-06
	x_3	6	16	0.046256	7.91E-06	479	1446	3.4612	9.79E-06	4	11	0.03066	4.87E-08
	x_4	6	17	0.0451	8.10E-06	725	2186	5.2517	9.97E-06	5	14	0.036304	1.84E-07
	x_5	8	21	0.058008	2.05E-06	723	2179	5.2287	9.84E-06	6	17	0.043019	4.01E-10
100000	x_1	6	17	0.1012	4.08E-06	146	444	1.9984	9.42E-06	3	8	0.0418	5.40E-08
	x_2	6	17	0.089258	5.05E-06	221	670	2.9995	9.72E-06	3	8	0.040817	4.27E-06
	x_3	6	17	0.089481	8.11E-06	701	2113	9.4877	9.99E-06	4	11	0.05554	4.05E-08
	x_4	7	19	0.092316	1.18E-06	739	2227	9.9448	9.82E-06	5	14	0.069756	1.80E-07
	x_5	8	21	0.11838	2.90E-06	745	2245	10.072	9.98E-06	6	17	0.081469	2.71E-10

TABLE 5. Results for the three algorithms on Problem 3.

		HYBRIDSCG				DFPB1				MFRM			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	208	634	0.05439	9.87E-06	624	1882	0.158	9.87E-06	6	24	0.002396	3.11E-06
	x_2	208	634	0.043583	9.87E-06	624	1882	0.13109	9.87E-06	6	24	0.002149	5.94E-06
	x_3	208	634	0.04418	9.86E-06	64	196	0.01487	8.62E-06	6	24	0.002174	9.94E-06
	x_4	9	28	0.003061	3.59E-06	300	909	0.076095	1.00E-05	11	46	0.002925	2.71E-06
	x_5	8	26	0.003225	5.51E-06	618	1865	0.12764	9.99E-06	16	68	0.004583	8.38E-06
5000	x_1	219	667	0.17978	9.90E-06	43	133	0.037317	9.85E-06	6	24	0.00564	6.96E-06
	x_2	219	667	0.20085	9.90E-06	65	199	0.054585	8.55E-06	7	28	0.006942	1.33E-06
	x_3	219	667	0.18565	9.90E-06	146	444	0.12493	9.45E-06	7	28	0.007307	2.22E-06
	x_4	9	28	0.008754	8.03E-06	654	1972	0.55188	9.86E-06	11	46	0.011082	6.06E-06
	x_5	9	28	0.008722	3.28E-06	659	1987	0.56525	9.99E-06	17	72	0.016643	7.67E-06
10000	x_1	224	682	0.34196	9.73E-06	41	126	0.066791	8.91E-06	6	24	0.01097	9.85E-06
	x_2	224	682	0.32324	9.73E-06	97	296	0.15224	9.44E-06	7	28	0.012606	1.88E-06
	x_3	224	682	0.32965	9.73E-06	216	655	0.35641	9.67E-06	7	28	0.013306	3.14E-06
	x_4	9	29	0.015087	5.10E-06	665	2006	1.1222	9.98E-06	11	46	0.019798	8.58E-06
	x_5	9	28	0.023411	4.64E-06	673	2029	1.1502	9.85E-06	18	76	0.033463	4.44E-06
50000	x_1	235	715	1.2938	9.76E-06	98	299	0.62404	8.90E-06	7	28	0.046645	2.20E-06
	x_2	235	715	1.6854	9.76E-06	215	652	1.3631	9.55E-06	7	28	0.045009	4.20E-06
	x_3	235	715	1.3124	9.76E-06	471	1422	2.9649	9.81E-06	7	28	0.044088	7.03E-06
	x_4	10	31	0.061176	3.04E-06	707	2131	4.444	9.89E-06	12	50	0.088998	5.20E-06
	x_5	9	29	0.055521	4.66E-06	704	2122	4.4704	9.89E-06	18	76	0.12249	9.93E-06
100000	x_1	240	730	4.2127	9.59E-06	145	441	1.7	9.81E-06	7	28	0.084886	3.11E-06
	x_2	240	730	3.3075	9.59E-06	316	957	3.6007	9.94E-06	7	28	0.084564	5.94E-06
	x_3	240	730	3.7609	9.59E-06	690	2080	7.743	9.94E-06	7	28	0.084681	9.94E-06
	x_4	10	31	0.10698	4.30E-06	5	26	0.13981	0	12	50	0.15095	7.35E-06
	x_5	9	29	0.1052	6.59E-06	718	2164	8.1766	9.88E-06	19	80	0.24534	5.75E-06

TABLE 6. Results for the three algorithms on Problem 4.

		HYBRIDSCG				**DFPB1**				**MFRM**			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	1	4	0.001312	0	38	118	0.007735	9.79E-06	6	24	0.002023	1.65E-06
	x_2	1	4	0.000811	0	48	149	0.011013	9.16E-06	5	20	0.001592	2.32E-06
	x_3	5	17	0.002061	8.04E-06	93	284	0.017588	9.51E-06	10	42	0.003416	6.42E-06
	x_4	1	7	0.001015	0	426	1287	0.071075	9.90E-06	16	71	0.004065	8.48E-06
	x_5	1	9	0.001308	0	613	1849	0.096644	9.94E-06	1	15	0.001776	0
5000	x_1	1	4	0.001705	0	3	15	0.004367	0	6	24	0.00492	3.68E-06
	x_2	1	4	0.001533	0	63	193	0.037773	8.74E-06	5	20	0.004287	5.20E-06
	x_3	6	19	0.005213	4.78E-06	208	631	0.12208	9.55E-06	11	46	0.008255	3.89E-06
	x_4	1	7	0.003002	0.00E+00	645	1945	0.39337	9.81E-06	18	79	0.015013	6.15E-06
	x_5	1	9	0.003058	0	3	20	0.006331	0	1	15	0.003854	0
10000	x_1	1	4	0.002513	0	34	105	0.035563	8.06E-06	6	24	0.008339	5.20E-06
	x_2	1	4	0.002308	0	96	293	0.10306	9.36E-06	5	20	0.006425	7.35E-06
	x_3	6	19	0.008479	6.76E-06	307	929	0.335	9.62E-06	11	46	0.014964	5.50E-06
	x_4	1	7	0.003576	0	5	26	0.012717	0	18	79	0.026799	8.69E-06
	x_5	1	9	0.004746	0.00E+00	3	20	0.010875	0	1	15	0.007209	0
50000	x_1	1	4	0.007637	0	97	296	0.40247	9.47E-06	7	28	0.035767	1.16E-06
	x_2	1	4	0.008306	0	213	646	0.88198	9.87E-06	6	24	0.02789	1.64E-06
	x_3	6	20	0.027349	6.78E-06	666	2008	2.8021	9.99E-06	12	50	0.058484	3.33E-06
	x_4	1	7	0.011779	0	3	20	0.03512	0	20	87	0.10175	6.31E-06
	x_5	1	9	0.013706	0	2	17	0.030306	0	1	15	0.026876	0
100000	x_1	1	4	0.012268	0.00E+00	145	441	1.1719	9.21E-06	7	28	0.062812	1.65E-06
	x_2	1	4	0.011989	0	314	950	2.4091	9.99E-06	6	24	0.055935	2.32E-06
	x_3	6	20	0.050065	9.58E-06	681	2053	5.2381	9.82E-06	12	50	0.11534	4.71E-06
	x_4	1	7	0.019024	0	3	20	0.067134	0	20	87	0.20414	8.92E-06
	x_5	1	9	0.024915	0	2	17	0.057477	0	1	15	0.052133	0

TABLE 7. Results for the three algorithms on Problem 5.

		HYBRIDSCG				**DFPB1**				**MFRM**			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	14	39	0.022763	7.75E-06	86	263	0.016227	9.77E-06	26	98	0.006234	3.51E-06
	x_2	14	38	0.003236	7.42E-06	86	262	0.016786	9.24E-06	46	178	0.011195	7.43E-06
	x_3	14	42	0.00418	4.72E-06	84	257	0.018012	9.89E-06	37	144	0.009441	7.11E-06
	x_4	16	51	0.004114	7.95E-06	345	1043	0.064886	9.76E-06	46	194	0.011187	7.06E-06
	x_5	30	95	0.007561	9.32E-06	481	1453	0.091225	9.98E-06	43	182	0.009821	8.70E-06
5000	x_1	47	142	0.033341	8.63E-06	197	597	0.11997	9.72E-06	38	147	0.030431	4.96E-06
	x_2	181	549	0.11761	9.92E-06	196	594	0.1291	9.75E-06	20	77	0.017908	4.98E-06
	x_3	14	42	0.014535	9.10E-06	193	585	0.12104	9.72E-06	41	157	0.029593	8.92E-06
	x_4	17	53	0.022852	8.18E-06	782	2356	0.51453	9.99E-06	-	-	-	-
	x_5	32	100	0.021863	9.62E-06	756	2278	0.49511	9.94E-06	45	190	0.043542	7.14E-06
10000	x_1	60	183	0.082076	9.86E-06	294	889	0.37295	9.80E-06	37	143	0.051113	9.28E-06
	x_2	34	102	0.045406	9.75E-06	293	886	0.35567	9.52E-06	22	84	0.032708	9.78E-06
	x_3	15	44	0.023447	3.99E-06	288	871	0.341	9.99E-06	39	149	0.050962	6.74E-06
	x_4	17	54	0.025686	8.74E-06	811	2443	0.99399	9.93E-06	44	186	0.064499	7.68E-06
	x_5	33	103	0.042758	8.89E-06	881	2653	1.0904	9.90E-06	46	194	0.065669	8.62E-06
50000	x_1	107	325	0.4712	9.86E-06	939	2827	4.4152	9.86E-06	56	218	0.2953	6.31E-06
	x_2	26	77	0.10972	6.50E-06	652	1965	3.0427	9.77E-06	69	280	0.38081	6.87E-06
	x_3	15	44	0.066827	8.84E-06	643	1938	3.0116	9.83E-06	-	-	-	-
	x_4	18	56	0.10153	8.73E-06	-	-	-	-	46	194	0.28237	8.47E-06
	x_5	35	109	0.15201	8.99E-06	-	-	-	-	50	210	0.2796	8.12E-06
100000	x_1	184	558	1.5203	9.49E-06	968	2914	7.9513	9.84E-06	31	121	0.32019	4.48E-06
	x_2	19	56	0.14931	9.22E-06	963	2900	8.2014	9.99E-06	-	-	-	-
	x_3	15	45	0.11472	8.71E-06	951	2863	8.1928	9.88E-06	46	178	0.46446	6.99E-06
	x_4	18	57	0.14127	9.26E-06	-	-	-	-	47	198	0.59746	8.31E-06
	x_5	36	112	0.29533	8.86E-06	-	-	-	-	52	218	0.58434	7.37E-06

TABLE 8. Results for the three algorithms on Problem 6.

		HYBRIDSCG				DFPB1				MFRM			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	8	24	0.028518	6.23E-06	463	1398	0.15017	9.80E-06	11	44	0.005115	8.32E-06
	x_2	8	24	0.00387	6.48E-06	321	971	0.094723	9.83E-06	11	44	0.004648	7.32E-06
	x_3	8	24	0.003359	7.01E-06	318	963	0.076505	9.99E-06	11	44	0.003925	8.83E-06
	x_4	8	24	0.004506	5.91E-06	147	447	0.038539	9.49E-06	9	36	0.003494	8.29E-06
	x_5	8	24	0.003804	4.28E-06	98	299	0.028484	9.01E-06	7	28	0.003089	8.25E-06
5000	x_1	8	25	0.011983	6.36E-06	698	2104	0.85174	9.80E-06	8	32	0.010689	1.87E-06
	x_2	8	25	0.012516	6.39E-06	696	2098	0.84443	9.90E-06	8	32	0.011528	1.80E-06
	x_3	8	25	0.011862	6.33E-06	691	2083	0.85148	9.84E-06	8	32	0.011359	1.59E-06
	x_4	8	25	0.010762	4.66E-06	464	1401	0.55128	9.86E-06	7	28	0.009012	8.62E-06
	x_5	8	24	0.011493	8.82E-06	216	656	0.27	9.96E-06	7	28	0.010534	5.08E-06
10000	x_1	8	24	0.020913	7.12E-06	712	2146	1.7802	9.87E-06	8	32	0.021535	2.62E-06
	x_2	8	24	0.020639	7.19E-06	710	2140	1.7344	9.97E-06	8	32	0.022687	2.52E-06
	x_3	8	24	0.019573	7.48E-06	705	2125	1.7163	9.91E-06	8	32	0.023337	2.22E-06
	x_4	8	24	0.028183	6.76E-06	680	2050	1.631	9.96E-06	8	32	0.022381	1.22E-06
	x_5	8	24	0.024112	4.77E-06	319	965	0.77825	9.83E-06	7	28	0.021931	7.18E-06
50000	x_1	8	25	0.069879	6.45E-06	744	2242	6.4162	9.87E-06	8	32	0.080613	5.85E-06
	x_2	8	25	0.078116	6.42E-06	742	2236	6.3846	1.00E-05	8	32	0.079199	5.63E-06
	x_3	8	25	0.078497	6.44E-06	737	2222	6.3313	9.99E-06	8	32	0.079298	4.96E-06
	x_4	7	21	0.068783	9.55E-06	713	2150	6.1378	9.98E-06	8	32	0.078776	2.72E-06
	x_5	7	21	0.060604	6.23E-06	692	2086	5.9485	9.83E-06	8	32	0.079353	1.61E-06
100000	x_1	7	21	0.12721	6.69E-06	753	2270	12.8687	9.97E-06	8	32	0.17703	8.28E-06
	x_2	7	21	0.12365	6.57E-06	754	2272	13.0305	9.87E-06	8	32	0.17207	7.96E-06
	x_3	7	21	0.12876	6.18E-06	750	2260	12.8745	9.97E-06	8	32	0.16946	7.01E-06
	x_4	7	21	0.12073	4.18E-06	727	2192	13.1911	9.99E-06	8	32	0.18228	3.85E-06
	x_5	7	21	0.12014	2.70E-06	706	2128	12.1153	9.91E-06	8	32	0.17072	2.27E-06

TABLE 9. Results for the three algorithms on Problem 7.

		HYBRIDSCG				**DFPB1**				**MFRM**			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	6	23	0.026961	8.90E-06	47	147	0.012069	9.51E-06	4	21	0.002293	3.24E-07
	x_2	7	25	0.002594	1.12E-06	45	140	0.011434	8.84E-06	4	21	0.001968	1.43E-07
	x_3	3	14	0.00209	2.43E-06	7	22	0.002668	8.05E-06	3	17	0.001705	5.81E-08
	x_4	4	17	0.00311	9.07E-07	74	228	0.020342	9.16E-06	7	34	0.002805	6.36E-06
	x_5	5	19	0.002793	1.87E-06	73	225	0.017418	8.97E-06	8	37	0.003184	1.90E-06
5000	x_1	7	25	0.008957	2.13E-06	113	346	0.10171	9.13E-06	4	21	0.007073	7.25E-07
	x_2	7	25	0.007984	2.50E-06	74	228	0.060023	9.60E-06	4	21	0.006495	3.20E-07
	x_3	3	14	0.005602	5.42E-06	6	20	0.00638	6.65E-06	3	17	0.006098	1.30E-07
	x_4	4	17	0.00679	2.03E-06	166	507	0.14053	1.00E-05	8	38	0.010315	1.49E-06
	x_5	5	19	0.007403	4.18E-06	241	732	0.21577	9.50E-06	8	37	0.010848	4.26E-06
10000	x_1	7	25	0.013733	3.01E-06	168	512	0.26911	9.42E-06	4	21	0.010077	1.02E-06
	x_2	7	25	0.017363	3.53E-06	113	346	0.18599	9.22E-06	4	21	0.010746	4.52E-07
	x_3	3	14	0.011848	7.67E-06	14	45	0.024651	9.20E-06	3	17	0.009026	1.84E-07
	x_4	4	17	0.012261	2.87E-06	246	747	0.39502	9.73E-06	8	38	0.017835	2.10E-06
	x_5	5	19	0.013002	5.91E-06	354	1072	0.58062	9.84E-06	8	37	0.017645	6.02E-06
50000	x_1	7	25	0.04846	6.73E-06	367	1112	2.2913	9.96E-06	4	21	0.043377	2.29E-06
	x_2	7	25	0.05098	7.90E-06	250	759	1.5387	9.80E-06	4	21	0.037317	1.01E-06
	x_3	4	16	0.031574	2.55E-06	67	208	0.43005	9.24E-06	3	17	0.031092	4.11E-07
	x_4	4	17	0.039941	6.42E-06	371	1124	2.2876	9.99E-06	8	38	0.063828	4.70E-06
	x_5	6	21	0.048487	1.79E-06	371	1123	2.2984	9.76E-06	9	41	0.081229	1.41E-06
100000	x_1	7	25	0.09203	9.52E-06	375	1135	4.2041	9.77E-06	4	21	0.069675	3.24E-06
	x_2	7	26	0.099234	7.70E-06	368	1114	4.1018	9.88E-06	4	21	0.071007	1.43E-06
	x_3	4	16	0.070669	3.60E-06	101	311	1.1614	9.67E-06	3	17	0.061634	5.81E-07
	x_4	4	17	0.075186	9.07E-06	378	1145	4.2028	9.99E-06	8	38	0.12589	6.65E-06
	x_5	6	21	0.078324	2.53E-06	389	1177	4.3844	9.88E-06	9	41	0.15559	1.99E-06

TABLE 10. Results for the three algorithms on Problem 8.

		HYBRIDSCG				**DFPB1**				**MFRM**			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	5	21	0.012929	3.33E-06	-	-	-	-	9	42	0.003772	9.49E-06
	x_2	5	21	0.001766	2.01E-06	-	-	-	-	9	42	0.004332	9.49E-06
	x_3	5	21	0.001681	1.92E-06	-	-	-	-	13	62	0.004369	5.02E-06
	x_4	6	24	0.003185	1.20E-06	-	-	-	-	13	62	0.004391	5.02E-06
	x_5	6	24	0.002362	1.72E-06	-	-	-	-	13	62	0.004568	5.02E-06
5000	x_1	5	21	0.004363	7.44E-06	-	-	-	-	10	50	0.013257	3.97E-06
	x_2	5	21	0.005446	4.50E-06	-	-	-	-	12	58	0.014512	3.50E-06
	x_3	5	21	0.006948	4.30E-06	-	-	-	-	12	58	0.015315	3.50E-06
	x_4	6	24	0.005116	2.68E-06	127	392	0.092756	9.86E-06	12	58	0.015189	3.50E-06
	x_5	6	24	0.004529	3.85E-06	-	-	-	-	12	58	0.014247	3.50E-06
10000	x_1	6	24	0.007889	8.38E-07	-	-	-	-	11	54	0.030046	7.99E-06
	x_2	5	21	0.007325	6.37E-06	-	-	-	-	11	54	0.030193	7.99E-06
	x_3	5	21	0.006432	6.08E-06	113	349	0.17688	9.79E-06	11	54	0.036489	7.99E-06
	x_4	6	24	0.008236	3.79E-06	-	-	-	-	11	54	0.029971	7.99E-06
	x_5	6	24	0.010118	5.44E-06	-	-	-	-	11	54	0.027405	7.99E-06
50000	x_1	6	24	0.024719	1.87E-06	76	238	0.50184	9.01E-06	27	118	0.20019	8.75E-06
	x_2	6	24	0.024973	1.13E-06	76	238	0.4899	9.01E-06	27	118	0.20591	8.75E-06
	x_3	6	24	0.027511	1.08E-06	-	-	-	-	27	118	0.21152	8.75E-06
	x_4	6	24	0.026514	8.47E-06	-	-	-	-	27	118	0.19623	8.75E-06
	x_5	7	27	0.027801	1.15E-06	-	-	-	-	27	118	0.21923	8.75E-06
100000	x_1	6	24	0.051423	2.65E-06	59	188	0.70717	9.94E-06	-	-	-	-
	x_2	6	24	0.05418	1.60E-06	-	-	-	-	-	-	-	-
	x_3	6	24	0.064822	1.53E-06	-	-	-	-	-	-	-	-
	x_4	7	27	0.066902	1.13E-06	-	-	-	-	-	-	-	-
	x_5	7	27	0.05231	1.63E-06	-	-	-	-	-	-	-	-

TABLE 11. Results for the three algorithms on Problem 9.

		HYBRIDSCG				**DFPB1**				**MFRM**			
DIM	INP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1000	x_1	7	25	0.024428	3.31E-06	39	123	0.006235	9.49E-06	4	25	0.001891	8.20E-07
	x_2	7	25	0.001179	2.00E-06	71	221	0.008798	9.82E-06	4	25	0.00165	4.96E-07
	x_3	7	25	0.00113	1.91E-06	55	174	0.006965	9.77E-06	4	25	0.00171	4.73E-07
	x_4	8	28	0.001269	1.68E-06	161	492	0.020052	9.32E-06	4	25	0.001679	3.71E-06
	x_5	8	28	0.001734	2.41E-06	238	724	0.030846	9.83E-06	4	25	0.002033	5.32E-06
5000	x_1	7	25	0.003592	7.40E-06	69	215	0.032179	9.77E-06	4	25	0.004306	1.83E-06
	x_2	7	25	0.003092	4.48E-06	39	123	0.017953	9.59E-06	4	25	0.004595	1.11E-06
	x_3	7	25	0.002703	4.27E-06	43	135	0.021851	8.70E-06	4	25	0.004466	1.06E-06
	x_4	8	28	0.004574	3.75E-06	244	742	0.11336	9.72E-06	4	25	0.004989	8.29E-06
	x_5	8	28	0.003543	5.39E-06	247	751	0.11355	9.67E-06	5	29	0.005801	1.64E-07
10000	x_1	8	28	0.006141	1.17E-06	107	330	0.099953	9.86E-06	4	25	0.008901	2.59E-06
	x_2	7	25	0.005568	6.34E-06	70	217	0.066935	9.99E-06	4	25	0.01048	1.57E-06
	x_3	7	25	0.006774	6.04E-06	70	217	0.085183	9.95E-06	4	25	0.009931	1.50E-06
	x_4	8	28	0.006288	5.31E-06	247	751	0.32871	9.75E-06	5	29	0.012762	1.62E-07
	x_5	8	28	0.007178	7.62E-06	-	-	-	-	5	29	0.010457	2.32E-07
50000	x_1	8	28	0.024964	2.62E-06	239	727	0.89519	9.95E-06	4	25	0.029792	5.80E-06
	x_2	8	28	0.02207	1.59E-06	160	489	0.57925	9.81E-06	4	25	0.029347	3.51E-06
	x_3	8	28	0.023477	1.52E-06	160	489	0.5803	9.40E-06	4	25	0.030874	3.35E-06
	x_4	9	31	0.024584	1.33E-06	-	-	-	-	5	29	0.034249	3.61E-07
	x_5	9	31	0.024464	1.91E-06	-	-	-	-	5	29	0.032112	5.19E-07
100000	x_1	8	28	0.044541	3.71E-06	244	742	1.5579	9.63E-06	4	25	0.073174	8.20E-06
	x_2	8	28	0.044001	2.25E-06	237	721	1.5175	9.86E-06	4	25	0.057577	4.96E-06
	x_3	8	28	0.04765	2.14E-06	236	719	1.559	9.98E-06	4	25	0.058631	4.73E-06
	x_4	9	31	0.050755	1.88E-06	-	-	-	-	5	29	0.071601	5.11E-07
	x_5	9	31	0.067252	2.70E-06	-	-	-	-	5	29	0.063296	7.34E-07

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