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Gradient Projection Method for Quasiconvex Equilibrium Problems

QAMRUL HASAN ANSARI[1,2], MUZAFFAR SARKAR RAJU[1] AND FEEROZ BABU[3]

ABSTRACT. This paper proposes a gradient projection method to solve the equilibrium problems where the bifunction is quasiconvex in its second argument. We use the Greenberg-Pierskalla quasi-subdifferential of quasiconvex functions. We prove the convergence of the sequence generated by the proposed algorithm under some mild assumptions. Some examples and their numerical evolutions are illustrated where previous methods are not applicable.

1. INTRODUCTION

Let $\mathscr{H}$ be a real Hilbert space, C be a nonempty closed and convex subset of $\mathscr{H}$, and $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (in short, EP) is to find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the solution set of the EP defined by the bifunction f over the set C by $S(f,C)$.

The EP has been extensively studied in the fields such as optimization and nonlinear analysis. We refer to Chapter 1 in [6] for historical details and applications. Over the past few decades, numerous methods have been proposed to solve different types of EPs; see [1–5, 7, 9, 12, 15–17, 19] and the references therein.

Recently, Hai [11] introduced a gradient projection method to solve strongly pseudomonotone equilibrium problems. However, this method assumes the convexity of the bifunction in the second argument, and therefore, it cannot be employed to solve equilibrium problems involving bifunctions which are quasiconvex but not convex with respect to the second variable. In this paper, we propose a method that can handle such equilibrium problems subject to certain assumptions. Since convex functions are also quasiconvex, the technique mentioned in [11] can be seen as a specific case of our method.

Another approach, described by Yen et al. [20], tackles an EP where the bifunction may be quasiconvex in the second argument. However, their method assumes the bifunction to be pseudomonotone as well as paramonotone. They also assumed that the bifunction needs to be upper semicontinuous with respect to both the arguments. In our approach, we relax these assumptions and consider a strongly pseudomonotone bifunction which is upper semicontinuous with respect to the second variable only, allowing us to solve a broader class of equilibrium problems.

The sections of this paper are arranged as follows. In the next section, we give some preliminaries and basic results which will be required throughout the paper. We propose a projection algorithm to solve EP (1.1) and prove the convergence of the sequence generated by our algorithm in Section 3. We justify the proposed method with numerical examples in the last section.

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Corresponding author: Qamrul Hasan Ansari; qhansari@gmail.com

2. Preliminary Results

Let $\mathscr{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. It is well-known that for any $x,y,z \in \mathscr{H}$, the following relation holds:

$$\big\langle z-x, x-y \big\rangle = \frac{1}{2}\|y-z\|^2 - \frac{1}{2}\|z-x\|^2 - \frac{1}{2}\|x-y\|^2. \tag{2.2}$$

A nonempty subset C of a real vector space X is said to form a cone if for all $x \in C$ and $\lambda > 0$, $\lambda x \in C$.

Let C be a nonempty convex subset of $\mathscr{H}$. A function $g : C \to \mathbb{R}$ is said to be

(a) convex on C if

$$g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda) g(y), \quad \forall x,y \in C \text{ and } \forall \lambda \in [0,1];$$

(b) quasiconvex on C if

$$g(\lambda x + (1-\lambda)y) \le \max\{g(x), g(y)\}, \quad \forall x,y \in C \text{ and } \forall \lambda \in [0,1].$$

Clearly, every convex function is quasiconvex, but the converse is not true in general. For any $\alpha \in \mathbb{R}$, the sublevel set of $g : \mathscr{H} \to \mathbb{R}$ is defined by

$$\mathrm{lev}_{<\alpha} g := \{x \in \mathscr{H} : g(x) < \alpha\}.$$

It is well-known that g is quasiconvex if and only if $\mathrm{lev}_{<\alpha} g$ is convex for all $\alpha \in \mathbb{R}$.

For a quasiconvex function $g : \mathscr{H} \to \mathbb{R}$, the Greenberg-Pierskalla quasi-subdifferential [10] at $x \in \mathscr{H}$ is defined by

$$\partial^{GP} g(x) = \big\{u \in \mathscr{H} : \langle u, y-x\rangle < 0,\ \forall y \in \mathrm{lev}_{<g(x)} g\big\}. \tag{2.3}$$

The following lemma guarantees the existence of the Greenberg-Pierskalla quasi-subdifferential of a quasiconvex function.

Lemma 2.1. [13, Lemma 3(e)] *If $g : \mathscr{H} \to \mathbb{R}$ is quasiconvex and upper semicontinuous on $\mathscr{H}$, then $\partial^{GP} g(x) \neq \emptyset$ for all $x \in \mathscr{H}$.*

Proposition 2.1. [10, Theorem 6] *Let $g : \mathscr{H} \to \mathbb{R}$ be a quasiconvex and upper semicontinuous function. Then $\partial^{GP} g(x)$ forms a cone and hence is unbounded for every $x \in \mathscr{H}$.*

Let C be a nonempty set in $\mathscr{H}$. Then for each $x \in \mathscr{H}$, the mapping $P_C : x \mapsto P_C(x)$ defined by

$$P_C(x) = \underset{z\in C}{\operatorname{argmin}} \|x-z\|,$$

is called the projection map onto C. When C is nonempty, closed and convex, the mapping P_C is single-valued and well defined for every $x \in \mathscr{H}$.

Proposition 2.2. [8] *Let C be a nonempty, closed and convex set in $\mathscr{H}$. Then, the projection map P_C satisfies the following inequalities:*

(a) $\|P_C(x) - P_C(y)\| \le \|x-y\|, \quad \forall x,y \in \mathscr{H}$.
(b) $\big\langle y - P_C(x), x - P_C(x)\big\rangle \le 0, \quad \forall x \in \mathscr{H},\ y \in C$.

Let C be a nonempty set in $\mathscr{H}$. A mapping $F : C \to \mathscr{H}$ is said to be γ-strongly pseudomonotone on C if there exists a constant $\gamma \in (0,\infty)$ such that for all $x,y \in C$,

$$\langle F(y), x-y\rangle \ge 0 \ \Rightarrow\ \langle F(x), x-y\rangle \ge \gamma\|x-y\|^2.$$

Furthermore, a bifunction $f : C \times C \to \mathbb{R}$ is called

(a) pseudomonotone on C if for all $x,y \in C$,

$$f(x,y) \ge 0 \quad \Rightarrow \quad f(y,x) \le 0;$$

(b) γ-strongly pseudomonotone on C if there exists $\gamma \in (0,\infty)$ such that for all $x,y \in C$,

$$f(x,y) \geq 0 \quad \Rightarrow \quad f(y,x) \leq -\gamma\|x-y\|^2;$$

(c) paramonotone on C with respect to $S(f,C)$ if

$$x \in S(f,C),\ y \in C \text{ and } f(x,y) = f(y,x) = 0 \quad \Rightarrow \quad y \in S(f,C).$$

For a bifunction $f : C \times C \to \mathbb{R}$, we use the notation $\partial^{GP} f(x,x)$ for the Greenberg-Pierskalla quasi-subdifferential of the function $y \mapsto f(x,y)$ at x for all $x \in C$.

Let C be a nonempty, closed and convex subset of $\mathscr{H}$. We make the following assumptions on the bifunction $f : C \times C \to \mathbb{R}$.

Assumption 1. (A0) $f(x,x) = 0$ *for all* $x \in C$.

(A1) $f(x,\cdot)$ *is quasiconvex for all* $x \in C$.

(A2) $f(x,\cdot)$ *is upper semicontinuous on* C.

(A3) f *is* γ*-strongly pseudomonotone on* C.

(A4) *The solution set* $S(f,C)$ *is nonempty.*

Now we give an example of a bifunction which is strongly pseudomonotone and upper semicontinuous with respect to the second argument, but it is not upper semicontinuous with respect to both arguments.

Example 2.1. *Let* $C = [0,2] \subset \mathbb{R}$. *Define* $f : C \times C \to \mathbb{R}$ *by*

$$f(x,y) = \begin{cases} \left(\frac{1}{|x-1|} - \frac{1}{2}\right) x(y-x), & \text{when } x \neq 1; \\ y-1, & \text{when } x = 1. \end{cases} \tag{2.4}$$

When $x = 1$, *then* $f(1,y) = y-1$ *for every* $y \in C$. *So,* $f(1,\cdot)$ *is continuous and hence, upper semicontinuous on* C.

When $x \neq 1$, *then obviously* $f(x,\cdot)$ *is continuous and, hence, upper semicontinuous on* C.

To check the upper semicontinuity of $f(\cdot,\cdot)$ *at* $(1,1.5)$, *let us consider the sequence* $(x_k,y_k)_{k\in\mathbb{N}}$ *in* $C \times C$ *with* $x_k = 1+\frac{1}{k+1}$ *and* $y_k = 1.5+\frac{1}{k+1}$ *for all* $k \in \mathbb{N}$. *Then* $\lim_{k\to\infty}(x_k,y_k) = (1,1.5)$. *But* $\limsup_{k\to\infty} f(x_k,y_k) = \limsup_{k\to\infty}\left(k+1-\frac{1}{2}\right)\left(1+\frac{1}{k+1}\right)(1.5-1) = +\infty > 0.5 = f(1,1.5)$. *So,* $f(x,y)$ *is not upper semicontinuous at* $(1,1.5)$.

We observe that $0 \leq x \leq 2$. *Hence,* $-1 \leq x-1 \leq 1$, *i.e.,* $|x-1| \leq 1$. *So, for* $x \neq 1$ *we have* $\frac{1}{|x-1|} \geq 1$, *and hence,* $\left(\frac{1}{|x-1|} - \frac{1}{2}\right) \geq \frac{1}{2}$ *for all* $x(\neq 1) \in C$. *Let* $x,y \in C$ *and* $f(x,y) \geq 0$. *Consider the following two cases:*

Case (i): *Suppose* $x = 1$. *Then,* $f(x,y) = y-1$ *and for* $y = 1$, $f(y,x) = 0 = -\gamma|x-y|^2$ *holds for any* $\gamma > 0$. *Assume that* $y \neq 1$ *and* $f(x,y) \geq 0 \Rightarrow y-1 \geq 0$, *then*

$$\begin{aligned} f(y,1) &= \left(\frac{1}{|y-1|} - \frac{1}{2}\right) y(1-y) \\ &\leq \left(\frac{1}{|y-1|} - \frac{1}{2}\right) (y(1-y) + (y-1)) \\ &\leq -\frac{1}{2}(1-y)^2. \end{aligned}$$

Case (ii): *Suppose* $x \neq 1$. *Then for* $x,y \in C$ *satisfying* $f(x,y) \geq 0$, *we have* $x(y-x) \geq 0$. *Now, if* $y = 1$, *then* $x(1-x) \geq 0$. *This implies that*

$$f(1,x) = x-1 \leq (x-1) + x(1-x) \leq -\frac{1}{2}(x-1)^2.$$

If $y \neq 1$, we have

$$\begin{aligned} f(y,x) = \left(\frac{1}{|y-1|} - \frac{1}{2}\right) y(x-y) &\leq \left(\frac{1}{|y-1|} - \frac{1}{2}\right) \{y(x-y) + x(y-x)\} \\ &= -\left(\frac{1}{|y-1|} - \frac{1}{2}\right) (x-y)^2 \\ &\leq -\frac{1}{2}(x-y)^2 \\ &= -\gamma \|x-y\|^2, \quad \text{with } \gamma = \frac{1}{2}. \end{aligned}$$

Therefore, f is γ-strongly pseudomonotone. Furthermore, observe that $f(x,.)$ is quasiconvex for every $x \in C$ and $f(0,y) = 0$ for all $y \in C$, i.e., $0 \in S(f,C)$. Thus, f satisfies all the conditions of Assumption 1.

The following result, which is related to a certain divergent series, will be used to discuss the convergence analysis.

Lemma 2.2. [18] *If $\{\xi_n\} \subseteq [0,1)$ is a sequence, then*

$$\sum_{n=1}^{\infty} \xi_n = +\infty \quad \Leftrightarrow \quad \prod_{n=1}^{\infty} (1 \pm \xi_n) = 0.$$

3. An Algorithm and Its Convergence Analysis

In this section, we extend the algorithm introduced in [11] for solving convex equilibrium problems to quasiconvex equilibrium problems. We use Greenberg-Pierskalla quasi-subgradient to propose our algorithm. We analyze that if the sequence generated by our algorithm terminates after a finite number of iterates, then the final iterated point is a solution of the EP (1.1). We further establish that the sequence generated by Algorithm 1 converges to a solution of EP (1.1) under some mild conditions.

Algorithm 1 (Projection Method for Quasiconvex EP)**.**

Step 0: *Fix $M > 1$. Take $x_0 \in C$, and a sequence $\{\beta_k\} \subset (0,+\infty)$ satisfying*

$$\beta_k \to 0 \quad \text{and} \quad \sum_{k=0}^{\infty} \beta_k = +\infty.$$

Set $k = 0$.

Step 1: *Take $u_k \in \partial^{GP} f(x_k,x_k)$ such that $\|u_k\| \leq M$ and*

$$\lambda_k = \frac{\beta_k}{\max\{1, \|u_k\|^2\}}, \tag{3.5}$$

Compute x_{k+1} as

$$x_{k+1} = P_C\big(x_k - \lambda_k u_k\big). \tag{3.6}$$

Step 2: *If $x_{k+1} = x_k$, then STOP: Otherwise update $k := k+1$ and go to Step 1.*

As the Greenberg-Pierskalla subdifferential encompasses the convex subdifferential, the algorithm presented in [11, Algorithm 4.1] can be viewed as a specific instance of the aforementioned algorithm.

Remark 3.1. *(a) It is clear that if $u_k = 0$, then obviously $x_{k+1} = x_k$, and hence, by Lemma 3.3, x_k is a solution of EP (1.1).*

(b) *If we normalize u_k, that is, $\|u_k\| = 1$, then from* (3.5)*, we have $\lambda_k = \beta_k$. In this case, Algorithm 1 is considered by Yen and Muu [20], and the convergence of the sequence is studied under the assumption that $f(\cdot,\cdot)$ is upper semicontinuous, and f is pseudomonotone as well as paramonotone.*

(c) *If we take $\beta_k = \frac{1}{k}$, then we see that $\sum_{k=1}^{\infty}\beta_k = +\infty$ and $\sum_{k=1}^{\infty}\beta_k^2 < +\infty$, which are the same conditions considered in [20]. Furthermore, $\beta_k = \frac{1}{\sqrt{k}}$ satisfies the parametric condition in Algorithm 1, but it does not satisfy the condition of the algorithm in [20]. So, in Algorithm 1, we have the opportunity to consider a larger class of parameters.*

Lemma 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$ and $f : C \times C \to \mathbb{R}$ be a bifunction that satisfies the assumptions* (A0)*,* (A1) *and* (A2)*. If $x_{k+1} = x_k$ for some $k \in \mathbb{N}$ in Algorithm 1, then $x_k \in S(f,C)$.*

Proof. Since x_{k+1} is the projection of $x_k - \lambda_k u_k$ onto the closed convex set C, we have

$$\left\langle y - x_{k+1}, x_k - \lambda_k u_k - x_{k+1} \right\rangle \leq 0, \quad \forall y \in C.$$

If $x_{k+1} = x_k$, then the above inequality reduces to

$$\left\langle y - x_k, -\lambda_k u_k \right\rangle \leq 0, \quad \forall y \in C,$$

that is,

$$\left\langle y - x_k, u_k \right\rangle \geq 0, \quad \forall y \in C. \tag{3.7}$$

On the other hand, $u_k \in \partial^{GP} f(x_k, x_k)$ implies that

$$\left\langle u_k, z - x_k \right\rangle < 0, \quad \forall z \in \mathrm{lev}_{<f(x_k,x_k)} f(x_k,\cdot) = \mathrm{lev}_{<0} f(x_k,\cdot).$$

Hence, it is observed from (3.7) that $y \notin \mathrm{lev}_{<0} f(x_k,\cdot)$ for all $y \in C$. This implies that $f(x_k,y) \geq 0$ for all $y \in C$. Thus, $x_k \in S(f,C)$. □

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathscr{H}$. If the bifunction $f : C \times C \to \mathbb{R}$ satisfies the assumptions* (A0)-(A4)*, then the sequence $\{x_k\}$ generated by Algorithm 1 converges to a solution of the equilibrium problem* (1.1).

Proof. From the definition of x_{k+1}, we have

$$\left\langle x_k - \lambda_k u_k - x_{k+1}, y - x_{k+1} \right\rangle \leq 0, \quad \forall y \in C,$$

that is,

$$\left\langle x_k - x_{k+1}, y - x_{k+1} \right\rangle \leq \lambda_k \left\langle u_k, y - x_{k+1} \right\rangle, \quad \forall y \in C. \tag{3.8}$$

Taking $y = x_k$ and using Schwarz inequality, the above inequality reduces to

$$\|x_{k+1} - x_k\|^2 \leq \lambda_k \left\langle u_k, x_k - x_{k+1} \right\rangle \leq \lambda_k \|u_k\| \|x_{k+1} - x_k\|.$$

Hence,

$$\|x_{k+1} - x_k\| \leq \lambda_k \|u_k\|. \tag{3.9}$$

Now consider two cases:

Case (i): Suppose $\|u_k\|^2 \leq 1$. Then, (3.5) gives $\lambda_k = \beta_k$. Hence, (3.9) turns to

$$\|x_{k+1} - x_k\| \leq \beta_k \|u_k\| \leq \beta_k.$$

Case (ii): If $\|u_k\| > 1$, then (3.5) implies $\lambda_k = \frac{\beta_k}{\|u_k\|^2}$. Therefore, (3.9) becomes

$$\|x_{k+1} - x_k\| \leq \frac{\beta_k}{\|u_k\|^2}\|u_k\| = \frac{\beta_k}{\|u_k\|} < \beta_k.$$

Thus, in both the cases, we obtain from (3.9) that

$$\|x_{k+1} - x_k\| \leq \beta_k, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{3.10}$$

Moreover, from (2.2), we have

$$\begin{aligned}\left\langle x_{k+1}-x_k, x_{k+1}-x^*\right\rangle &= -\left\langle x_k-x_{k+1}, x_{k+1}-x^*\right\rangle \\ &= -\left[\frac{1}{2}\|x_k-x^*\|^2-\frac{1}{2}\|x_{k+1}-x_k\|^2-\frac{1}{2}\|x_{k+1}-x^*\|^2\right].\end{aligned}$$

This refers to

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &= \|x_k-x^*\|^2-\|x_{k+1}-x_k\|^2+2\left\langle x_{k+1}-x_k, x_{k+1}-x^*\right\rangle \\ &\leq \|x_k-x^*\|^2-\|x_{k+1}-x_k\|^2+2\lambda_k\left\langle u_k, x^*-x_{k+1}\right\rangle, \qquad (3.11)\end{aligned}$$

where the last inequality follows from (3.8).

Furthermore, consider a set

$$I=\left\{k\in\mathbb{N}:\left\langle u_k, x^*-x_{k+1}\right\rangle \geq \frac{\gamma}{2}\|x_k-x^*\|^2\right\}. \qquad (3.12)$$

For all $i\in I$, we analyze from (3.5) that

$$\beta_i=\lambda_i\max\{1,\|u_i\|^2\}\geq\lambda_i\|u_i\|^2=\lambda_i\|u_i\|\|u_i\|.$$

Using (3.9) and the definition of set I, we get

$$\begin{aligned}\beta_i \geq \|x_{i+1}-x_i\|\|u_i\| &\geq \left\langle u_i, x_i-x_{i+1}\right\rangle \\ &= \left\langle u_i, x_i-x^*\right\rangle+\left\langle u_i, x^*-x_{i+1}\right\rangle \\ &\geq \left\langle u_i, x_i-x^*\right\rangle+\frac{\gamma}{2}\|x_i-x^*\|^2. \qquad (3.13)\end{aligned}$$

Since $u_i\in\partial^{GP}f(x_i,x_i)$, we have

$$\left\langle u_i, y-x_i\right\rangle<0, \quad \forall y\in \operatorname{lev}_{<0}f(x_i,\cdot). \qquad (3.14)$$

Also, since $x^*\in S(f,C)$, it follows that $f(x^*,x_i)\geq 0$, and hence, strong pseudomonotonicity of f implies that

$$f(x_i,x^*)\leq-\gamma\|x_i-x^*\|^2<0.$$

So, $x^*\in\operatorname{lev}_{<0}f(x_i,\cdot)$. Then from (3.14), we obtain

$$\left\langle u_i, x^*-x_i\right\rangle<0, \quad \text{i.e.,} \quad \left\langle u_i, x_i-x^*\right\rangle>0.$$

This together with (3.13) implies that

$$\beta_i>\frac{\gamma}{2}\|x_i-x^*\|^2,$$

and hence,

$$\|x_i-x^*\|<\sqrt{\frac{2\beta_i}{\gamma}}, \quad \forall i\in I. \qquad (3.15)$$

Since the set I defined by (3.12) could be finite or infinite, we consider two cases.

Case 1. I is a finite set. Then, $m=\max\left\{i:i\in I\right\}+1\notin I$, and hence,

$$\left\langle u_m, x^*-x_{m+1}\right\rangle<\frac{\gamma}{2}\|x_m-x^*\|^2.$$

Then, for all $k\geq m$, this implies with (3.11) that

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &< (1+\lambda_k\gamma)\|x_k-x^*\|^2 \\ &< (1+\lambda_k\gamma)(1+\lambda_{k-1}\gamma)\cdots(1+\lambda_m\gamma)\|x_m-x^*\|^2 \\ &= \prod_{i=m}^{k}(1+\lambda_i\gamma)\|x_m-x^*\|^2.\end{aligned}$$

Then, the sequence $\{x_k\}$ is bounded. Moreover, by the choice of u_k in Step 1, we have $\|u_k\| \leq M$ for all $k \in \mathbb{N}$. Therefore, from (3.5), we have

$$\sum_{k=0}^{\infty} \lambda_k \geq \frac{1}{\max\{1; M^2\}} \sum_{k=0}^{\infty} \beta_k = \frac{1}{M^2} \sum_{k=0}^{\infty} \beta_k.$$

Since $\sum\limits_{k=0}^{\infty} \beta_k = \infty$, it follows from the above inequality that $\sum\limits_{k=0}^{\infty} \lambda_k = \infty$. Thus, from Lemma 2.2, we obtain

$$\lim_{k\to\infty} \prod_{i=m}^{k} (1+\lambda_i \gamma) = 0,$$

and hence,

$$\lim_{k\to\infty} x_k = x^*.$$

Case 2. I is an infinite set. Then, we show that for an arbitrary $\varepsilon > 0$, there exists $k_0 \in I$ such that $\|x_k - x^*\| < \varepsilon$ for all $k \geq k_0$.

Now, consider two cases:

Case (a): If $k \notin I$ and let $m(k) = \max\{i \in I : i < k\}$, then $k > m(k) \geq k_0$. Let $\varepsilon > 0$ be arbitrary. Since $\beta_k \to 0$ and I is an infinite set, there exists $k_0 \in I$ such that

$$\max\left\{\beta_k, \sqrt{\frac{2\beta_k}{\gamma}}\right\} \leq \frac{\varepsilon}{2\sqrt{(1+\lambda_{m(k)+1}\gamma)}}, \quad \forall k \geq k_0.$$

Furthermore, from (3.11), we observe that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\leq \|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2 + \lambda_k \gamma \|x_k - x^*\|^2 \\ &= (1+\lambda_k\gamma)\|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2, \quad \forall k \notin I.\end{aligned}$$

Using (3.10), (3.15) and the above inequality, we get

$$\begin{aligned}\|x_k - x^*\|^2 &\leq \left(1+\lambda_{m(k)+1}\gamma\right)\|x_{m(k)+1} - x^*\|^2 - \|x_k - x_{m(k)+1}\|^2 \\ &\leq \left(1+\lambda_{m(k)+1}\gamma\right)\|x_{m(k)+1} - x^*\|^2,\end{aligned}$$

and so,

$$\begin{aligned}\|x_k - x^*\| &= \sqrt{\left(1+\lambda_{m(k)+1}\gamma\right)}\|x_{m(k)+1} - x^*\| \\ &\leq \sqrt{\left(1+\lambda_{m(k)+1}\gamma\right)}\left\{\|x_{m(k)+1} - x_{m(k)}\| + \|x_{m(k)} - x^*\|\right\} \\ &\leq \sqrt{\left(1+\lambda_{m(k)+1}\gamma\right)}\left\{\beta_{m(k)} + \sqrt{\frac{2\beta_{m(k)}}{\gamma}}\right\} \\ &\leq \sqrt{\left(1+\lambda_{m(k)+1}\gamma\right)}\left\{\frac{\varepsilon}{2\sqrt{(1+\lambda_{m(k)+1}\gamma)}} + \frac{\varepsilon}{2\sqrt{(1+\lambda_{m(k)+1}\gamma)}}\right\} \\ &= \varepsilon.\end{aligned}$$

Case (b): If $k \in I$, then from (3.15), we have

$$\|x_k - x^*\| < \sqrt{\frac{2\beta_k}{\gamma}} \leq \frac{\varepsilon}{2\sqrt{(1+\lambda_{m(k)+1}\gamma)}} < \varepsilon.$$

Hence, $x_k \to x^*$. This completes the desired result. $\square$

4. Numerical Examples

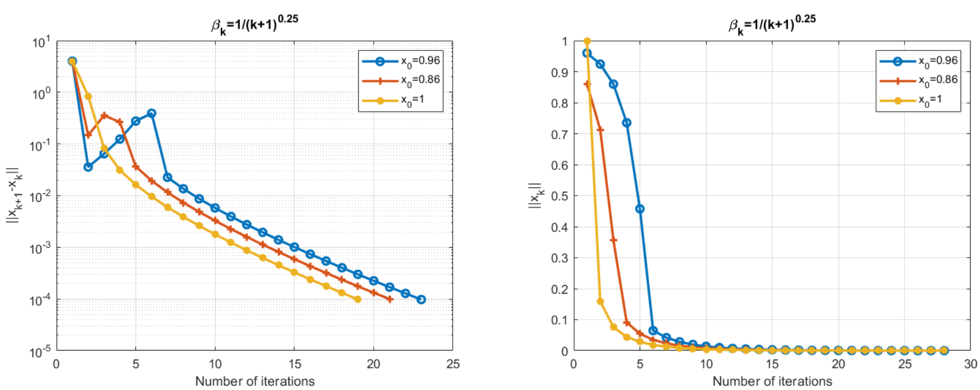
This section illustrates the functionality of the composed Algorithm 1 through large-scale examples. The computational programming is performed in MATLAB R2024a running on Acer Swift 3 PC with 11th Gen Intel(R) Core(TM) i5 @ 2.42 GHz processor and RAM 16.0 GB.

Example 4.2. *Consider the nonempty closed and convex set $C = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$ in $\mathbb{R}$ and a bifunction $f : C \times C \to \mathbb{R}$ defined as in Example 2.1 which is quasiconvex. Note that zero is the unique solution of the EP* (1.1) *whose bifunction is given by* (2.4)*.*

As we have seen in Example 2.1 all the conditions of Algorithm 1 are satisfied. Consider the parameter $\beta_k = \frac{1}{(k+1)^\alpha}$, $\quad 0 < \alpha < 1$ for the Algorithm 1. We analyze Algorithm 1 with random initial points $x_0 = 0.96,\ 0.86,\ 1$ and $\alpha = 0.25$. We take $\|x_{k+1} - x_k\| < 10^{-4}$ and $\|x_k - \bar{x}\| < 10^{-4}$ as the stopping criterion to check the convergence of the sequence $\{x_k\}$ to the unique solution $\bar{x} = 0$ of EP (1.1)*. The results are shown in Figure 1. The CPU times and number of iterations corresponding to different initial points are shown in Table 1.*

Error	*Initial Point* (x_0)	*No. of Iterations*	*CPU Times (Secs)*
$\|x_{k+1} - x_k\| \leq 10^{-4}$	0.96	*22*	*6.91*$\times 10^{-5}$
	0.86	*20*	*5.33*$\times 10^{-5}$
	1	*18*	*5.13*$\times 10^{-5}$
$\|x_k - \bar{x}\| < 10^{-4}$	0.96	*28*	*9.79*$\times 10^{-5}$
	0.86	*26*	*6.69*$\times 10^{-5}$
	1	*24*	*6.43*$\times 10^{-5}$

Table 1. Table of CPU Times for Example 4.2



(A) $\|x_{k+1} - x_k\| \to 0$ (B) $\|x_k\| \to 0$

Figure 1. Convergence of Example 4.2

Example 4.3. *Let E, P be two $n \times n$ symmetric and positive definite matrices, $e^\top$, $c^\top$, $b^\top \in \mathbb{R}^n$, and d, $g \in \mathbb{R}$. Consider a nonempty closed convex subset C of $\mathbb{R}^n$ defined by*

$$C = \{X \in \mathbb{R}^n : m < \langle c, X\rangle + d < M\},$$

with $0 < m < M < +\infty$. Define $F : C \to \mathbb{R}^n$ and $G : C \to \mathbb{R}$ by

$$F(X) = \frac{EX + e}{\langle c, X\rangle + d}, \qquad G(X) = \frac{\frac{1}{2}\langle PX, X\rangle + \langle b, X\rangle + g}{\langle c, X\rangle + d}.$$

Let the bifunction $f : C \times C \to \mathbb{R}$ be defined by

$$f(X,Y) = \langle X, F(Y) - F(X)\rangle + G(Y) - G(X).$$

Note that the function $f(X,\cdot)$ is quasiconvex for all $X \in C$ (see, [14, Example 4.1]). Furthermore, it is clear that $f(X,\cdot)$ is upper semicontinuous on C. We show that the operator F is strongly pseudomonotone on C. Suppose $\langle F(Y), X-Y\rangle \geq 0$ for any $X,Y \in C$, then we have

$$\left\langle \frac{EY+e}{\langle c,Y\rangle + d}, X-Y \right\rangle \geq 0,$$

that is,

$$\langle EY+e, X-Y\rangle \geq 0. \tag{4.16}$$

This implies that

$$\begin{aligned}
\langle F(X), X-Y\rangle &= \left\langle \frac{EX+e}{\langle c,X\rangle + d}, X-Y \right\rangle \\
&= \frac{1}{\langle c,X\rangle + d}\langle EX+e, X-Y\rangle \\
&\geq \frac{1}{M}\langle EX+e, X-Y\rangle \\
&= \frac{1}{M}\langle E(X-Y), X-Y\rangle + \frac{1}{M}\langle EY+e, X-Y\rangle \\
&\geq \frac{1}{M}\langle E(X-Y), X-Y\rangle \\
&\geq \gamma\|X-Y\|^2, \text{ for some } \gamma \in (0,+\infty).
\end{aligned}$$

The last inequality follows from the fact that E is positive definite. Using strong pseudomonotonicity of F, we can easily show that the bifunction f is γ-strongly pseudomonotone on C. Therefore, f satisfies all conditions of Assumption 1.

Using different initial choices X_0 and stopping conditions $\|X_{k+1} - X_k\| < 10^{-4}$, we investigate Algorithm 1. The convergence of the composed Algorithm 1 to a solution of EP (1.1) *is shown for $n = 2,\ 5,\ 10,\ 50,\ 250$ and $\beta_k = \frac{1}{(k+1)^\alpha}$, $(0 < \alpha \leq 1)$ with $\alpha = 1,\ 0.9$ and 0.7.*

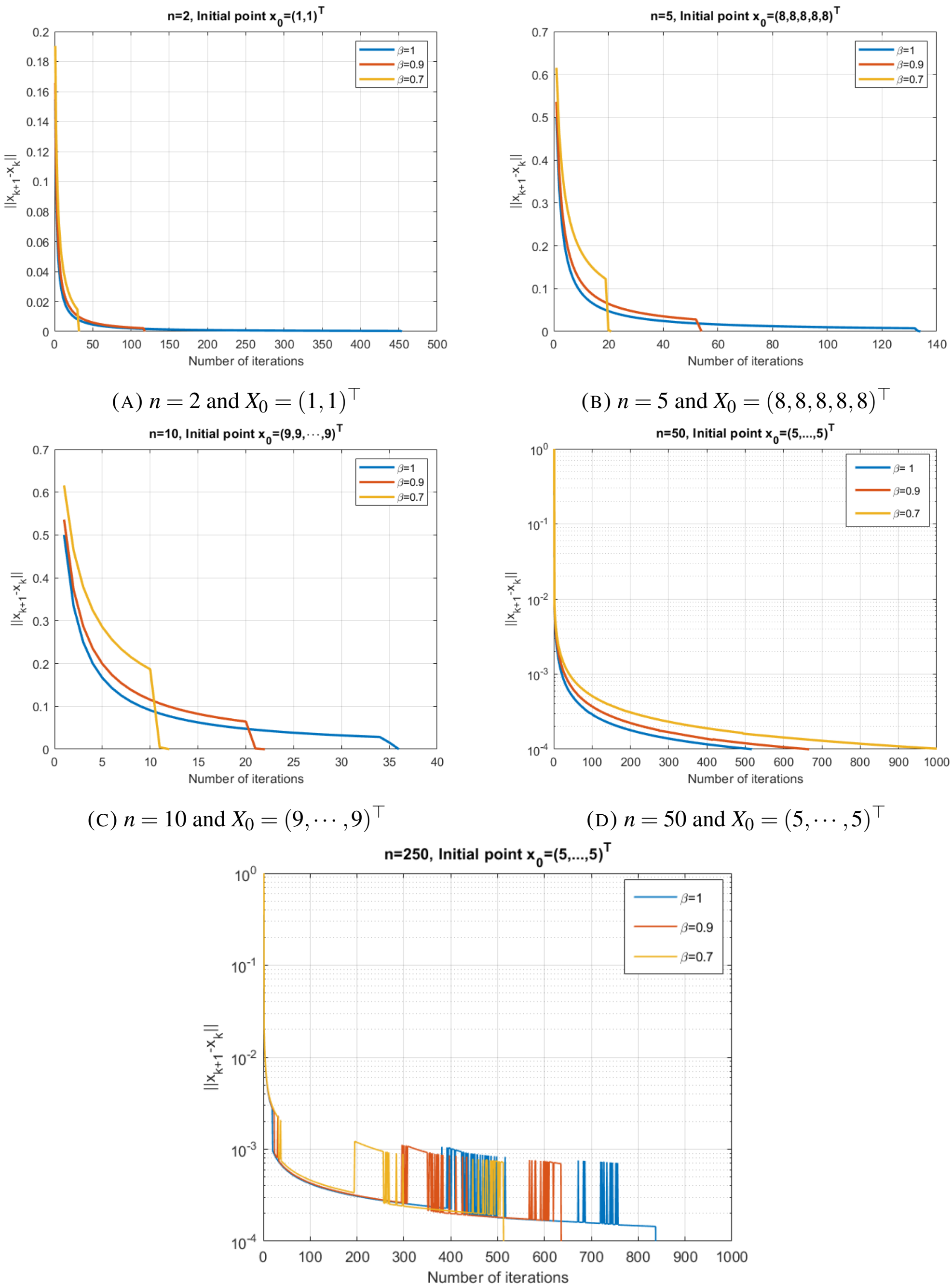
For $n = 2$, we take $C = [0,10]^2 \subset \mathbb{R}^2$,

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad e = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, \qquad c = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, \qquad d = 6,$$
$$P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, \qquad g = 1.$$

Figure 2, (A) illustrates the convergence result for the initial point $X_0 = (1,1)^\top$. When $n > 2$, we consider $C = [0,10]^n \subset \mathbb{R}^n$ and E, P, which are symmetric, positive definite, randomly generated matrices with nonnegative elements. In the interval $(0,1)$, the vectors e, c, b and the scalars g, d are considered at random. The convergence results for $n = 5$, 10, 50, and 250 are displayed in subfigures (B), (C), (D) and (E) of Figure 2, in that order. Table 2 displays the CPU times and number of iterations for various values of n, which correspond to various initial locations and step sizes.

Remark 4.2. *In Example 4.2 and Example 4.3, $f(x,\cdot)$ is assumed quasiconvex and upper semicontinuous for all $x \in C$, while in* [11] *the convexity of $f(x,\cdot)$ is assumed, in* [20] *the upper semicontinuity of $f(\cdot,\cdot)$ is assumed. So, by Algorithm 1 we can solve a larger class of equilibrium problems which cannot be solved by other methods proposed in* [11,20].

Remark 4.3. *From Table 2, it is clear that the number of iterations depends on both the dimension* (n) *and the choice of the initial point* (X_0).



(A) $n = 2$ and $X_0 = (1,1)^\top$

(B) $n = 5$ and $X_0 = (8,8,8,8,8)^\top$

(C) $n = 10$ and $X_0 = (9,\cdots,9)^\top$

(D) $n = 50$ and $X_0 = (5,\cdots,5)^\top$

(E) $n = 250$ and $X_0 = (5,\cdots,5)^\top$

FIGURE 2. Convergence of Example 4.3.

n	Initial Point	α	No. of Iterations	CPU Times (Secs)
2	$X_0 = (1,1)^\top$	1.0	453	6.7189
		0.9	118	1.5788
		0.7	32	0.3374
5	$X_0 = (8,8,8,8,8)^\top$	1.0	134	0.5470
		0.9	54	0.1752
		0.7	20	0.1203
10	$X_0 = (9,\cdots,9)^\top$	1.0	36	1.0790
		0.9	22	0.6127
		0.7	12	0.2761
50	$X_0 = (5,\cdots,5)^\top$	1.0	500	13.9800
		0.9	690	16.9100
		0.7	1000	18.0100
250	$X_0 = (5,\cdots,5)^\top$	1.0	830	23.1900
		0.9	635	30.4300
		0.7	510	34.1600

TABLE 2. Table of CPU Times for Example 4.3

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[1]ALIGARH MUSLIM UNIVERSITY
DEPARTMENT OF MATHEMATICS
ALIGARH, INDIA
Email address: qhansari@gmail.com; msarkar704@gmail.com

[2]COLLEGE OF SCIENCES
CHONGQING UNIVERSITY OF TECHNOLOGY
CHONGQING 400054, CHINA
Email address: qhansari@gmail.com

[3]MATHEMATICS DIVISION, SCHOOL OF ADVANCED SCIENCES & LANGUAGES
VIT BHOPAL UNIVERSITY
BHOPAL-INDORE HIGHWAY, KOTHRIKALAN, SEHORE
MADHYA PRADESH 466 114, INDIA
Email address: firoz77b@gmail.com; feerozbabu@vitbhopal.ac.in