CARPATHIAN J. MATH.
Volume **41** (2025), No. 1,
Pages 205-222

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2025.01.14

Two new extragradient methods for solving pseudomonotone the equilibrium problem in Hilbert spaces

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ABSTRACT. This paper introduces two new extragradient methods designed to solve pseudomonotone equilibrium problems subject to a Lipschitz-type condition. These methods incorporate a variable stepsize criterion that dynamically adjusts with each iteration based on prior iterations. A distinguishing feature of these methods is their independence from prior knowledge of Lipschitz-type constants or any line-search method. The convergence theorems for the proposed methods are established under mild conditions, without requiring the knowledge of Lipschitz-type constants. Additionally, the paper includes several investigations demonstrating the numerical efficacy of the methods and facilitating comparisons with other approaches. This paper contributes to the advancement of computational methods for addressing pseudomonotone equilibrium problems across various applications.

1. INTRODUCTION

This research focuses on developing new iterative methods to solve the equilibrium problem, represented by (EP). Here, $\mathcal{H}$ represents a real Hilbert space, whereas $\mathcal{C}$ denotes a nonempty closed convex subset of $\mathcal{H}$. The bifunction $\mathcal{R} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ plays a central role, satisfying the condition $\mathcal{R}(z_1, z_1) = 0$ for every $z_1 \in \mathcal{C}$.

Formally, the equilibrium problem associated with the bifunction $\mathcal{R}$ on $\mathcal{C}$ is articulated as follows: Our objective is to find $s^* \in \mathcal{C}$ such that

$$\text{(EP)} \qquad \mathcal{R}(s^*, z_1) \geq 0 \quad \text{for all} \quad z_1 \in \mathcal{C}.$$

To put it simply, we want to find an element s^* in the closed convex set $\mathcal{C}$ that satisfies the inequality shown in (EP) for any z_1 in $\mathcal{C}$. This fundamental problem, described by (EP), represents an important task in the realm of real Hilbert spaces. Consequently, our research aims to move forward by introducing new iterative methods specifically designed to address this problem.

This study is devoted to the numerical investigation of the equilibrium problem under specific conditions outlined as follows. These conditions provide a comprehensive framework for understanding the equilibrium problem and serve as the foundation for the numerical characterization pursued in this research.

($\mathcal{R}1$) The set $Sol(\mathcal{R}, \mathcal{C})$, representing the solution set of the problem (EP), and it is assumed to be nonempty.

($\mathcal{R}2$) The bifunction $\mathcal{R}$ is *pseudomonotone* [6, 4], indicating that:

$$\mathcal{R}(z_1, z_2) \geq 0 \Longrightarrow \mathcal{R}(z_2, z_1) \leq 0 \; \forall \; z_1, z_2 \in \mathcal{C}.$$

Received: 16.01.2024. In revised form: 20.06.2024. Accepted: 10.07.2024
2010 *Mathematics Subject Classification.* 47J25, 47H09, 47H06, 47J05.
Key words and phrases. *Equilibrium problem; Pseudomonotone bifunction; Weak convergence theorem; Variational inequality problem.*
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($\mathcal{R}3$) Furthermore, $\mathcal{R}$ is *Lipschitz-like continuous* [17] on $\mathcal{C}$ if there exist positive constants c_1 and c_2 such that:

$$\mathcal{R}(z_1, z_3) \leq \mathcal{R}(z_1, z_2) + \mathcal{R}(z_2, z_3) + c_1\|z_1 - z_2\|^2 + c_2\|z_2 - z_3\|^2 \ \forall \ z_1, z_2, z_3 \in \mathcal{C}.$$

($\mathcal{R}4$) Moreover, for each sequence $\{z_k\} \subset \mathcal{C}$ satisfying $z_k \rightharpoonup z^*$, the following inequality holds:

$$\limsup_{k \to +\infty} \mathcal{R}(z_k, z_1) \leq \mathcal{R}(z^*, z_1) \ \forall \ z_1 \in \mathcal{C}.$$

($\mathcal{R}5$) $\mathcal{R}(z_1, \cdot)$ is convex and subdifferentiable on $\mathcal{H}$ for any fixed $z_1 \in \mathcal{H}$.

The equilibrium problem is very important in academic research because it provides a unifying framework for addressing a wide range of mathematical problems. These problems include a diverse array of problem types, including vector and scalar minimization problems, fixed-point problems, complementarity problems, variational inequalities, saddle point problems, inverse optimization problems, and Nash equilibrium problems in non-cooperative games (for more details, see [6, 18, 10, 15, 5]). However, the relevance of the equilibrium problem extends beyond its mathematical domain, finding practical applications across various economic contexts. Notably, it has been employed in seminal economic investigations such as Cournot's research [9], elucidating the dynamics of supply and demand [2], and forming a foundational concept in the theoretical framework of non-cooperative games and Nash equilibrium models [20, 19]. The term "equilibrium problem" was first used in the academic literature by Muu and Oettli in 1992 [18], and subsequently subjected to more examination by Blum in 1994 [6]. This interpretation of the equilibrium problem has proved adaptive, allowing it to include a wide range of mathematical and economic phenomena, highlighting its importance in both theoretical and practical research.

The extragradient method, initially proposed by Flåm and Antipin [11] and subsequently refined by Tran et al. [23], has emerged as a valuable numerical approach for tackling equilibrium problems. This iterative scheme involves determining the next iteration, s_{k+1}, based on the current iteration s_k. The method is delineated by the following steps:

$$\text{(1.1)} \qquad \begin{cases} t_k = \arg\min_{t \in \mathcal{C}} \{\lambda \mathcal{R}(s_k, t) + \frac{1}{2}\|s_k - t\|^2\}, \\ s_{k+1} = \arg\min_{t \in \mathcal{C}} \{\lambda \mathcal{R}(t_k, t) + \frac{1}{2}\|s_k - t\|^2\}, \end{cases}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, with c_1 and c_2 denoting previously defined Lipschitz-type constants. The iterative nature of the method entails solving two minimization problems on the feasible set $\mathcal{C}$ using the prescribed stepsize λ. The technique described in Equation (1.1) is also referred to as the two-step extragradient method, attributed to Korpelevich's pioneering work [16] for solving saddle point problems. However, it is worth noting two significant limitations of this approach. Firstly, it relies on a fixed step size, necessitating prior knowledge or estimation of the Lipschitz constant associated with the relevant bifunction. Secondly, it only converges weakly in Hilbert spaces, given that Lipschitz constants are often unknown or challenging to compute. Recent research endeavors have explored various avenues to effectively address these shortcomings in the context of equilibrium problems, as extensively reviewed in [26, 27, 25, 28, 13, 12, 24].

From a computational viewpoint, determining the Lipschitz constant before use can be difficult, limiting the method's applicability in scenarios where these constants are unknown. As a result, a pertinent question arises.

Is it possible to introduce an extragradient method with an adaptive stepsize rule for solving pseudomonotone equilibrium problems (EP)?

This research aims to devise explicit-type methods that yield weak convergence sequences akin to the gradient approach when addressing equilibrium problems involving pseudomonotone bifunctions. We propose new extragradient-type methods tailored for solving equilibrium problems within an infinite-dimensional real Hilbert space, drawing upon the foundational work of Censor et al. [8] and Hieu et al. [14]. Our contributions to this study are as follows:

- *Subgradient Extragradient Approach*: We introduce a subgradient extragradient method tailored for solving equilibrium problems within a real Hilbert space. Notably, our approach features a monotone variable stepsize rule, a key aspect that we thoroughly analyze to establish the weak convergence of the generated sequence.
- *Solving Variational Inequality and Fixed Point Problems*: Our proposed method's versatility extends beyond equilibrium problems, proving beneficial for variational inequality and fixed point problems as well. This adaptability significantly broadens its impact across related problem domains.
- *Numerical Validation*: We conduct extensive numerical experiments to validate our theoretical findings and showcase the practical effectiveness of our proposed methods. These experiments include comparisons with previously reported results, enabling a comprehensive assessment of our approaches. Our numerical results affirm that the proposed methods not only align with theoretical expectations but also surpass existing methodologies in terms of performance.

The paper is structured as follows: Section 2 presents fundamental identities and lemmas crucial for conducting convergence analysis. Section 3 introduces new methods and explores their convergence properties. In Section 4, we assess the efficacy of our proposed method through numerical experiments and comparative analyses against existing methodologies, demonstrating its superior performance.

2. Preliminaries

In this section, we establish the foundation by presenting essential identities, key lemmas, and fundamental definitions. We commence by defining the *metric projection* $P_{\mathcal{C}}(z_1)$ of $z_1 \in \mathcal{H}$, formulated as follows:

$$P_{\mathcal{C}}(z_1) = \underset{z_2 \in \mathcal{C}}{\arg\min}\{\|z_1 - z_2\|\}.$$

The subsequent lemma elaborates on the fundamental properties of the projection mapping.

Lemma 2.1. [3] *Let $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ represent a metric projection. Then, it satisfies the following properties:*

(i)
$$\|z_1 - P_{\mathcal{C}}(z_2)\|^2 + \|P_{\mathcal{C}}(z_2) - z_2\|^2 \leq \|z_1 - z_2\|^2, \ z_1 \in \mathcal{C}, z_2 \in \mathcal{H};$$

(ii) $z_3 = P_{\mathcal{C}}(z_1)$ *if and only if*
$$\langle z_1 - z_3, z_2 - z_3 \rangle \leq 0, \quad \forall\, z_2 \in \mathcal{C};$$

(iii)
$$\|z_1 - P_{\mathcal{C}}(z_1)\| \leq \|z_1 - z_2\|, \quad z_2 \in \mathcal{C}, z_1 \in \mathcal{H}.$$

Another lemma from [3] is as follows:

Lemma 2.2. [3] *For any $z_1, z_2 \in \mathcal{H}$ and $\ell \in \mathbb{R}$, the following properties hold:*

(i)
$$\|\ell z_1 + (1-\ell) z_2\|^2 = \ell\|z_1\|^2 + (1-\ell)\|z_2\|^2 - \ell(1-\ell)\|z_1 - z_2\|^2.$$

(ii)

$$\|z_1 + z_2\|^2 \le \|z_1\|^2 + 2\langle z_2, z_1 + z_2 \rangle.$$

Continuing, the *normal cone of* $\mathcal{C}$ at $z_1 \in \mathcal{C}$ is defined as

$$N_{\mathcal{C}}(z_1) = \{z_3 \in \mathcal{H} : \langle z_3, z_2 - z_1 \rangle \le 0 \ \forall \ z_2 \in \mathcal{C}\}.$$

Consider a convex function $\mho : \mathcal{C} \to \mathbb{R}$. The *subdifferential of* $\mho$ at $z_1 \in \mathcal{C}$ is defined as

$$\partial\mho(z_1) = \{z_3 \in \mathcal{H} : \mho(z_2) - \mho(z_1) \ge \langle z_3, z_2 - z_1 \rangle \ \forall \, z_2 \in \mathcal{C}\}.$$

Each element in the set $\partial\mho(z_1)$ is referred to as a *subgradient* of the function $\mho$ at the point z_1. If a function $\mho$ has at least one subgradient at z_1, it is considered to be *subdifferentiable* at z_1.

Lemma 2.3. [22] *Let $\mho : \mathcal{C} \to \mathbb{R}$ be a convex, lower semicontinuous function that is subdifferentiable on $\mathcal{C}$. An element $s \in \mathcal{C}$ is a minimizer of the function $\mho$ if and only if*

$$0 \in \partial\mho(s) + N_{\mathcal{C}}(s),$$

where $\partial\mho(s)$ represents the subdifferential of $\mho$ at $s \in \mathcal{C}$, and $N_{\mathcal{C}}(s)$ represents the normal cone of $\mathcal{C}$ at s.

Lemma 2.4. [21] *Let $\mathcal{C}$ be a nonempty subset of $\mathcal{H}$ and $\{s_k\}$ be a sequence in $\mathcal{H}$ satisfying the following conditions:*

(i) *For any $s \in \mathcal{C}$, $\lim_{k\to+\infty} \|s_k - s\|$ exists;*

(ii) *Any sequentially weak cluster element of $\{s_k\}$ belongs to $\mathcal{C}$.*

Then, $\{s_k\}$ converges weakly to an element in $\mathcal{C}$.

3. Main Results

In this section, we introduce a numerical iterative approach aimed at enhancing the efficiency of the extragradient method (1.1). This method utilizes a monotone stepsize rule to facilitate its operation. Below, we offer a detailed explanation of the method.

Algorithm 1 (Improved Subgradient Extragradient Method)

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(1) **Input:** Provide initial parameters $\lambda_1 > 0$, $s_1 \in \mathcal{H}$, $\mu \in (0, 1)$, $\theta \in (0, 2 - \sqrt{2})$.
(2) **Output:** Obtain a convergent sequence $\{s_k\}$.
(3) **Initialization:** Set $k = 0$.
(4) **Iteration:**

Step 1: Compute $t_k = \arg\min_{t\in\mathcal{C}} \left\{\lambda_k \mathcal{R}(s_k, t) + \frac{1}{2}\|s_k - t\|^2\right\}$.
If $s_k = t_k$, terminate.

Step 2: Take $\omega_k \in \partial_2\mathcal{R}(s_k, t_k)$ such that $s_k - \lambda_k\omega_k - t_k \in N_{\mathcal{C}}(t_k)$. Create a set

$$\mathcal{H}_k = \{z \in \mathcal{H} : \langle s_k - \lambda_k\omega_k - t_k, z - t_k \rangle \le 0\}.$$

Step 3: Compute $s_{k+1} = \arg\min_{t\in\mathcal{H}_k} \left\{\lambda_k \mathcal{R}(t_k, t) + \frac{1}{2}\|s_k - t\|^2\right\}$.

Step 4: Compute the stepsize λ_{k+1} as follows:

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \ \frac{\mu}{2} \cdot \frac{(2-\sqrt{2}-\theta)\|s_k - t_k\|^2 + (2-\sqrt{2}-\theta)\|s_{k+1} - t_k\|^2}{[\mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) - \mathcal{R}(t_k, s_{k+1})]}\right\} \\ \quad \text{if} \quad \mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) - \mathcal{R}(t_k, s_{k+1}) > 0, \\ \lambda_k \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Otherwise.} \end{cases} \tag{3.2}$$

Step 5: Set $k := k + 1$ and return to **Step 1**.

Lemma 3.5. *The sequence $\{\lambda_k\}$, as defined in* (3.2)*, converges to λ and is bounded by*

$$\min\left\{\frac{\mu(2-\sqrt{2}-\theta)}{\max\{2c_1,2c_2\}},\lambda_1\right\}\leq\lambda\leq\lambda_1.$$

Proof. Consider the condition $\mathcal{R}(s_k,s_{k+1})-\mathcal{R}(s_k,t_k)-\mathcal{R}(t_k,s_{k+1})>0$, we have:

$$\begin{aligned}&\frac{\mu(2-\sqrt{2}-\theta)(\|s_k-t_k\|^2+\|s_{k+1}-t_k\|^2)}{2[\mathcal{R}(s_k,s_{k+1})-\mathcal{R}(s_k,t_k)-\mathcal{R}(t_k,s_{k+1})]}\\&\geq\frac{\mu(2-\sqrt{2}-\theta)(\|s_k-t_k\|^2+\|s_{k+1}-t_k\|^2)}{2[c_1\|s_k-t_k\|^2+c_2\|s_{k+1}-t_k\|^2]}\\&\geq\frac{\mu(2-\sqrt{2}-\theta)}{2\max\{c_1,c_2\}}.\end{aligned}\tag{3.3}$$

From (3.3), it is evident that the sequence is bounded. Additionally, since $\{\lambda_k\}$ is monotonically decreasing, we deduce that $\lim_{k\to+\infty}\lambda_k=\lambda$. □

Lemma 3.6. *Algorithm 1 produces the following important inequality:*

$$\lambda_k\mathcal{R}(t_k,t)-\lambda_k\mathcal{R}(t_k,s_{k+1})\geq\langle s_k-s_{k+1},t-s_{k+1}\rangle\ \forall\,t\in\mathcal{H}_k.$$

Proof. By employing the expression for s_{k+1} from Algorithm 1 and insights from Lemma 2.3, we establish that

$$0\in\partial_2\Big\{\lambda_k\mathcal{R}(t_k,\cdot)+\frac{1}{2}\|s_k-\cdot\|^2\Big\}(s_{k+1})+N_{\mathcal{H}_k}(s_{k+1}).$$

This implies that for $\upsilon\in\partial_2\mathcal{R}(t_k,s_{k+1})$, there exists a vector $\overline{\upsilon}\in N_{\mathcal{H}_k}(s_{k+1})$ such that

$$\lambda_k\upsilon+s_{k+1}-s_k+\overline{\upsilon}=0.$$

Hence, we obtain

$$\langle s_k-s_{k+1},t-s_{k+1}\rangle=\lambda_k\langle\upsilon,t-s_{k+1}\rangle+\langle\overline{\upsilon},t-s_{k+1}\rangle\ \forall\,t\in\mathcal{H}_k.$$

Since $\overline{\upsilon}\in N_{\mathcal{H}_k}(s_{k+1})$, it follows that $\langle\overline{\upsilon},t-s_{k+1}\rangle\leq0$ for all $t\in\mathcal{H}_k$. Therefore,

$$\langle s_k-s_{k+1},t-s_{k+1}\rangle\leq\lambda_k\langle\upsilon,t-s_{k+1}\rangle\ \forall\,t\in\mathcal{H}_k.\tag{3.4}$$

Given $\upsilon\in\partial_2\mathcal{R}(t_k,s_{k+1})$, we have

$$\mathcal{R}(t_k,t)-\mathcal{R}(t_k,s_{k+1})\geq\langle\upsilon,t-s_{k+1}\rangle\ \forall\,t\in\mathcal{H}.\tag{3.5}$$

Combining (3.4) and (3.5), we arrive at

$$\lambda_k\mathcal{R}(t_k,t)-\lambda_k\mathcal{R}(t_k,s_{k+1})\geq\langle s_k-s_{k+1},t-s_{k+1}\rangle\ \forall\,t\in\mathcal{H}_k.$$

□

Lemma 3.7. *An insightful inequality, deduced from Algorithm 1, is expressed as follows:*

$$\lambda_k\big\{\mathcal{R}(s_k,s_{k+1})-\mathcal{R}(s_k,t_k)\big\}\geq\langle s_k-t_k,s_{k+1}-t_k\rangle.$$

Proof. Given that $s_{k+1}\in\mathcal{H}_k$, the condition

$$\langle s_k-\lambda_k\omega_k-t_k,s_{k+1}-t_k\rangle\leq0,$$

implies

$$\langle s_k-t_k,s_{k+1}-t_k\rangle\leq\lambda_k\langle\omega_k,s_{k+1}-t_k\rangle.\tag{3.6}$$

Considering $\omega_k\in\partial_2\mathcal{R}(s_k,t_k)$ and utilizing the subdifferential definition, we derive

$$\mathcal{R}(s_k,t)-\mathcal{R}(s_k,t_k)\geq\langle\omega_k,t-t_k\rangle\ \forall\,t\in\mathcal{H}.$$

Setting $t = s_{k+1}$, we obtain

$$\mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) \geq \langle \omega_k, s_{k+1} - t_k \rangle. \tag{3.7}$$

Combining (3.6) and (3.7), we deduce

$$\lambda_k \big\{ \mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) \big\} \geq \langle s_k - t_k, s_{k+1} - t_k \rangle. \tag{3.8}$$

□

Theorem 3.1. *Let $\{s_k\}$ be a sequence generated by Algorithm 1, satisfying items ($\mathcal{R}1$)–($\mathcal{R}5$). Under these conditions, the sequence $\{s_k\}$ converges weakly to a point $s^* \in Sol(\mathcal{R}, \mathcal{C})$. Furthermore, it holds that $\lim_{k \to +\infty} P_{Sol(\mathcal{R},\mathcal{C})}(s_k) = s^*$.*

Proof. By substituting $t = s^*$ into Lemma 3.6, we obtain

$$\lambda_k \mathcal{R}(t_k, s^*) - \lambda_k \mathcal{R}(t_k, s_{k+1}) \geq \langle s_k - s_{k+1}, s^* - s_{k+1} \rangle. \tag{3.9}$$

Given that $s^* \in Sol(\mathcal{R}, \mathcal{C})$ with $\mathcal{R}(s^*, t_k) > 0$, according to condition ($\mathcal{R}2$), we have

$$\mathcal{R}(t_k, s^*) < 0.$$

This leads to the transformation of expression (3.9) into the following form:

$$\langle s_k - s_{k+1}, s_{k+1} - s^* \rangle \geq \lambda_k \mathcal{R}(t_k, s_{k+1}). \tag{3.10}$$

Using relation (3.2), we find

$$\mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) - \mathcal{R}(t_k, s_{k+1}) \leq \frac{(2 - \sqrt{2} - \theta)\mu\big(\|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2\big)}{2\lambda_{k+1}},$$

which implies

$$\begin{aligned} &\lambda_k \mathcal{R}(t_k, s_{k+1}) \\ &\geq \lambda_k \mathcal{R}(s_k, s_{k+1}) - \lambda_k \mathcal{R}(s_k, t_k) \\ &- \frac{(2 - \sqrt{2} - \theta)\lambda_k \mu\big(\|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2\big)}{2\lambda_{k+1}}. \end{aligned} \tag{3.11}$$

Combining equations (3.10) and (3.11), we derive the inequality:

$$\begin{aligned} \langle s_k - s_{k+1}, s_{k+1} - s^* \rangle &\geq \lambda_k \{ \mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) \} \\ &\quad - \frac{(2 - \sqrt{2} - \theta)\lambda_k \mu\big(\|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2\big)}{2\lambda_{k+1}}. \end{aligned} \tag{3.12}$$

By combining expressions (3.8) and (3.12), we deduce:

$$\begin{aligned} \langle s_k - s_{k+1}, s_{k+1} - s^* \rangle &\geq \langle s_k - t_k, s_{k+1} - t_k \rangle \\ &\quad - \frac{(2 - \sqrt{2} - \theta)\lambda_k \mu\big(\|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2\big)}{2\lambda_{k+1}}. \end{aligned} \tag{3.13}$$

Additionally, we can use the following relations:

$$2\langle s_k - s_{k+1}, s_{k+1} - s^* \rangle = \|s_k - s^*\|^2 - \|s_{k+1} - s_k\|^2 - \|s_{k+1} - s^*\|^2,$$

$$2\langle t_k - s_k, t_k - s_{k+1} \rangle = \|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2 - \|s_k - s_{k+1}\|^2.$$

Thus, we can derive:

$$\begin{aligned} \|s_{k+1} - s^*\|^2 &\leq \|s_k - s^*\|^2 - \|s_k - t_k\|^2 - \|s_{k+1} - t_k\|^2 \\ &\quad + \frac{(2 - \sqrt{2} - \theta)\lambda_k \mu\big(\|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2\big)}{\lambda_{k+1}}. \end{aligned} \tag{3.14}$$

Given that $\lambda_k \to \lambda$ as $k \to \infty$, there exists a fixed $k_1 \in \mathbb{N}$ such that:

$$\frac{\mu \lambda_k}{\lambda_{k+1}} \leq 1, \quad \forall k \geq k_1.$$

Thus, we have:

$$\begin{aligned} \|s_{k+1} - s^*\|^2 \leq \|s_k - s^*\|^2 - \|s_k - t_k\|^2 - \|s_{k+1} - t_k\|^2 \\ + (2 - \sqrt{2} - \theta)\big(\|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2\big). \end{aligned} \tag{3.15}$$

Furthermore, this implies:

$$\begin{aligned} \|s_{k+1} - s^*\|^2 \leq \|s_k - s^*\|^2 - (\sqrt{2} - 1)\|s_k - t_k\|^2 - (\sqrt{2} - 1)\|s_{k+1} - t_k\|^2 \\ - \theta\big(\|s_k - t_k\|^2 + \|s_{k+1} - t_k\|^2\big). \end{aligned} \tag{3.16}$$

From equation (3.16), we deduce:

$$\|s_{k+1} - s^*\|^2 \leq \|s_k - s^*\|^2 \ \forall\, k \geq k_1. \tag{3.17}$$

Thus, we can conclude that the sequence $\{s_k\}$ is bounded. Let $m \geq k_1$, and consider equation (3.16) for $k_1, k_1 + 1, \cdots, m$. Summing these equations up, we obtain:

$$\begin{aligned} \|s_{m+1} - s^*\|^2 &\leq \|s_{k_1} - s^*\|^2 - \sum_{k=k_1}^{m} (\sqrt{2} - 1)\|s_k - t_k\|^2 \\ &\quad - \sum_{k=k_1}^{m} (\sqrt{2} - 1)\|s_{k+1} - t_k\|^2 \\ &\leq \|s_{k_1} - s^*\|^2. \end{aligned} \tag{3.18}$$

This leads to:

$$\begin{aligned} &\sum_{k=k_1}^{m} (\sqrt{2} - 1)\|s_k - t_k\|^2 + \sum_{k=k_1}^{m} (\sqrt{2} - 1)\|s_{k+1} - t_k\|^2 \\ &\leq \|s_{k_1} - s^*\|^2 - \|s_{m+1} - s^*\|^2. \end{aligned} \tag{3.19}$$

By taking the limit as $k \to +\infty$ in equation (3.19), we get:

$$\sum_{k=1}^{+\infty} \|s_k - t_k\|^2 < +\infty \Longrightarrow \lim_{k \to +\infty} \|s_k - t_k\| = 0, \tag{3.20}$$

and:

$$\sum_{k=1}^{+\infty} \|s_{k+1} - t_k\|^2 < +\infty \Longrightarrow \lim_{k \to +\infty} \|s_{k+1} - t_k\| = 0. \tag{3.21}$$

From equations (3.20), (3.21), and applying Cauchy's inequality, we conclude:

$$\lim_{k \to +\infty} \|s_{k+1} - s_k\| = 0. \tag{3.22}$$

Let $\hat{s}$ denote a weak limit point of $\{s_k\}$, signifying that a subsequence, denoted by $\{s_{k_j}\}$, of $\{s_k\}$ converges weakly to $\hat{s}$. Consequently, the sequence $\{t_{k_j}\}$ also converges weakly to $\hat{s}$, and $\hat{s}$ is an element of the set $\mathcal{C}$. Considering relations (3.11), the definition of λ_{k+1}, and inequality (3.13), we derive the following inequality:

$$\lambda_{k_j}\mathcal{R}(t_{k_j},t) \geq \lambda_{k_j}\mathcal{R}(t_{k_j},s_{k_j+1}) + \langle s_{k_j}-s_{k_j+1}, t-s_{k_j+1}\rangle$$
$$\geq \lambda_{k_j}\mathcal{R}(s_{k_j},s_{k_{k+1}}) - \lambda_{k_j}\mathcal{R}(s_{k_j},t_{k_j})$$
$$-\frac{(2-\sqrt{2}-\theta)\mu\lambda_{k_j}}{2\lambda_{k_j+1}}\|s_{k_j}-t_{k_j}\|^2 - \frac{(2-\sqrt{2}-\theta)\mu\lambda_{k_j}}{2\lambda_{k_j+1}}\|t_{k_j}-s_{k_j+1}\|^2$$
$$+\langle s_{k_j}-s_{k_j+1}, t-s_{k_j+1}\rangle$$
$$\geq \langle s_{k_j}-t_{k_j}, s_{k_j+1}-t_{k_j}\rangle - \frac{(2-\sqrt{2}-\theta)\mu\lambda_{k_j}}{2\lambda_{k_j+1}}\|s_{k_j}-t_{k_j}\|^2$$
$$-\frac{(2-\sqrt{2}-\theta)\mu\lambda_{k_j}}{2\lambda_{k_j+1}}\|t_{k_j}-s_{k_j+1}\|^2 + \langle s_{k_j}-s_{k_j+1}, t-s_{k_j+1}\rangle, \tag{3.23}$$

where $t \in \mathcal{H}_k$. It is evident from expressions (3.20), (3.21), and (3.22) that the right-hand side of the above inequality converges to zero due to the boundedness of $\{s_k\}$. Given $\lambda_{k_j} > 0$ and the condition ($\mathcal{R}3$), along with $t_{k_j} \rightharpoonup \hat{s}$, we establish:

$$0 \leq \limsup_{j\to+\infty} \mathcal{R}(t_{k_j},t) \leq \mathcal{R}(\hat{s},t)\ \forall\, t \in \mathcal{C}.$$

Since $\mathcal{C} \subset \mathcal{H}_k$ implies $\hat{s} \in \mathcal{C}$ and $\mathcal{R}(\hat{s},t) \geq 0$, for all $t \in \mathcal{C}$, it follows that $\hat{s} \in Sol(\mathcal{R},\mathcal{C})$. Consequently, Lemma 2.4 guarantees that $\{s_k\}$ and $\{t_k\}$ converge weakly to s^* as $k \to +\infty$.

The final step involves showing that $\lim_{k\to+\infty} P_{Sol(\mathcal{R},\mathcal{C})}(s_k) = s^*$. Let $\Im_k := P_{Sol(\mathcal{R},\mathcal{C})}(s_k)$ for every $k \in \mathbb{N}$. The following inequality illustrates the boundedness of $\Im_k$:

$$\|\Im_k\| \leq \|\Im_k - s_k\| + \|s_k\| \leq \|s^* - s_k\| + \|s_k\|. \tag{3.24}$$

Given the definition of $\{\Im_k\}$ as a bounded sequence, we have:

$$\|s_{k+1} - \Im_{k+1}\|^2 \leq \|s_{k+1} - \Im_k\|^2 \leq \|s_k - \Im_k\|^2\ \forall\, k \geq k_1. \tag{3.25}$$

According to expression (3.25), the sequence $\|s_k - \Im_k\|$ is convergent. For $m > k \geq k_1$ and using (3.16), we have:

$$\|\Im_k - s_m\|^2 \leq \|\Im_k - s_{m-1}\|^2 \leq \cdots \leq \|\Im_k - s_k\|^2. \tag{3.26}$$

Assuming $\Im_m, \Im_k \in Sol(\mathcal{R},\mathcal{C})$, and applying Lemma 2.1 (i) along with (3.26) for $m > k \geq k_1$, we obtain:

$$\|\Im_k - \Im_m\|^2 \leq \|\Im_k - s_m\|^2 - \|\Im_m - s_m\|^2 \leq \|\Im_k - s_k\|^2 - \|\Im_m - s_m\|^2. \tag{3.27}$$

As $\lim_{k\to+\infty}\|\Im_k - s_k\|$ indicates that $\lim_{m,k\to+\infty}\|\Im_k - \Im_m\| = 0$. Since the solution set $Sol(\mathcal{R},\mathcal{C})$ is closed and $\{\Im_k\}$ is a Cauchy sequence, we have $\{\Im_k\} \to \hat{\Pi} \in Sol(\mathcal{R},\mathcal{C})$. From Lemma 2.1 (ii) and $s^*, \hat{\Pi} \in Sol(\mathcal{R},\mathcal{C})$, we obtain:

$$\langle s_k - \Im_k, s^* - \Im_k\rangle \leq 0. \tag{3.28}$$

Given that $\Im_k \to \hat{\Pi}$ and $s_k \rightharpoonup s^*$, implies:

$$\langle s^* - \hat{\Pi}, s^* - \hat{\Pi}\rangle \leq 0,$$

which gives $s^* = \hat{\Pi} = \lim_{k\to+\infty} P_{Sol(\mathcal{R},\mathcal{C})}(s_k)$. Moreover, $\|s_k - t_k\| \to 0$ implies $\lim_{k\to+\infty} P_{Sol(\mathcal{R},\mathcal{C})}(t_k) = s^*$. Therefore, the theorem is proved.

□

We introduce an iterative approach that relies on a monotone variable stepsize rule and involves two strongly convex minimization problems. The specific details of the second primary method are presented below.

Algorithm 2 (Improved Extragradient Method)

(1) **Input:** Provide initial parameters $\lambda_1 > 0$, $s_1 \in \mathcal{H}$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$.
(2) **Output:** Obtain a convergent sequence $\{s_k\}$.
(3) **Initialization:** Set $k = 0$.
(4) **Iteration:**
Step 1: Compute $t_k = \arg\min_{t \in \mathcal{C}} \left\{ \lambda_k \mathcal{R}(s_k, t) + \frac{1}{2}\|s_k - t\|^2 \right\}.$
If $s_k = t_k$, terminate.
Step 2: Compute $s_{k+1} = \arg\min_{t \in \mathcal{C}} \left\{ \lambda_k \mathcal{R}(t_k, t) + \frac{1}{2}\|s_k - t\|^2 \right\}.$
Step 3: Compute the stepsize λ_{k+1} as follows:

$$\lambda_{k+1} = \begin{cases} \min\left\{ \lambda_k,\ \frac{\mu}{2} \cdot \frac{(2-\sqrt{2}-\theta)\|s_k - t_k\|^2 + (2-\sqrt{2}-\theta)\|s_{k+1} - t_k\|^2}{[\mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) - \mathcal{R}(t_k, s_{k+1})]} \right\} \\ \quad \text{if} \quad \mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, t_k) - \mathcal{R}(t_k, s_{k+1}) > 0, \\ \lambda_k \qquad\qquad\qquad \text{Otherwise.} \end{cases} \tag{3.29}$$

Step 4: Set $k := k + 1$ and return to **Step 1**.

(i) Let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be an operator. The *variational inequality problem* $\mathcal{A}$ is formulated as follows: Find $s^* \in \mathcal{C}$ such that

$$\left\langle \mathcal{A}(s^*), t_1 - s^* \right\rangle \geq 0 \ \forall\, t_1 \in \mathcal{C}. \tag{VIP}$$

Define the bifunction $\mathcal{R}$ as follows:

$$\mathcal{R}(t_1, t_2) := \left\langle \mathcal{A}(t_1), t_2 - t_1 \right\rangle \ \forall\, t_1, t_2 \in \mathcal{C}. \tag{3.30}$$

The equilibrium problem is then reformulated into a problem of variational inequality defined by (VIP), considering Lipschitz constants of the mapping. The construction of two new methods for solving variational inequalities employs the formula (3.30).

Corollary 3.1. *Let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be weakly continuous, pseudomonotone, and L-Lipschitz continuous. Choose $\lambda_1 > 0$, $s_1 \in \mathcal{H}$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ such that the solution set $Sol(\mathcal{A}, \mathcal{C}) \neq \emptyset$. Evaluate*

$$t_k = P_{\mathcal{C}}(s_k - \lambda_k \mathcal{A}(s_k)).$$

Given s_k, t_k, construct the set

$$\mathcal{H}_k = \{ z \in \mathcal{H} : \langle s_k - \lambda_k \mathcal{A}(s_k) - t_k, z - t_k \rangle \leq 0 \} \text{ for each } k \geq 0.$$

Evaluate

$$s_{k+1} = P_{\mathcal{H}_k}(s_k - \lambda_k \mathcal{A}(t_k)).$$

Modify the stepsize as follows:

$$\lambda_{k+1} = \begin{cases} \min\left\{ \lambda_k,\ \frac{(2-\sqrt{2}-\theta)\mu\|s_k - t_k\|^2 + (2-\sqrt{2}-\theta)\mu\|s_{k+1} - t_k\|^2}{2\left\langle \mathcal{A}(s_k) - \mathcal{A}(t_k),\ s_{k+1} - t_k \right\rangle} \right\} \\ \qquad \textit{if} \quad \left\langle \mathcal{A}(s_k) - \mathcal{A}(t_k),\ s_{k+1} - t_k \right\rangle > 0, \\ \lambda_k \qquad\qquad\qquad \textit{Otherwise.} \end{cases}$$

Then, the sequence $\{s_k\}$ converges weakly to $s^* \in Sol(\mathcal{A}, \mathcal{C})$.

Corollary 3.2. *Let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be weakly continuous, pseudomonotone, and L-Lipschitz continuous. Choose $\lambda_1 > 0$, $s_1 \in \mathcal{H}$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ such that the solution set $Sol(\mathcal{A}, \mathcal{C}) \neq \emptyset$. Evaluate*

$$t_k = P_{\mathcal{C}}(s_k - \lambda_k \mathcal{A}(s_k)),$$

and

$$s_{k+1} = P_{\mathcal{C}}(s_k - \lambda_k \mathcal{A}(t_k)).$$

Modify the stepsize as follows:

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k,\ \frac{(2-\sqrt{2}-\theta)\mu\|s_k-t_k\|^2+(2-\sqrt{2}-\theta)\mu\|s_{k+1}-t_k\|^2}{2\left\langle \mathcal{A}(s_k)-\mathcal{A}(t_k),\ s_{k+1}-t_k\right\rangle}\right\} \\ \qquad \textit{if} \quad \left\langle \mathcal{A}(s_k)-\mathcal{A}(t_k),\ s_{k+1}-t_k\right\rangle > 0, \\ \lambda_k \qquad \textit{Otherwise}. \end{cases}$$

Hence, the sequence $\{s_k\}$ converges weakly to $s^* \in Sol(\mathcal{A}, \mathcal{C})$.

(ii) Let us define a mapping $\mathcal{B} : \mathcal{C} \to \mathcal{C}$ as a κ-strict pseudocontraction if, according to Browder and Petryshyn [7], there exists a constant $\kappa \in (0,1)$ satisfying the following inequality:

$$\|\mathcal{B}t_1 - \mathcal{B}t_2\|^2 \leq \|t_1 - t_2\|^2 + \kappa\|(t_1 - \mathcal{B}t_1) - (t_2 - \mathcal{B}t_2)\|^2 \quad \text{for all} \quad t_1, t_2 \in \mathcal{C}. \tag{3.31}$$

Here, $\mathcal{R}$ is a bifunction defined as follows:

$$\mathcal{R}(t_1, t_2) = \langle t_1 - \mathcal{B}t_1, t_2 - t_1\rangle \quad \text{for all} \quad t_1, t_2 \in \mathcal{C}. \tag{3.32}$$

As established in [29], the expression (3.32) satisfies constraints ($\mathcal{R}$1)-($\mathcal{R}$5), and the Lipschitz constants take values $c_1 = c_2 = \frac{3-2\kappa}{2-2\kappa}$. The two main results are utilized to devise two new methods for solving fixed-point problems using the formula (3.32).

Corollary 3.3. *Let a mapping $\mathcal{B} : \mathcal{C} \to \mathcal{H}$ be weakly continuous and a κ-strict pseudocontraction. Choose $\lambda_1 > 0$, $s_1 \in \mathcal{H}$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ with $Sol(\mathcal{B}, \mathcal{C}) \neq \emptyset$. Evaluate*

$$t_k = P_{\mathcal{C}}\big[s_k - \lambda_k(s_k - \mathcal{B}(s_k))\big].$$

Given s_k, t_k, and

$$\mathcal{H}_k = \{z \in \mathcal{E} : \langle (1-\lambda_k)s_k + \lambda_k\mathcal{B}(s_k) - t_k, z - t_k\rangle \leq 0\}.$$

Evaluate

$$s_{k+1} = P_{\mathcal{H}_k}\big[s_k - \lambda_k(t_k - \mathcal{B}(t_k))\big],$$

and let

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \frac{(2-\sqrt{2}-\theta)\mu\|s_k-t_k\|^2+(2-\sqrt{2}-\theta)\mu\|s_{k+1}-t_k\|^2}{2\left\langle (s_k-t_k)-[\mathcal{B}(s_k)-\mathcal{B}(t_k)], s_{k+1}-t_k\right\rangle}\right\} & \textit{if } \langle (s_k - t_k) - [\mathcal{B}(s_k) - \mathcal{B}(t_k)], s_{k+1} - t_k\rangle > 0, \\ \lambda_k & \textit{else}. \end{cases}$$

Then, the sequence $\{s_k\}$ converges weakly to $s^* \in Sol(\mathcal{B}, \mathcal{C})$.

Corollary 3.4. *Let a mapping $\mathcal{B} : \mathcal{C} \to \mathcal{H}$ be weakly continuous and a κ-strict pseudocontraction. Choose $\lambda_1 > 0$, $s_1 \in \mathcal{H}$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ with $Sol(\mathcal{B}, \mathcal{C}) \neq \emptyset$. Evaluate*

$$t_k = P_{\mathcal{C}}\big[s_k - \lambda_k(s_k - \mathcal{B}(s_k))\big].$$

Evaluate

$$s_{k+1} = P_{\mathcal{C}}\big[s_k - \lambda_k(t_k - \mathcal{B}(t_k))\big].$$

Evaluate

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \frac{(2-\sqrt{2}-\theta)\mu\|s_k-t_k\|^2+(2-\sqrt{2}-\theta)\mu\|s_{k+1}-t_k\|^2}{2\left\langle (s_k-t_k)-[\mathcal{B}(s_k)-\mathcal{B}(t_k)], s_{k+1}-t_k\right\rangle}\right\} & \textit{if } \langle (s_k - t_k) - [\mathcal{B}(s_k) - \mathcal{B}(t_k)], s_{k+1} - t_k\rangle > 0, \\ \lambda_k & \textit{else}. \end{cases}$$

Then, the sequence $\{s_k\}$ converges weakly to $s^* \in Sol(\mathcal{B}, \mathcal{C})$.

4. Numerical Illustrations

This section provides a detailed account of a series of numerical experiments conducted to evaluate the effectiveness of the proposed methodologies. These experiments serve two primary purposes: first, they offer valuable insights into the selection of optimal control parameters; second, they provide empirical evidence showcasing the superiority of the proposed approaches over those previously documented in the literature. All computational analyses were performed using MATLAB version 9.5 (R2018b) on a computational platform featuring an Intel(R) Core(TM) i5-6200 Processor CPU, operating at a base frequency of 2.30GHz (with a maximum turbo frequency of 2.40GHz), and equipped with 8.00 GB of RAM.

Example 4.1. *The Nash-Cournot Oligopolistic Equilibrium model, outlined in* [23]*, serves as the basis for formulating the initial test problem. Specifically, in this context, we define the bifunction* $\mathcal{R}$ *as:*

$$\mathcal{R}(s,t) = \langle Ps + Qt + c, t - s \rangle.$$

Here, $c \in \mathbb{R}^M$*, and the matrices* P *and* Q *are of size* $M \times M$*. Matrix* P *is symmetric and positive semi-definite, while the difference matrix* $Q - P$ *is symmetric and negative semi-definite. The Lipschitz-like criteria are characterized by* $c_1 = c_2 = \frac{1}{2}\|P - Q\|$ *(for further clarification, refer to* [23]*).*

Let us examine Example 4.1 to observe the numerical behavior of Algorithm 1 as the control parameter μ *varies. This experimentation aids in identifying the optimal value for the potential control parameter* μ*. The numerical investigation commences with the initialization of* $s_1 = (1, 1, \cdots, 1)$*, followed by setting* $M = 5$*, and defining the error term* $D_k = \|s_k - t_k\|$*. Below are the specifications for the matrices* P *and* Q*, as well as the vector* c*. Consider the matrices* P*,* Q*, and* c *defined as follows:*

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

Furthermore, let the set $\mathcal{C} \subset \mathbb{R}^M$ *be defined as:*

$$\mathcal{C} := \{s \in \mathbb{R}^M : -2 \le s_i \le 5\}.$$

We present several numerical experiments based on Example 4.1 to assess the efficacy of Algorithm 1 in terms of both execution time (t in seconds) and iteration count (k) required for convergence. Our objective is to explore how the performance of Algorithm 3.1 is influenced by variations in:

(i) The parameter μ *(Experiment 1),*
(ii) The parameter θ *(Experiment 2), and*
(iii) A comparative analysis of Algorithm 1 with existing methods (Experiment 3).

In Experiment 1, we investigate the effect of different values of μ *on the algorithm's performance. Experiment 2 focuses on analyzing how adjustments to the parameter* θ *impact the behavior of Algorithm 1. Finally, Experiment 3 involves comparing the performance of our algorithm with other existing methods. Through these experiments, we aim to gain insights into the behavior and effectiveness of Algorithm 1 under various parameter settings and in comparison to alternative approaches.*

Experiment 1. *(Impact of the parameter μ).*

In this experiment, we investigate the influence of the parameter μ on the performance of Algorithm 1, where the chosen parameters are $D_k = \|s_k - t_k\|$, $\lambda_1 = 0.275$, and $\theta = 0.16$.

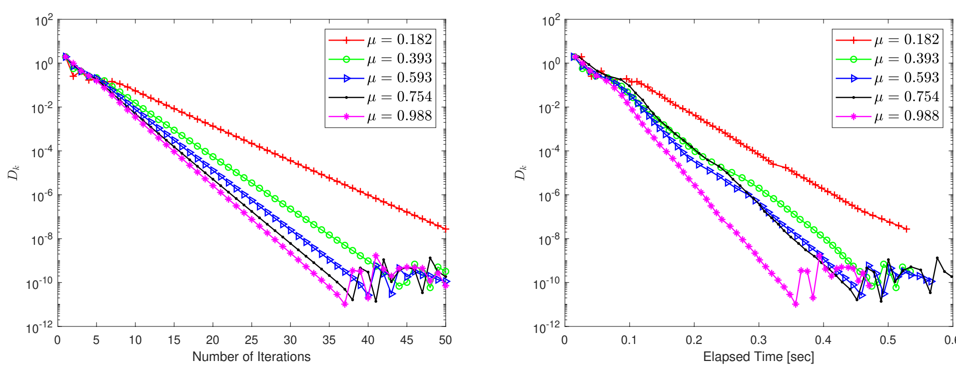
The numerical results obtained for five different initial points μ values are presented in Figure 1 and Table 1. Analysis of these results yields the following observations:

- *(i) The computational efficiency, as measured by the number of iterations required for convergence, demonstrates sensitivity to the selection of μ. Specifically, Algorithm 1 exhibits an increase in iteration count as μ approaches zero.*
- *(ii) Additionally, the computational efficiency in terms of CPU time shows dependency on the choice of μ. Algorithm 1 displays prolonged CPU time as μ tends to zero.*

These findings suggest that the parameter μ significantly impacts both the convergence behavior and computational cost of Algorithm 1. As μ decreases, the algorithm requires more iterations and consumes more CPU time, indicating a trade-off between precision and computational resources. Further investigation into optimal parameter selection may yield improvements in algorithmic efficiency.

TABLE 1. Numerical data corresponding to Figure 1.

μ	METHOD-1	
	(k)	(t)
0.182	50	0.5283
0.393	50	0.53417
0.593	50	0.56618
0.754	50	0.5982
0.988	50	0.47054



(A) Number of iterations (k)

(B) Elapsed time (t)

FIGURE 1. Numerical performance of Algorithm 1 with different values of μ.

Experiment 2. *(Impact of the parameter θ).*

In Experiment 2, we investigate the influence of the parameter θ on the performance of Algorithm 1. The parameters selected for this experiment are $D_k = \|s_k - t_k\|$, $\lambda_1 = 0.12$, and $\mu = 0.55$.

The numerical results obtained for five different initial points of θ are presented in Figure 2 and Table 2. Analysis of the graphical representation and the tabulated data yields the following observations:

(i) *The computational efficiency, as indicated by the number of iterations, is notably affected by the choice of θ. Specifically, a decrease in the value of θ leads to a reduction in the iteration count for Algorithm 1.*

(ii) *Similarly, the computational efficiency in terms of CPU time exhibits dependency on the parameter θ. Algorithm 1 demonstrates reduced CPU time as θ tends towards zero.*

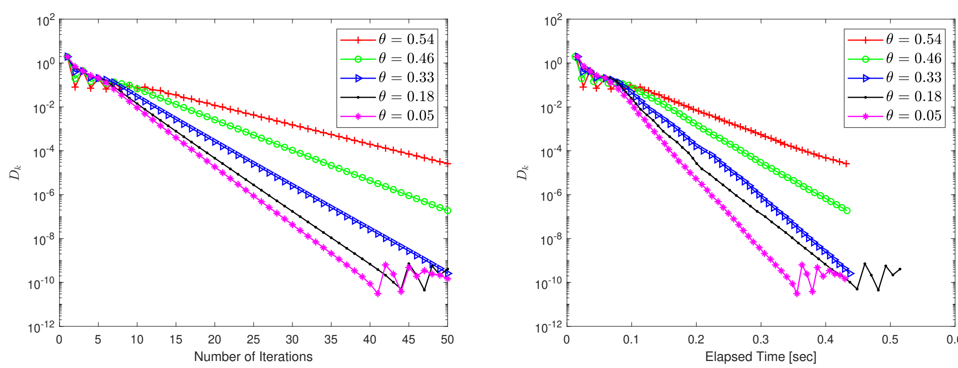
Table 2 provides the corresponding numerical data for the plotted results in Figure 2. Each row corresponds to a different value of μ, while the columns represent the number of iterations (k) and the elapsed time (t).

Figure 2 illustrates the numerical performance of Algorithm 1 with varying values of θ. The left subplot demonstrates the relationship between the number of iterations and θ, while the right subplot illustrates the impact of θ on the elapsed time.

These findings underscore the significance of parameter selection, particularly θ, in optimizing the computational efficiency of the algorithm. Lower values of θ are associated with improved performance metrics, such as reduced iteration counts and CPU time, highlighting its pivotal role in algorithmic optimization.

TABLE 2. Numerical data corresponding to Figure 2.

μ	METHOD-1	
	(k)	(t)
0.54	50	0.431722100000000
0.46	50	0.432941200000000
0.33	50	0.437452300000000
0.18	50	0.514797300000000
0.05	50	0.429655900000000



(A) Number of iterations (k)

(B) Elapsed time (t)

FIGURE 2. Numerical performance of Algorithm 1 with different values of θ.

Experiment 3. *(Algorithm 1 comparison with existing methods).*

In Experiment 3, we conduct a numerical comparison of Algorithm 1 with previously established methods using Example 4.1. The initial conditions for these numerical experiments are set as $s_1 = (1, 1, \cdots, 1)$, and the error term is defined as $D_k = \|s_k - t_k\|$. Figure 3 present a comprehensive set of results for the initial 50 iterations. We outline the control settings for each method as follows:

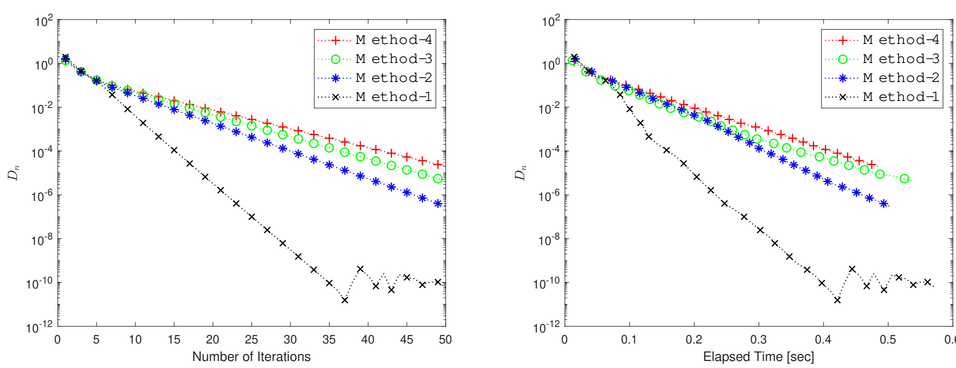
- METHOD-1 *(Algorithm 1):*
$$\lambda_1 = 0.275, \quad \mu = 0.55, \quad \theta = 0.05$$
- METHOD-2 *(Algorithm 1 in* [14]*):*
$$\lambda_1 = 0.275, \quad \mu = 0.55$$
- METHOD-3 *(Algorithm 2a in* [23]*):*
$$\alpha = 0.5, \quad \theta = 0.5, \quad \rho = 1$$
- METHOD-4 *(Algorithm 1 in* [1]*):*
$$\lambda_k = \frac{1}{k}, \quad \alpha = 0.5, \quad \theta = 0.5, \quad \rho = 1$$

Figure 3 presents the numerical results obtained from comparing Algorithm 1 with existing methods. Analysis of both the graphs and the accompanying table leads to the following observations:

(i) Algorithm 1 consistently outperforms previously established algorithms in terms of the number of iterations required to converge.

(ii) Similarly, Algorithm 1 consistently outperforms previously established algorithms in terms of the required CPU time in most cases, notably demanding less CPU time for convergence.

This comparison underscores the superior performance of Algorithm 1 over existing methods, emphasizing its potential as an efficient optimization technique.



(A) Number of iterations (k) (B) Elapsed time (t)

FIGURE 3. Numerical performance of Algorithm 1 compared to other existing algorithms.

Example 4.2. *Let $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ represent an operator defined as follows:*

$$\mathcal{A}(s) = \begin{pmatrix} 0.5s_1s_2 - 2s_2 - 10^7 \\ -4s_1 - 0.1s_2^2 - 10^7 \end{pmatrix} \tag{4.33}$$

where

$$\mathcal{C} = \{s \in \mathbb{R}^2 : (s_1 - 2)^2 + (s_2 - 2)^2 \leq 1\}.$$

It is evident that $\mathcal{A}$ is Lipschitz continuous and pseudomonotone with $L = 5$. By setting the bifunction as $\mathcal{R}(s,t) = \langle \mathcal{A}(s), t - s \rangle$ and $c_1 = c_2 = \frac{5}{2}$. In this experiment, we utilize the same parameters as in Example 4.1. The numerical results for various initial values, such as $s_1 = (1.5, 1.7)^T$, $s_1 = (2.0, 3.0)^T$, $s_1 = (1.0, 2.0)^T$, and $s_1 = (2.7, 2.6)^T$, are presented in Figures 4-7 and summarized in Tables 3-4. The comparative findings are as follows:

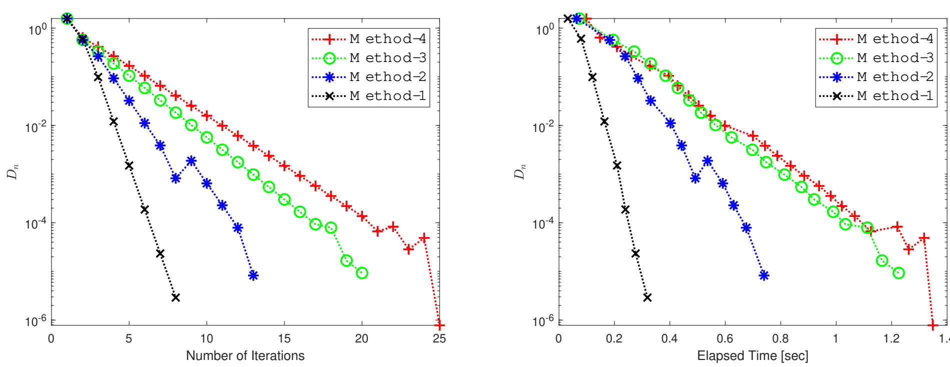
(i) *Algorithm 1 consistently exhibits superior performance compared to previously existing algorithms. It demonstrates advantages in terms of both the number of iterations required to converge and the associated CPU time.*
(ii) *Interestingly, our analysis reveals that altering the initial points has a minimal impact on the iteration count. However, it significantly affects the CPU time required for convergence. This highlights the intricate relationship between initial conditions and computational efficiency in iterative algorithms.*

TABLE 3. Numerical data for Figures 4-7 in terms of the number of iterations.

		Number of iterations		
s_1	*Algorithm 1 in* [1]	*Algorithm 2a in* [23]	*Algorithm 1 in* [14]	*Algorithm 1*
$(1.5, 1.7)^T$	*25*	*20*	*13*	*8*
$(2.0, 3.0)^T$	*26*	*21*	*13*	*8*
$(1.0, 2.0)^T$	*26*	*22*	*14*	*8*
$(2.7, 2.6)^T$	*18*	*14*	*10*	*7*

TABLE 4. Numerical data for Figures 4–7 in terms of elapsed time.

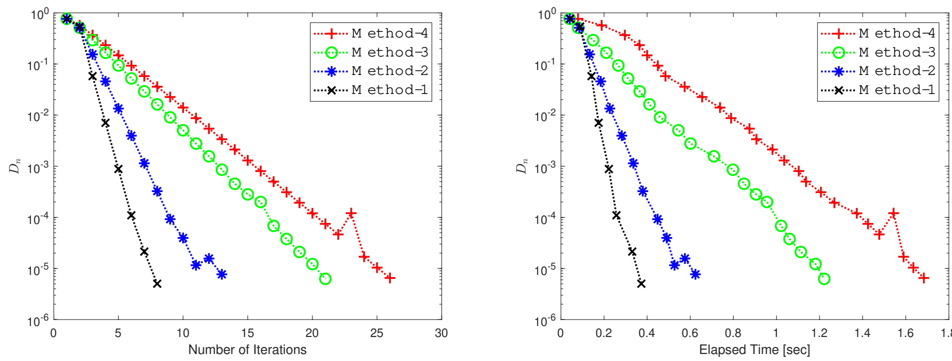
		CPU time		
s_1	*Algorithm 1 in* [1]	*Algorithm 2a in* [23]	*Algorithm 1 in* [14]	*Algorithm 1*
$(1.5, 1.7)^T$	*1.3488092*	*1.2253712*	*0.7397098*	*0.3190929*
$(2.0, 3.0)^T$	*1.6837978*	*1.2216390*	*0.6241229*	*0.3730259*
$(1.0, 2.0)^T$	*1.4884408*	*1.1326319*	*0.8129079*	*0.3580586*
$(2.7, 2.6)^T$	*0.9886659*	*0.6064355*	*0.5215287*	*0.3997974*



(A) Number of iterations (k)

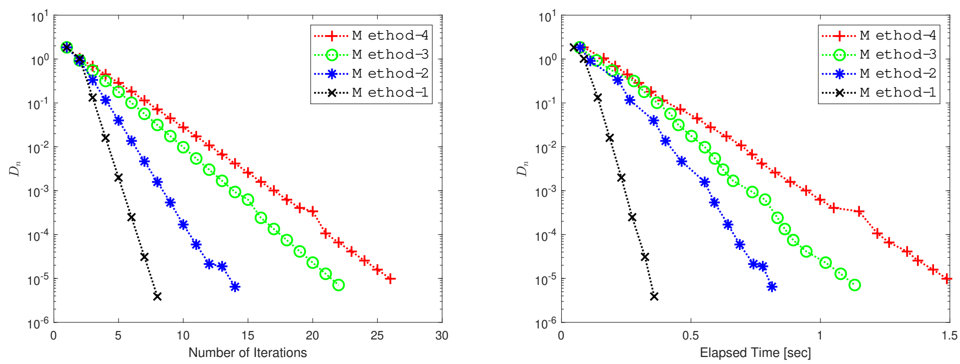
(B) Elapsed time (t)

FIGURE 4. Computational comparison of Algorithm 1 with $s_1 = (1.5, 1.7)^T$.



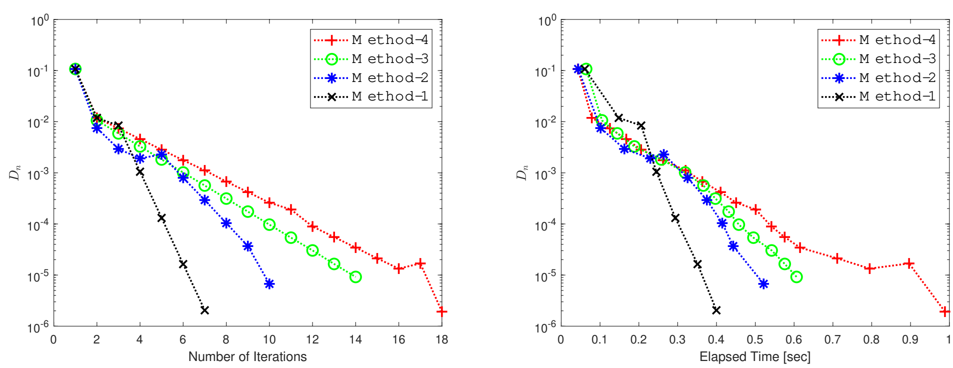
(A) Number of iterations (k) (B) Elapsed time (t)

FIGURE 5. Computational comparison of Algorithm 1 with $s_1 = (2.0, 3.0)^T$.



(A) Number of iterations (k) (B) Elapsed time (t)

FIGURE 6. Computational comparison of Algorithm 1 with $s_1 = (1.0, 2.0)^T$.



(A) Number of iterations (k) (B) Elapsed time (t)

FIGURE 7. Computational comparison of Algorithm 1 with $s_1 = (2.7, 2.6)^T$.

5. Conclusions

The paper describes two explicit extragradient-like methods for solving an equilibrium problem in a real Hilbert space. These methods use a pseudomonotone and a Lipschitz-type bifunction. Furthermore, a new stepsize rule independent of Lipschitz-type parameter information has been established, and the convergence of the proposed methods has been proven. Several experiments were carried out to demonstrate the numerical performance of the proposed method and to allow comparisons to commonly used methods in the literature.

Acknowledgements

The first author would like to thank Faculty of Science and Technology and Research and Development Institute, Valaya Alongkorn Rajabhat University under the Royal Patronage Pathum Thani Province. The second author was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2024). The third author was supported by Office of the Permanent Secretary, Ministry of Higher Education, Science, Research and Innovation (OPS MHESI), Thailand Science Research and Innovation (TSRI) and Phetchabun Rajabhat University (Grant no. RGNS 65-168). The fourth author was supported by Chiang Mai University and Fundamental Fund 2024, Chiang Mai University. The authors would like to thank the anonymous referees for their valuable comments and suggestions. The authors also thank Professor Dr. Poom Kumam from King Mongkut's University of Technology Thonburi, Thailand for his advice and comments to improve quality the results of this manuscript.

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