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Characterizations of amenable gyrogroups related to Tarski's Theorem

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ABSTRACT. Amenable groups, the notion due to J. von Neumann and found responsible for the counter-intuitive Banach–Tarski paradox, are groups that admit a left-invariant mean. In this article, we extend the notion of amenability to gyrogroups, which are non-associative group-like structures. We study their elementary properties and use them to construct an example of infinite amenable gyrogroup. We also give a few characterizations of amenable gyrogroups and extend Tarski's Theorem to the case of gyrogroups.

1. INTRODUCTION

Amenable groups, the notion due to J. von Neumann and found responsible for the counter-intuitive Banach–Tarski paradox, are groups that admit a left-invariant mean. Despite having its origin in measure theory, amenable groups have been studied in several fields of mathematics, and now play an important role in many areas of mathematics such as ergodic theory, harmonic analysis, representation theory, dynamical systems, geometric group theory, probability theory, and statistics, among others. We refer the reader to [4,7,14,20] for more details on the subject. As for the recent progress on the subject see, for example, [2,5,9,15,19].

Around 1990, via studying the parametrization of the Lorentz transformation group, A. Ungar discovered an interesting mathematical structure, which is now called a *gyrogroup*. Gyrogroups have been proven to be closely connected with groups in several ways, as various results on groups naturally extend to gyrogroups. For example, we can use gyrogroups to regulate hyperbolic geometry just like we use groups to regulate Euclidean geometry; see, for instance, [21]. Lately, gyrogroups have been studied from several perspectives such as algebraic and topological perspectives; see, for instance, [3,6,8,11–13,16,18,22–24]. These works suggest that groups and gyrogroups have a strong connection from various viewpoints. In addition, by studying gyrogroups one often gains a better understanding of groups, and vice versa. For example, see Theorem 3.8 below. It is a result obtained by trying to generalize Tarski's theorem to gyrogroup's case. However, Tarski's theorem is not only applicable to gyrogroups, but also a wider classes of actions.

In this work, we extend the notion of amenability to gyrogroups, study basic properties of amenable gyrogroups, and give characterizations of amenable gyrogroups.

2. PRELIMINARIES

In this section, we give a brief review on terminology and facts that will be used in later sections. See [10,16,21] for more details.

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2.1. **Basic knowledge of gyrogroups.** Recall that a *groupoid* $(G, \oplus)$ consists of a nonempty set G, together with a binary operation $\oplus$ on G. We denote the group of automorphisms of $(G, \oplus)$ by $\mathrm{Aut}\,(G, \oplus)$ or just $\mathrm{Aut}\,(G)$ if the operation is clear.

Definition 2.1. A groupoid $(G, \oplus)$ is called a *gyrogroup* if it satisfies the following axioms:

(1) there exists an element $e \in G$ such that $e \oplus a = a$ for all $a \in G$; (left identity)
(2) for each $a \in G$, there exists an element $b \in G$ such that $b \oplus a = e$; (left inverse)
(3) for all $a, b \in G$, there is an automorphism $\mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \tag{2.1}$$

for all $c \in G$; and (left gyroassociative law)
(4) for all $a, b \in G$, $\mathrm{gyr}\,[a \oplus b, b] = \mathrm{gyr}\,[a, b]$. (left loop property)

We remark that the axioms in Definition 2.1 imply the right counterparts: a right identity exists, a right inverse exists, and the right gyroassociative law and the right loop property are satisfied. In fact, any gyrogroup has a unique two-sided identity, denoted by e, and an element a of the gyrogroup has a unique two-sided inverse, denoted by $\ominus a$. The automorphism $\mathrm{gyr}\,[a, b]$ is called the *gyroautomorphism* generated by a and b. Notice that every group is a gyrogroup with all of its gyroautomorphisms being the identity function.

The *gyrogroup cooperation* of a gyrogroup G, denoted by $\boxplus$, is defined by the formula

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b, \qquad a, b \in G.$$

For elements a, b in a gyrogroup G, we define $a \ominus b = a \oplus (\ominus b)$ and $a \boxminus b = a \boxplus (\ominus b)$.

Due to lacking of associativity, the right cancellation law did not holds for gyrogroups. That is, if a and b are elements of a gyrogroup, we might have $(b \oplus a) \ominus a \neq b$. However, if one of the operation involved is the gyrogroup cooperation, the right cancellation law is restored, as shown in the following proposition.

Proposition 2.1 (see Table 2.2 in page 50 of [21]). *Let G be a gyrogroup, and let $a, b, c \in G$.*

(1) $\ominus a \oplus (a \oplus b) = b$. (left cancellation)
(2) $(b \oplus a) \boxminus a = b$. (right cancellation)
(3) $(b \boxplus a) \ominus a = b$. (right cancellation)
(4) $\ominus(a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a)$. (gyrosum inversion law)
(5) $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$. (right gyroassociative law)

For an element a of a gyrogroup G, we define the *left gyrotranslation by a*, denoted by L_a, to be the function from G to G given by $L_a(x) = a \oplus x$ for all $x \in G$. Note that all left gyrotranslations are bijective, and $L_a^{-1} = L_{\ominus a}$ according to the left cancellation.

Now, we give a few facts related to gyroautomorphisms. The gyrator identity given in the following theorem shows that the gyroautomorphism generated by a and b is completely determined by a and b.

Theorem 2.1 (see Table 2.2 in page 50 of [21]). *Let a, b, and c be elements of a gyrogroup G. Then,*

(1) $\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$. (gyrator identity)
(2) $\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a]$. (inversive symmetry)
(3) $\mathrm{gyr}\,[\ominus \mathrm{gyr}\,[a, b]b, a] = \mathrm{gyr}\,[a, b]$.

The next proposition follows directly from the gyroassociative laws. Here, if A and B are subsets of a gyrogroup G, then $A \oplus B$ is defined as $A \oplus B = \{a \oplus b \mid a \in A, b \in B\}$.

Proposition 2.2. *Let G be a gyrogroup. If V is a subset of G such that $\mathrm{gyr}[a, b](V) = V$ for all $a, b \in G$, then*

$$A \oplus (B \oplus V) = (A \oplus B) \oplus V$$

for all subsets A and B of G.

Definition 2.2. Let G be a gyrogroup. A non-empty subset H of G is a *subgyrogroup* if H is a gyrogroup under the operation inherited from G, and the restriction of $\mathrm{gyr}[a,b]$ to H becomes an automorphism of H for all $a, b \in H$. A subgyrogroup H of a gyrogroup G is an *L-subgyrogroup* if $\mathrm{gyr}\,[a,h](H) = H$ for all $a \in G, h \in H$.

Proposition 2.3. *A non-empty subset H of a gyrogroup G is a subgyrogroup if and only if* $\ominus a \in H$ *and* $a \oplus b \in H$ *for all* $a, b \in H$.

For a subgyrogroup H of a gyrogroup G, set $G/H = \{a \oplus H \mid a \in G\}$, where $a \oplus H$ is called a *left coset* defined by $a \oplus H = \{a \oplus h \mid h \in H\}$. We remark that a generic subgyrogroup H of a gyrogroup G does not partition G into left cosets. However, if H is an L-subgyrogroup of G, then G/H forms a partition of G, and in particular, two distinct left cosets of H are disjoint (see Theorem 27 of [16]).

We now proceed with the construction of quotient gyrogroups. Let G and H be gyrogroups, and let $\phi : G \to H$ be a homomorphism. Define the *kernel* of ϕ to be the set $\ker\phi = \{a \in G \mid \phi(a) = e\}$. It can be checked that $\ker\phi$ is a subgyrogroup of G.

Definition 2.3. A subgyrogroup N of a gyrogroup G is *normal* in G if it is the kernel of a homomorphism defined on G.

Observe that if N is a normal subgyrogroup of a gyrogroup G, then N is an L-subgyrogroup of G. In fact, $\mathrm{gyr}\,[a,b](N) = N$ for all $a, b \in G$.

Theorem 2.2 (Theorem 29, [16])**.** *If N is a normal subgyrogroup of a gyrogroup G, then the collection*

$$G/N = \{a \oplus N \mid a \in G\}$$

forms a gyrogroup under the operation $\oplus$ *defined by*

$$(a \oplus N) \oplus (b \oplus N) = (a \oplus b) \oplus N$$

for all $a, b \in G$. *The gyrogroup G/N is called a quotient gyrogroup.*

2.2. **Notations in $l^1(X)$ and $l^\infty(X)$ spaces.** Let X be a non-empty set. For a function $f : X \to [0, \infty]$, define

$$\sum_{x \in X} f(x) = \sup\left\{\sum_{x \in A} f(x) \mid A \text{ is a finite subset of } X\right\}.$$

Also, define

$$l^1(X) = \left\{f : X \to \mathbb{R} \mid \sum_{x \in X} |f(x)| < \infty\right\},$$

and define a function $\|\cdot\|_1 : l^1(X) \to \mathbb{R}$ by the formula

$$\|f\|_1 = \sum_{x \in X} |f(x)|$$

for all $f \in l^1(X)$.

Furthermore, we define

$$l^\infty(X) = \{f : X \to \mathbb{R} \mid f \text{ is bounded}\},$$

and define a function $\|\cdot\|_\infty : l^\infty(X) \to \mathbb{R}$ by the formula

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}.$$

Then, $(l^1(X), \|\cdot\|_1)$ and $(l^\infty(X), \|\cdot\|_\infty)$ are Banach spaces. We remark that the subspace of $l^\infty(X)$ spanning by $\{1_A \mid A \subseteq X\}$ is dense in $l^\infty(X)$.

2.3. **Means and finitely additive probability measures.**

Definition 2.4. Let X be a non-empty set. A function $\mu : \mathcal{P}(X) \to [0,1]$ is called a *finitely additive probability measure* if

(1) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{P}(X)$ are disjoint;
(2) $\mu(X) = 1$.

We denote the set of all finitely additive probability measures on X by $PM(X)$.

Definition 2.5. Let X be a non-empty set. A *mean* on X is a linear functional m in $l^\infty(X)^*$ such that

(1) $m(1_X) = 1$;
(2) $m(f) \geq 0$ whenever $f \geq 0$ and $f \in l^\infty(X)$.

We denote the set of all means on X by $M(X)$.

By identifying any subset A of X with its characteristic function 1_A, we obtain a one-to-one correspondence between $M(X)$ and $PM(X)$ as described in the following theorem.

Theorem 2.3 (Theorem A.4 of [10]). *The function* $\hat{\ }: M(X) \to PM(X)$, *defined by*

$$\hat{m}(A) = m(1_A)$$

for all $A \in \mathcal{P}(X)$ *and for all* $m \in M(X)$, *is bijective.*

Let X be a non-empty set, and define

$$Prob(X) = \{\mu \in l^1(X) \mid \|\mu\|_1 = 1 \text{ and } \mu \geq 0\}.$$

Also, define a function $l^1(X) \to l^\infty(X)^*$ by the formula

$$f \mapsto m_f, \text{ where } m_f(g) = \sum_{x \in X} f(x)g(x) \text{ for all } g \in l^\infty(X).$$

Note that the function $f \mapsto m_f$ is injective, and it sends $f \in Prob(X)$ to $m_f \in M(X)$.

Theorem 2.4 (Fact A.3 (iv) of [10]). *The set* $Prob(X)$ *is weak*-dense in* $M(X)$.

Define a function $l^1(X)^* \to l^\infty(X)$ by the formula

$$\phi \mapsto n_\phi, \text{ where } n_\phi(x) = \phi(1_x) \text{ for all } x \in X.$$

Theorem 2.5. *Let* $f \in l^1(X)$, *and let* $\phi \in l^1(X)^*$. *Then,* $\phi(f) = m_f(n_\phi)$.

3. Main results

3.1. **Basic properties of amenable gyrogroups.** In this section, we extend the concept of amenabily for gyrogroups, and prove some elementary properties of amenable gyrogroups.

Let G be a gyrogroup. A finitely additive probability measure μ on G is said to be *left-invariant* if $\mu(g \oplus A) = \mu(A)$ for all $A \subseteq G$ and for all $g \in G$.

Definition 3.6. Let G be a gyrogroup. We say that G is *amenable* if G admits a left-invariant finitely additive probability measure.

Theorem 3.6. *Let G be a gyrogroup.*

(1) Finite gyrogroups are amenable.
(2) Any non-zero measure subgyrogroup of an amenable gyrogroup is amenable.
(3) If N is a normal subgyrogroup of G and G is amenable, then G/N is amenable.
(4) If N is a normal subgyrogroup of G such that N and G/N are amenable, then G is amenable.

(5) *Let $\{G_i\}_{i\in I}$ be a collection of amenable subgyrogroups of a gyrogroup H. If for all $i, j \in I$, there is an index $k \in I$ such that $G_i, G_j \subseteq G_k$, then the gyrogroup $G = \bigcup_{i\in I} G_i$ is amenable.*

Proof.

(1) The function defined by the formula $\mu(A) = \frac{|A|}{|G|}$ for all $A \subseteq G$ is a left-invariant finitely additive probability measure on G.

(2) Suppose that G is an amenable gyrogroup with left-invariant finitely additive probability measure μ. Let H be a subgyrogroup of G with $\mu(H) > 0$. Then, the function $\nu : \mathcal{P}(H) \to [0,1]$, defined by the formula

$$\nu(A) = \frac{\mu(A)}{\mu(H)}$$

for all $A \in \mathcal{P}(H)$, is a left-invariant finitely additive probability measure on H.

(3) Suppose that G is an amenable gyrogroup with left-invariant finitely additive probability measure μ. Let N be a normal subgyrogroup of G. Then, the function $\nu : \mathcal{P}(G/N) \to [0,1]$, defined by the formula

$$\nu(A) = \mu\left(\bigcup A\right)$$

for all $A \in \mathcal{P}(G/N)$, is a finitely additive probability measure on G/N. It follows from Proposition 2.2 that ν is left-invariant.

(4) Let ν_1 and ν_2 be left-invariant finitely additive probability measures on N and G/N, respectively. For each $A \subseteq G$, define $f_A : G \to [0,1]$ by the formula

$$f_A(x) = \nu_1(N \cap (\ominus x \oplus A))$$

for all $x \in G$. Let $x, y \in G$ such that $x \oplus N = y \oplus N$. Then, $\ominus y \oplus x = n \in N$. It follows that

$$\begin{aligned}
\ominus y \oplus A &= (n \boxminus x) \oplus A \\
&= (n \boxplus (\ominus x)) \oplus A \\
&= (n \oplus \text{gyr}[n, x](\ominus x)) \oplus A \\
&= n \oplus (\text{gyr}[n, x](\ominus x) \oplus \text{gyr}[\text{gyr}[n, x](\ominus x), n](A)) \\
&= n \oplus (\text{gyr}[n, x](\ominus x) \oplus \text{gyr}[\ominus \text{gyr}[n, x]x, n](A)) \\
&\overset{(\star)}{=} n \oplus (\text{gyr}[n, x](\ominus x) \oplus \text{gyr}[n, x](A)) \\
&= n \oplus \text{gyr}[n, x](\ominus x \oplus A),
\end{aligned}$$

where $(\star)$ follows from part (3) of Theorem 2.1. Then,

$$\begin{aligned}
f_A(y) &= \nu_1(N \cap (\ominus y \oplus A)) \\
&= \nu_1(N \cap (n \oplus \text{gyr}[n, x](\ominus x \oplus A)) \\
&= \nu_1(n \oplus (N \cap (\text{gyr}[n, x](\ominus x \oplus A))) \\
&= \nu_1(N \cap (\text{gyr}[n, x](\ominus x \oplus A)) \\
&= \nu_1(\text{gyr}[n, x](N \cap (\ominus x \oplus A))) \\
&= \nu_1(N \cap (\ominus x \oplus A)) \\
&= f_A(x).
\end{aligned}$$

This shows that the function $\hat{f_A} : G/N \to [0,1]$, defined by the formula

$$\hat{f_A}(x \oplus N) = f_A(x)$$

for all $x \in G$, is well-defined. Define a function $\mu : \mathcal{P}(G) \to \mathbb{R}$ by the formula

$$\mu(A) = \int \hat{f}_A d\nu_2$$

for all $A \in \mathcal{P}(G)$ (see, for example, [1] for a treatment on integration with respect to finitely additive measures). Since $0 \leq \hat{f}_A \leq 1$ for all $A \subseteq G$, μ is a function from $\mathcal{P}(G)$ to $[0,1]$. Moreover, $\mu(G) = \int \hat{f}_G d\nu_2 = \int 1_{G/N} d\nu_2 = 1$. To see finite additivity, let A and B be disjoint subsets of G. Then, for each $x \in G$,

$$\begin{aligned} f_{A\cup B}(x) = \nu_1(N \cap (\ominus x \oplus (A \cup B))) &= \nu_1((N \cap (\ominus x \oplus A)) \cup (N \cap (\ominus x \oplus B))) \\ &= \nu_1(N \cap (\ominus x \oplus A)) + \nu_1(N \cap (\ominus x \oplus B)) \\ &= f_A(x) + f_B(x). \end{aligned}$$

It follows that $\hat{f}_{A\cup B} = \hat{f}_A + \hat{f}_B$, and so

$$\mu(A \cup B) = \int \hat{f}_{A\cup B} d\nu_2 = \int \hat{f}_A + \hat{f}_B d\nu_2 = \int \hat{f}_A d\nu_2 + \int \hat{f}_B d\nu_2 = \mu(A) + \mu(B).$$

Finally, let $x \in G$, and let $A \subseteq G$. Then, for each $y \in G$,

$$\begin{aligned} f_{x\oplus A}(y) &= \nu_1(N \cap (\ominus y \oplus (x \oplus A))) \\ &= \nu_1(N \cap ((\ominus y \oplus x) \oplus \mathrm{gyr}\,[\ominus y, x](A)) \\ &= \nu_1(N \cap ((\ominus \mathrm{gyr}\,[\ominus y, x](\ominus x \oplus y)) \oplus \mathrm{gyr}\,[\ominus y, x](A)) \\ &= \nu_1(N \cap (\ominus(\ominus x \oplus y) \oplus A)) \\ &= f_A(\ominus x \oplus y). \end{aligned}$$

It follows that

$$\begin{aligned} \hat{f}_{x\oplus A}(y \oplus N) = f_{x\oplus A}(y) = f_A(\ominus x \oplus y) &= \hat{f}_A((\ominus x \oplus N) \oplus (y \oplus N)) \\ &= \hat{f}_A \circ L_{\ominus x \oplus N}(y \oplus N). \end{aligned}$$

This implies that $\mu(x \oplus A) = \int \hat{f}_{x\oplus A} d\nu_2 = \int \hat{f}_A \circ L_{\ominus x\oplus N} d\nu_2 = \int \hat{f}_A d\nu_2 = \mu(A)$.

(5) For each $i \in I$, let μ_i denote a left-invariant finitely additive probability measure on G_i, and define

$$M_i = \{\mu \in PM(G) \mid \mu(x \oplus A) = \mu(A) \text{ for all } a \in G_i \text{ and } A \in \mathcal{P}(G)\}.$$

Notice that M_i is non-empty since the function $A \mapsto \mu_i(A \cap G_i)$ is in M_i. Recall that a net $(f_i)_{i\in J}$ in $[0,1]^{\mathcal{P}(G)}$ converges to a function $f \in [0,1]^{\mathcal{P}(G)}$ if and only if $f_i(A) \to f(A)$ for all $A \in \mathcal{P}(G)$. It follows that M_i is closed in $[0,1]^{\mathcal{P}(G)}$ for all $i \in I$. Observe that if $G_i, G_j \subseteq G_k$, then $M_k \subseteq M_i \cap M_j$. Thus, the collection $\{M_i\}_{i\in I}$ has the finite intersection property. Since $[0,1]^{\mathcal{P}(G)}$ is compact by Tychonoff's Theorem, there exists an element $\mu \in \bigcap_{i\in I} M_i$. This proves that G is amenable. □

It is well known that every subgroup of an amenable group is amenable, and every abelian group is amenable. Unfortunately, it is not clear whether these results are true in the case of gyrogroups. Therefore, we propose the following questions.

Question. *Is every subgyrogroup of an amenable gyrogroup amenable?*

Question. *Is every gyrocommutative gyrogroup amenable?*

We conclude this section with an example of an infinite amenable gyrogroup.

Example 3.1. Let G be any non-trivial finite gyrogroup. For each $n \in \mathbb{N}$, define

$$G^n = \prod_{i=1}^{n} G \times \prod_{i=n+1}^{\infty} \{e\}.$$

Then, G^n is a finite subgyrogroup of the direct product $\prod_{i=1}^{\infty} G$. Now, the indexed collection $\{G^n\}_{n\in\mathbb{N}}$ forms a direct system of amenable gyrogroups whose direct union is

$$\bigcup_{n\in\mathbb{N}} G^n = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{i=1}^{\infty} G \mid (x_n)_{n\in\mathbb{N}} \text{ has finite non-identity terms} \right\},$$

which is an amenable gyrogroup by part (5) of Theorem 3.6.

3.2. **Tarski's Theorem for gyrogroups.** Suppose that a group G acts on a non-empty set X. Let E be a subset of X. We say that E is *G-paradoxical* if there are pairwise disjoint subsets $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m$ of E and elements $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_m$ of G such that

$$E = \bigcup_{i=1}^{n} g_i \cdot A_i = \bigcup_{i=1}^{m} h_i \cdot B_i.$$

Theorem 3.7 (Tarski's Theorem)**.** *Suppose that a group G acts on a non-empty set X, and let $E \subseteq X$. Then, there is a finitely additive G-invariant measure $\mu : \mathcal{P}(X) \to [0, \infty]$ with the property that $\mu(E) = 1$ if and only if E is not G-paradoxical.*

Recall that every group acts on itself by left-translation. This leads to an immediate consequence of Tarski's Theorem: a group G is amenable if and only if G is not G-paradoxical. Although each gyrogroup does not act on itself by left-gyrotranslation in general (see, for instance, [17]), it does admit a weaker action on itself. This action is strong enough to allow us to generalize Tarski's Theorem and its consequence to gyrogroups.

Definition 3.7. Let G be a gyrogroup, and let X be a non-empty set. A *gyrogroup semi-action* of G on X is a function $\cdot : G \times X \to X$ such that the following properties hold:

(1) $e \cdot x = x$ for all $x \in X$.
(2) $\ominus g \cdot (g \cdot x) = x$ for all $g \in G, x \in X$.

A few concrete examples of gyrogroup semi-actions are given as follows.

Example 3.2. Let G be an arbitrary gyrogroup.

(1) Then, the map defined by $g \cdot x = g \oplus x$ for all $g, x \in G$ is a gyrogroup semi-action of G on itself.
(2) Let X be a set of functions from G to a set such that if $f \in X$, then $f \circ L_a \in X$ for all $a \in G$ (for example, X can be chosen to be $\mathrm{Sym}\,(G)$, $l^\infty(G)$, or $Prob(G)$). Then, the map defined by $a \cdot f = f \circ L_a$ for all $a \in G, f \in X$ is gyrogroup semi-action of G on X.

Observe that every gyrogroup semi-action of G on X gives rise to a function $\phi : G \to \mathrm{Sym}(X)$ defined by

$$\phi(g)(x) = g \cdot x$$

for all $g \in G$ and for all $x \in X$. This function satisfies the property that (1) $\phi(e) = \mathrm{id}_X$ and (2) $\phi(\ominus g) = (\phi(g))^{-1}$ for all $g \in G$. Clearly, any function from G to $\mathrm{Sym}\,(X)$ satisfying the above properties also gives rise to a gyrogroup semi-action of G on X. From now on, we will simply write g instead of $\phi(g)$, and let $\langle G\rangle$ denote the subgroup of $\mathrm{Sym}(X)$ generated by $\phi(G)$. Then, it follows that

$$\langle G\rangle = \{g_1 \circ g_2 \circ \cdots \circ g_n \mid g_1, g_2, \ldots, g_n \in G, n \in \mathbb{N}\}.$$

Note that the group $\langle G\rangle$ acts on X in the usual way.

Theorem 3.8. *[Tarski's Theorem for gyrogroup semi-actions] Suppose that a gyrogroup G semi-acts on a set X, and let $E \subseteq X$. Then, there is a finitely additive G-invariant measure $\mu : \mathcal{P}(X) \to [0, \infty]$ with the property that $\mu(E) = 1$ if and only if E is not $\langle G\rangle$-paradoxical.*

Proof. It is easy to see that if $\mu : \mathcal{P}(X) \to [0, \infty]$ is a finitely additive measure with $\mu(E) = 1$, then μ is G-invariant if and only if μ is $\langle G\rangle$-invariant. Hence, the theorem follows directly from Tarski's Theorem. □

We end this section with a few remarks on the notion of gyrogroup semi-actions. Suppose that a gyrogroup G semi-acts on a set X. In contrast to the case of gyrogroup actions, the relation

$$x \sim y \iff y = g \cdot x \qquad \text{for some element } g \in G$$

does not define an equivalence relation on X. We can remedy the situation by defining $x \sim y$ if and only if there are element $g_1, g_2, \ldots, g_n \in G$ such that

$$y = g_n \cdot (\cdots (g_2 \cdot (g_1 \cdot x)) \cdots).$$

It is easy to see that $\sim$ is an equivalence relation on X. Now, we can define the *orbit* of $x \in X$, written $\mathrm{orb}(x)$, to be the equivalence class of x under $\sim$. However, the orbit of x we have just defined is actually the same thing as the orbit of x induced by the action of $\langle G\rangle$ on X.

Example 3.3. The group $\mathbb{R}$ semi-acts on $l^\infty(\mathbb{R})$, the group of real-valued bounded functions on $\mathbb{R}$, by defining

$$a \cdot f = \begin{cases} f \circ L_{a+1} & \text{if } a > 0 \\ f & \text{if } a = 0 \\ f \circ L_{a-1} & \text{if } a < 0 \end{cases}$$

for all $a \in \mathbb{R}$ and for all $f \in l^\infty(\mathbb{R})$. For each real number x, let $\lfloor x \rfloor_e$ denote the biggest even number that less than or equal to x. Define a function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = x - \lfloor x \rfloor_e.$$

Clearly, f is bounded on $\mathbb{R}$. Moreover, f is a periodic function with period 2. It follows that $1 \cdot f = f = 3 \cdot f$, but $(1 + 3) \cdot f \neq f$. This shows that the set $\{a \in \mathbb{R} \mid a \cdot f = f\}$ is not a subgroup of $\mathbb{R}$.

The next example demonstrates that the Orbit-Stabilizer Theorem (cf. Theorem 3.9 of [17]) cannot be generalized to the case of gyrogroup semi-actions, at least not in an obvious way.

Example 3.4. Consider the group of integers modulo 10, denoted by $\mathbb{Z}_{10}$. The group $\mathbb{Z}_{10}$ semi-acts on the set of real-valued functions on $\mathbb{Z}_{10}$ by the formula

$$a \cdot f = \begin{cases} f & \text{if } a = 0 \\ f \circ L_{a+5} & \text{if } a \neq 0. \end{cases}$$

Then,

$$\mathrm{orb}(1_0) = \{1_0, 1_1, \ldots, 1_9\},$$
$$\{a \cdot 1_0 \mid a \in \mathbb{Z}_{10}\} = \mathrm{orb}(1_0) \setminus \{1_5\},$$

and

$$\{a \in \mathbb{Z}_{10} \mid a \cdot 1_0 = 1_0\} = \{1_0, 1_5\}.$$

Here, 1_a is the characteristic function of $\{a\}$.

3.3. **Characterizations of amenable gyrogroups.** In this section, we give a few characterizations of amenability on gyrogroups. Recall that for each function f with a gyrogroup G as its domain, we define $a \cdot f$ to be the function $f \circ L_a$ for all $a \in G$.

Definition 3.8. A mean m on a gyrogroup G is said to be *left-invariant* if

$$m(f \circ L_a) = m(f)$$

for all $f \in l^\infty(G)$ and for all $a \in G$.

Theorem 3.9. *Let G be a gyrogroup. Then, the following statements are equivalent:*

(1) G is amenable.
(2) G is not $\langle G\rangle$-paradoxical.
(3) G admits a left-invariant mean.
(4) (Reiter's condition) For each finite subset E of G and for each $\epsilon > 0$, there exists a function $\nu \in Prob(G)$ such that $\|s \cdot \nu - \nu\|_1 \leq \epsilon$ for all $s \in E$.
(5) (Fϕlner's condition) For each finite subset E of G and for each $\epsilon > 0$, there exists a nonempty finite subset F of G such that

$$|(g \oplus F)\Delta F| \leq \epsilon|F|$$

for all $g \in E$.

Proof. The equivalence of (1) and (2) follows from Tarski's Theorem for gyrogroup semiactions.

(1) $\implies$ (3): Let μ be a left-invariant finitely additive probability measure on G. By Theorem 2.3, there is a mean $m \in M(G)$ such that

$$\mu(A) = m(1_A)$$

for all $A \subseteq G$. It follows that for each $g \in G$,

$$m(1_A \circ L_g) = m(1_{\ominus g \oplus A}) = \mu(\ominus g \oplus A) = \mu(A) = m(1_A).$$

It follows that $m(f \circ L_g) = m(f)$ for all $f \in \mathcal{E} = \operatorname{span}(\{1_A \mid A \subseteq X\})$ and for all $g \in G$. Suppose that $f \in l^\infty(X)$. Since $\mathcal{E}$ is dense in $l^\infty(X)$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $f_n \to f$. It can be seen that $f_n \circ L_g \to f \circ L_g$ for all $g \in G$. Thus,

$$m(f) = \lim_{n\to\infty} m(f_n) = \lim_{n\to\infty} m(f_n \circ L_g) = m(f \circ L_g)$$

for all $g \in G$. This shows that m is a left-invariant mean on G.

(3) $\implies$ (4): Let μ be a left-invariant mean on G. Suppose that E is a finite subset of G and $\epsilon > 0$. Since $Prob(G)$ is weak*-dense in $M(G)$, there exists a net $(\mu_i)_{i\in I}$ in $Prob(G)$ such that m_{μ_i} weak*-converges to μ. Then, for each $s \in G$, $m_{s\cdot\mu_i - \mu_i}$ weak*-converges to 0 in $l^\infty(G)^*$. It follows from Theorem 2.5 that $s \cdot \mu_i - \mu_i$ weakly converges to 0 in $l^1(G)$. This means that the weak closure of the set

$$\left\{\bigoplus_{s\in E} s \cdot \nu - \nu \mid \nu \in Prob(X)\right\}$$

in $\bigoplus_{s\in E} l^1(G)$ contains 0, where $\bigoplus_{s\in E} l^1(G)$ is equipped with the maximum norm given by $\left\|\bigoplus_{s\in E} \nu_s\right\|_{\max} = \max_{s\in E} \|\nu_s\|_1$. By definition, this weak closure is also norm closed. Thus, there exists a function $\nu \in Prob(G)$ such that $\|s \cdot \nu - \nu\|_1 \leq \epsilon$ for all $s \in E$.

(4) $\Longrightarrow$ (5): Let E be a finite subset of G, and let $\epsilon > 0$.Without loss of generality, we may assume that E is symmetric. Suppose that $\nu \in Prob(G)$ with $\|s \cdot \nu - \nu\|_1 \leq \frac{\epsilon}{2|E|}$ for all $s \in E$. Observe that if $f, h \in l^1(G)$ with $0 \leq f, h \leq 1$, then for each $t \in G$,

$$|f(t) - h(t)| = \int_0^1 |1_{\{f>r\}}(t) - 1_{\{h>r\}}(t)|dr,$$

where $\{f > r\} = \{x \in G \mid f(x) > r\}$. It follows that

$$\begin{aligned}
\|s \cdot \nu - \nu\|_1 &= \sum_{t\in G} |s \cdot \nu(t) - \nu(t)| \\
&= \sum_{t\in G} \int_0^1 |1_{\{s\cdot\nu>r\}}(t) - 1_{\{\nu>r\}}(t)|dr \\
&= \int_0^1 \sum_{t\in G} |1_{\{s\cdot\nu>r\}}(t) - 1_{\{\nu>r\}}(t)|dr \\
&= \int_0^1 \sum_{t\in G} |1_{\{s\cdot\nu>r\}\Delta\{\nu>r\}}(t)|dr \\
&= \int_0^1 |\{s \cdot \nu > r\}\Delta\{\nu > r\}|dr \\
&= \int_0^1 | \ominus s \oplus \{\nu > r\}\Delta\{\nu > r\}|dr.
\end{aligned}$$

Consider

$$\begin{aligned}
\int_0^1 \sum_{s\in E} | \ominus s \oplus \{\nu > r\}\Delta\{\nu > r\}|dr &= \sum_{s\in E} \int_0^1 | \ominus s \oplus \{\nu > r\}\Delta\{\nu > r\}|dr \\
&= \sum_{s\in E} \|s \cdot \nu - \nu\|_1 \\
&< \epsilon \\
&= \epsilon \sum_{t\in G} |\nu(t)| \\
&= \epsilon \sum_{t\in G} \int_0^1 |1_{\{\nu>r\}}(t) - 1_{\{0>r\}}(t)|dr \\
&= \epsilon \sum_{t\in G} \int_0^1 |1_{\{\nu>r\}}(t) - 1_{\varnothing}(t)|dr \\
&= \epsilon \sum_{t\in G} \int_0^1 |1_{\{\nu>r\}}(t)|dr \\
&= \epsilon \int_0^1 \sum_{t\in G} |1_{\{\nu>r\}}(t)|dr \\
&= \epsilon \int_0^1 |\{\nu > r\}|dr.
\end{aligned}$$

It is easy to see that there exists a number $r \in (0, 1)$ such that $\{\nu > r\}$ is a non-empty finite set, and

$$\sum_{s\in E} | \ominus s \oplus \{\nu > r\}\Delta\{\nu > r\}| \leq \epsilon|\{\nu > r\}|.$$

This shows that

$$|\ominus s \oplus \{\nu > r\}\Delta\{\nu > r\}| \leq \epsilon|\{\nu > r\}|$$

for all $s \in E$.

(5) $\Longrightarrow$ (1): For $\epsilon > 0$ and a symmetric finite subset E of G, define

$$\mathcal{M}_{E,\epsilon} = \{\mu \in PM(G) \mid |\mu(A) - \mu(g \oplus A)| \leq \epsilon \text{ for all } g \in E, A \subseteq G\}.$$

We will show that each $\mathcal{M}_{E,\epsilon}$ is non-empty and closed in $[0,1]^{\mathcal{P}(G)}$, and the collection of such $\mathcal{M}_{E,\epsilon}$ has the finite intersection property. Thus, by compactness of $[0,1]^{\mathcal{P}(G)}$, there exists an element μ in $\bigcap\limits_{\epsilon>0, E \text{ symmetric finite}} \mathcal{M}_{E,\epsilon}$. The function μ is a left-invariant finitely additive probability measure on G.

Claim. *$\mathcal{M}_{E,\epsilon}$ is closed in $[0,1]^{\mathcal{P}(G)}$.*

Proof of the claim. Let $(\mu_i)_{i\in I}$ be a net in $\mathcal{M}_{E,\epsilon}$ that converges to $\mu \in [0,1]^{\mathcal{P}(G)}$. This means that $\mu_i(A) \to \mu(A)$ for all $A \in \mathcal{P}(G)$. It follows that $\mu \in \mathcal{M}_{E,\epsilon}$.

Claim. *$\mathcal{M}_{E,\epsilon}$ is non-empty.*

Proof of the claim. Let F be a non-empty finite subset of G such that

$$|(g \oplus F)\Delta F| \leq \epsilon|F|$$

for all $g \in E$. Define $\mu : \mathcal{P}(G) \to [0,1]$ by the formula

$$\mu(A) = \frac{|A \cap F|}{|F|}$$

for all $A \in \mathcal{P}(G)$. It is easy to see that $\mu \in PM(G)$. Now, let $g \in E$, and let $A \in \mathcal{P}(G)$. Consider the case when $|A \cap F| \geq |(g \oplus A) \cap F|$. Since

$$\begin{aligned} g \oplus (A \cap F) &= ((g \oplus (A \cap F)) \cap F) \cup ((g \oplus (A \cap F)) \setminus F) \\ &\subseteq ((g \oplus A) \cap F) \cup ((g \oplus F) \setminus F) \\ &\subseteq ((g \oplus A) \cap F) \cup ((g \oplus F)\Delta F) \end{aligned}$$

and

$$\begin{aligned} |A \cap F| &= |g \oplus (A \cap F)| \\ &\leq |(g \oplus A) \cap F| + |(g \oplus F)\Delta F| \\ &\leq |(g \oplus A) \cap F| + \epsilon|F|, \end{aligned}$$

it follows that $0 \leq |A \cap F| - |(g \oplus A) \cap F| \leq \epsilon|F|$. Suppose that $|(g \oplus A) \cap F| \geq |A \cap F|$. Since

$$\begin{aligned} \ominus g \oplus ((g \oplus A) \cap F) &= ((\ominus g \oplus ((g \oplus A) \cap F)) \cap F) \cup ((\ominus g \oplus ((g \oplus A) \cap F)) \setminus F) \\ &\subseteq ((\ominus g \oplus (g \oplus A)) \cap F) \cup ((\ominus g \oplus F) \setminus F) \\ &\subseteq (A \cap F) \cup ((\ominus g \oplus F)\Delta F) \end{aligned}$$

and

$$\begin{aligned} |(g \oplus A) \cap F| &= |\ominus g \oplus ((g \oplus A) \cap F)| \\ &\leq |A \cap F| + |(\ominus g \oplus F)\Delta F| \\ &\leq |A \cap F| + \epsilon|F|, \end{aligned}$$

it follows that $0 \leq |(g \oplus A) \cap F| - |A \cap F| \leq \epsilon|F|$. Therefore, $|\mu(A) - \mu(g \oplus A)| \leq \epsilon$.

This shows that the collection $\{\mathcal{M}_{E,\epsilon} \mid \epsilon > 0, E \text{ is a symmetric finite subset of } G\}$ has the finite intersection property because $\mathcal{M}_{E_1 \cup E_2, \min\{\epsilon_1,\epsilon_2\}} \subseteq \mathcal{M}_{E_1,\epsilon_1} \cap \mathcal{M}_{E_2,\epsilon_2}$ whenever E_1 and E_2 are symmetric finite subsets of G and $\epsilon_1, \epsilon_2 > 0$. This completes the proof. □

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