CARPATHIAN J. MATH.
Volume **41** (2025), No. 1,
Pages 237-251

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2025.01.16

Stability of new functional equations and partial multipliers in Banach $*$-algebras

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ABSTRACT. This paper mainly studies new additive functional equations

$$f\left(\frac{x+y}{n_0}\right)=\frac{f(x)+f(y)}{n_0}$$

$$f\left(\frac{\lambda x+\lambda y}{n_0}\right)=\frac{f(\lambda x)+f(\lambda y)}{n_0}$$

where n_0 is a positive integer with $n_0 \neq 1$, and $\lambda > 0$ is a fixed constant. We prove the Hyers-Ulam stability for the above additive functional equations by using fixed point method in complex normed spaces. Further, we establish some new results about partial multipliers related to additive functional equations in complex Banach $*$-algebras.

1. INTRODUCTION

The stability problem of functional equations was initial proposed by mathematician S.M. Ulam [53] in 1940, and in [15] Hyers gave the first positive answer to the Ulam problem in Banach spaces. Since then, the conclusion obtained was generalized and summarized in many different ways. In [13, 40] Rassias and Gajda considered the stability problem with unbounded Cauchy difference, which is called Hyers-Ulam stability. By considering unbounded Cauchy difference, Aoki [2] extended Hyers theorem to additive mapping, and then Rassias [40] extended Hyers theorem to linear mapping. In [14], by means of Rassias method, Găvruta obtained the generalization of Rassias theorem by replacing unbounded Cauchy difference with general control function. In [29, 30] Park defined additive ρ-functional inequalities, and proved their Hyers-Ulam stability in Banach spaces and non Archimedean Banach spaces.

The stability of functional equations is a very active field and the method used in the proofs of subsequents is always Hyers' method, that is, starting from a given function, the approximate function is explicitly constructed by a fixed formula. This method is called the direct method and it was proved that, for the stability of functional equations, it is the most significant and strong tool. There are other known methods, for example, people studied the stability of functional equations by using the sandwich theorem (see [38]). Later, some authors observed that the approximate function and its estimated value can be obtained from the fixed point substitution (see [41]), and this method is applicable to more cases, and the general stability theorem can be obtained in a simple way. Isac and Rassias [16] first used the stability theory of functional equations to study a new fixed point theorem. Using a fixed point method, some authors studied the stability of several

Received: 15.01.2023. In revised form: 13.12.2023. Accepted: 31.03.2024
1991 *Mathematics Subject Classification.* 39B82, 39B72, 38B52.
Key words and phrases. *stability; partial multipliers; Banach $*$-algebras; fixed point method.*
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functional equations. In [17], Jung discussed the following Jensen's functional equation

$$2f(\frac{x+y}{2}) = f(x) + f(y)$$

and established the Hyers-Ulam-Rassias stability. And then Jung [18] also considered the stability of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

Recently, in [49] the authors investigated the Hyers-Ulam stability of the general functional equation

$$f(x+y) + g(x-y) = h(x) + k(y)$$

on an unbounded restricted domain, which generalized some of the results in literature. For more fruits about the stability of such kind functional equations, we can see [19, 25, 26, 27, 28, 33, 42, 54]. For some miscellaneous functional equations, Hyers-Ulam stability problems were discussed by many authors in the spirit of Rassias approach, see for instance [1, 7, 8, 10, 36, 45, 51, 52] and other resources. In addition, many people have studied the Hyers–Ulam stability of some derivations in algebras and rings, see [3, 5, 22, 23, 35]. Further, more generalisations and variants on the stability problem have been widely studied by a large of authors in different directions, see [4, 9, 12, 31, 32, 34, 37, 48, 50, 51] for example. At the same time, there are also many researches on partial multipliers related to additive functional equations. In 2016, Taghavi [46] introduced partial multipliers into complex Banach algebras. In Banach spaces or Banach algebras, many authors have widely studied the stability and partial multipliers of various functional equations and functional inequalities (see [11, 20, 21, 24, 36, 39, 43, 44, 47]).

In this paper, we solve the Hyers-Ulam stability by using fixed point method for the following new additive functional equations

$$f\left(\frac{x+y}{n_0}\right) = \frac{f(x)+f(y)}{n_0} \tag{1.1}$$

$$f\left(\frac{\lambda x+\lambda y}{n_0}\right) = \frac{f(\lambda x)+f(\lambda y)}{n_0} \tag{1.2}$$

in complex normed spaces, where n_0 is a positive integer with $n_0 \neq 1$, and $\lambda > 0$ is a fixed constant. It can be used to study partial multipliers related to additive functional equations (1.1) and (1.2) in complex Banach $*$-algebras.

We review some basic lemmas and definitions.

Lemma 1.1 ([6]). *Suppose that (X, d) is a complete generalized metric space and $J : X \to X$ is a strictly contractive mapping, with the Lipschitz constant L. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \text{for all nonnegative integers } n,$$

or there exists a natural number N such that

1. $d(J^n x, J^{n+1} x) < +\infty, \forall n \geq N$;
2. *The sequence $\{J^n x\}$ convergents to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set $Y = \{y \in X, d(J^N x, y) < +\infty\}$;*
4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy), \forall y \in Y$.

Definition 1.1 ([55]). *We use the term "Banach algebra" to refer to a vector algebra in which the underlying vector space is a complex Banach space and in which the multiplication satisfies the condition $\|xy\| \leqq \|x\|\|y\|$. A Banach algebra is called a Banach $*$-algebras if, for each element x, there exists a unique element x^* such that*

1. $(x^*)^* = x$;

2. $(xy)^* = y^* x^*$*;*

3. $(\alpha x + \beta y)^* = \bar{\alpha}\, x^* + \bar{\beta}\, y^*$*, where* α *and* β *are complex numbers,* $\bar{\alpha}$ *and* $\bar{\beta}$ *are their complex conjugates;*

4. $\| x x^* \| = \| x \|^2$*.*

Definition 1.2 ([46])**.** *Let* A *be a complex Banach* $*$*-algebra. A C-linear mapping* $P : A \to A$ *is called a partial multiplier if* P *satisfies*

$$P \circ P(xy) = P(x)P(y),$$
$$P(x^*) = P(x)^*$$

for all $x, y \in A$*.*

2. Stability of the additive functional equation (1.1)

We prove the Hyers-Ulam stability of additive functional equation (1.1) by using the fixed point method in complex normed spaces.

Theorem 2.1. *Let* E *be a complex normed space and* F *be a Banach space, and assume that the mapping* $f : E \to F$ *satisfies* $f(0) = 0$ *and the following inequality*

$$\| n_0 f\left(\frac{x+y}{n_0}\right) - f(x) - f(y) \| \le \varphi(x, y)\,, \forall x, y \in E, \tag{2.1}$$

where n_0 *is a positive integer,* $\varphi : E \times E \to [0, \infty)$ *is a given function. Moreover, there exists* $L \in (0, n_0)$ *such that*

$$\varphi(x, 0) \le L\, \varphi\left(\frac{x}{n_0}, 0\right), \forall x \in E,$$

and the mapping φ *also satisfies*

$$\lim_{n\to\infty} \frac{\varphi(n_0^{n+1}\, x, n_0^{n+1}\, y)}{n_0^{n+1}} = 0, \forall x, y \in E. \tag{2.2}$$

Then there exists a unique additive mapping $c : E \to F$ *such that*

$$\| f(x) - c(x) \| \le \frac{L}{n_0 - L}\varphi(x, 0), \forall x \in E.$$

Proof. Consider the set

$$X := \{g : E \to F, g(0) = 0\}\,,$$

and introduce the generalized metric on X :

$$d(g, h) = \inf\{\delta \in R_+ : \| g(x) - h(x) \| \le \delta\varphi(x, 0), \forall x \in E\}\,,$$

where, as usual, $\inf \emptyset = \infty$.

First, we prove that (X, d) is complete.

It is easy to see that d is symmetrical and $d(f, f) = 0$ for all $f \in X$. If $d(g, h) = 0$, then for every $x \in E$, we have $\| g(x) - h(x) \| \le 0$, which implies $g = h$. Next, if $d(f, g) = a < \infty$ and $d(g, h) = b < \infty$, then $\| f(x) - g(x) \| \le a\varphi(x, 0)$ and $\| g(x) - h(x) \| \le b\varphi(x, 0)$ for all $x \in E$, thus $\| f(x) - h(x) \| \le \| f(x) - g(x) \| + \| g(x) - h(x) \| \le a\varphi(x, 0) + b\varphi(x, 0) \le (a + b)\varphi(x, 0)$ for all $x \in E$, which implies $d(f, h) \le d(f, g) + d(g, h)$.

Suppose that $\{f_n\} \subset X$ is a Cauchy sequence. For a fixed x and $\forall \varepsilon > 0$, there exists positive integer N so that when $n, m \ge N$, there is $d(f_n, f_m) < \varepsilon$, that is $\| f_n(x) - f_m(x) \| \le \varepsilon\varphi(x, 0), n, m \ge N$. Hence $\{f_n(x)\}$ is a Cauchy sequence in F, then exists a mapping $f : E \to F$ with $f(0) = 0$, such that $\{f_n(x)\}$ converges to $f(x)$. Let $m \to \infty$ in $\| f_n(x) - f_m(x) \| \le \varepsilon\varphi(x, 0)$ and we get $\| f_n(x) - f(x) \| \le \varepsilon\varphi(x, 0)$. It can be seen that $\{f_n(x)\}$ uniformly converges to $f(x)$, so $f \in X$ and f_n converges to f.

Second, we consider the linear mapping

$$J : X \to X, Jg(x) = \frac{1}{n_0} g(n_0\, x), \forall x \in E.$$

Letting $d(g, h) = \omega$ for all $g, h \in X$, we have

$$\| g(x) - h(x) \| \leq \omega\varphi(x, 0), \forall x \in E.$$

Therefore,

$$\| \frac{1}{n_0} g(n_0\, x) - \frac{1}{n_0} h(n_0\, x) \| \leq \frac{1}{n_0}\, \omega\varphi(n_0\, x, 0) \leq \frac{L}{n_0} \omega\varphi(x, 0), \forall x \in E,$$

that is

$$d(Jg, Jh) \leq \frac{L}{n_0} d(g, h), \forall g, h \in X.$$

Letting $x = n_0\, t$ and $y = 0$ in (2.1), we obtain

$$\| n_0\, f(t) - f(n_0\, t) \| \leq \varphi(n_0\, t, 0) \leq L\, \varphi(t, 0), \forall t \in E,$$

and thus

$$\| f(x) - \frac{1}{n_0} f(n_0\, x) \| \leq \frac{1}{n_0} \varphi(n_0\, x, 0) \leq \frac{L}{n_0} \varphi(x, 0), \forall x \in E,$$

so $d(f, Jf) \leq \frac{L}{n_0} < 1$.

According to Lemma 1.1, there exists a mapping $c : X \to X$ such that

1. c is a fixed point of J, that is

$$Jc(x) = c(x) = \frac{1}{n_0} c(n_0\, x), \forall x \in E. \tag{2.3}$$

The mapping c is a unique fixed point of J in the set

$$Y = \{g \in X, d(f, g) < \infty\}.$$

This shows that c is a unique mapping satisfying (2.3) and there is $\delta \in (0, \infty)$, such that

$$\| f(x) - c(x) \| \leq \delta\varphi(x, 0), \forall x \in E.$$

2. $d(J^n\, f, c) \to 0, n \to \infty$, which implies the equality

$$\lim_{n\to\infty} \frac{f(n_0^n\, x)}{n_0^n} = c(x), \forall x \in E.$$

3. $d(f, c) \leq \frac{n_0}{n_0 - L} d(f, Jf)$, which implies the inequality

$$\| f(x) - c(x) \| \leq \frac{L}{n_0 - L} \varphi(x, 0), \forall x \in E.$$

Finally, we prove that c is additive.

Letting $x = n_0^{n+1}\, u$ and $y = n_0^{n+1}\, v$ in (2.1), we get

$$\| \frac{f(n_0^n\, u + n_0^n\, v)}{n_0^n} - \frac{f(n_0^{n+1}\, u)}{n_0^{n+1}} - \frac{f(n_0^{n+1}\, v)}{n_0^{n+1}} \| \leq \frac{\varphi(n_0^{n+1}\, u, n_0^{n+1}\, v)}{n_0^{n+1}}, \forall u, v \in E,$$

taking into consideration (2.2) and letting $n \to \infty$, we obtain

$$c(u + v) = c(u) + c(v), \forall u, v \in E.$$

Consequently, the mapping $c : E \to F$ is additive. The proof is completed. □

Corollary 2.1. *Let E be a complex normed space and F be a Banach space, $0 < p < 1$ and θ be nonnegative real numbers, and suppose that the mapping $f : E \to F$ satisfies $f(0) = 0$ and the following inequality*

$$\| n_0 f\left(\frac{x+y}{n_0}\right) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| y \|^p), \forall x, y \in E, \tag{2.4}$$

where n_0 is a positive integer. Then there exists a unique additive mapping $c : E \to F$ such that

$$\| f(x) - c(x) \| \leq \frac{n_0^p \theta}{n_0 - n_0^p} \| x \|^p, \forall x \in E.$$

Proof. For $x, y \in E$, letting $L = n_0^p$ and $\varphi(x, y) = \theta(\| x \|^p + \| y \|^p)$ in Theorem 2.1, we get

$$\varphi(x, 0) = \theta \| x \|^p, \varphi\left(\frac{x}{n_0}, 0\right) = \frac{\theta}{n_0^p} \| x \|^p .$$

Since $\theta \| x \|^p = \frac{n_0^p \theta}{n_0^p} \| x \|^p$, we have

$$\varphi(x, 0) \leq L \varphi\left(\frac{x}{n_0}, 0\right), \forall x \in E.$$

Meanwhile,

$$\lim_{n\to\infty} \frac{\varphi(n_0^{n+1} x, n_0^{n+1} y)}{n_0^{n+1}} = \lim_{n\to\infty} n_0^{(n+1)(p-1)} \theta(\| x \|^p + \| y \|^p) = 0, \forall x, y \in E.$$

So the conclusion is established. □

Theorem 2.2. *Let E be a complex normed space and F be a Banach space, and assume that the mapping $f : E \to F$ satisfies $f(0) = 0$ and (2.1), where n_0 is a positive integer, $\varphi : E \times E \to [0, \infty)$ is a given function. Moreover, there exists $L \in (0, \frac{1}{n_0})$ such that*

$$\varphi(x, 0) \leq L\varphi(n_0 x, 0), \forall x \in E,$$

and the mapping φ also satisfies

$$\lim_{n\to\infty} \frac{\varphi(n_0^{1-n} x, n_0^{1-n} y)}{n_0^{1-n}} = 0, \forall x, y \in E.$$

Then there exists a unique additive mapping $c : E \to F$ such that

$$\| f(x) - c(x) \| \leq \frac{1}{1 - n_0 L} \varphi(x, 0), \forall x \in E.$$

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the mapping

$$J : X \to X, Jg(x) = n_0 g\left(\frac{1}{n_0} x\right), \forall x \in E.$$

By letting $y = 0$ in (2.1), we get

$$\| n_0 f\left(\frac{1}{n_0} x\right) - f(x) \| \leq \varphi(x, 0) \leq L\varphi(n_0 x, 0), \forall x \in E,$$

hence $d(f, Jf) \leq 1$.

The remaining proof is similar to the proof of Theorem 2.1. □

Corollary 2.2. *Let E be a complex normed space and F be a Banach space, $p > 1$ and θ be nonnegative real numbers, and suppose that the mapping $f : E \to F$ satisfies $f(0) = 0$ and (2.4), where n_0 is a positive integer. Then there exists a unique additive mapping $c : E \to F$ such that*

$$\| f(x) - c(x) \| \leq \frac{n_0^p \, \theta}{n_0^p - n_0} \| x \|^p, \forall x \in E.$$

Proof. For $\forall x, y \in E$, letting $L = n_0^{-p}$and $\varphi(x, y) = \theta(\| x \|^p + \| y \|^p)$ in Theorem 2.2, and the rest of the proof is similar to Corollary 2.1. □

Corollary 2.3 ([6], Theorem 3.1)**.** *Let E be a (real or complex) linear space and F be a Banach space, and $q_i = \begin{cases} 2, i = 0 \\ \frac{1}{2}, i = 1 \end{cases}$. Suppose that the mapping $f : E \to F$ satisfies the condition $f(0) = 0$ and an inequality of the form*

$$\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \| \leq \varphi(x, y), \forall x, y \in E,$$

where $\varphi : E \times E \to [0, \infty)$ is a given function.

If there exists $L = L(i) < 1$ such that the mapping

$$x \to \psi(x) = \varphi(x, 0)$$

has the property

$$\psi(x) \leq L \cdot q_i \cdot \psi\left(\frac{x}{q_i}\right), \forall x \in E,$$

and the mapping φ has the property

$$\lim_{n\to\infty} \frac{\varphi(2\, q_i^n \, x, 2\, q_i^n \, y)}{2\, q_i^n} = 0, \forall x, y \in E,$$

then there exists a unique additive mapping $j : E \to F$ such that

$$\| f(x) - j(x) \| \leq \frac{L^{1-i}}{1-L} \psi(x), \forall x \in E.$$

Proof. For $\forall x, y \in E$, take $n_0 = 2$ in Theorem 2.1 and Theorem 2.2, the conclusion holds immediately. □

3. Stability of the additive functional equation (1.2)

We prove the Hyers-Ulam stability of additive functional equation (1.2) by using the fixed point method in complex normed spaces.

Theorem 3.3. *Let E be a complex normed space and F be a Banach space, and assume that the mapping $f : E \to F$ satisfies $f(0) = 0$ and the following inequality*

$$\| n_0 \, f\left(\frac{\lambda x + \lambda y}{n_0}\right) - f(\lambda x) - f(\lambda y) \| \leq \varphi(\lambda x, \lambda y), , \forall x, y \in E, \tag{3.1}$$

where n_0 is a positive integer, $\lambda > 0$ is a fixed constant ,and $\varphi : E \times E \to [0, \infty)$ is a given function. Moreover, there exists $L \in (0, n_0)$ such that

$$\varphi(\lambda x, 0) \leq L\, \varphi\left(\frac{\lambda x}{n_0}, 0\right), \forall x \in E,$$

and the mapping φ also satisfies

$$\lim_{n\to\infty} \frac{\varphi(n_0^{n+1} \, \lambda x, n_0^{n+1} \, \lambda y)}{n_0^{n+1}} = 0, \forall x, y \in E. \tag{3.2}$$

Then there exists a unique additive mapping $c : E \to F$ *such that*

$$\| f(\lambda x) - c(\lambda x) \| \leq \frac{L}{n_0 - L}\varphi(\lambda x, 0), \forall x \in E.$$

Proof. Consider the set

$$X := \{g : E \to F, g(0) = 0\},$$

and introduce the generalized metric on X :

$$d(g, h) = \inf \{\delta \in R_+ :\| g(\lambda x) - h(\lambda x) \| \leq \delta\varphi(\lambda x, 0), \forall x \in E\},$$

where, as usual, $\inf \emptyset = \infty$. According to the proof of Theorem 2.1, it is easy to know that (X, d) is complete.

Now we consider the linear mapping

$$J : X \to X, Jg(\lambda x) = \frac{1}{n_0} g(n_0 \lambda x), \forall x \in E.$$

Letting $d(g, h) = \omega$ for all $g, h \in X$, we have

$$\| g(\lambda x) - h(\lambda x) \| \leq \omega\varphi(\lambda x, 0), \forall x \in E.$$

Therefore,

$$\| \frac{1}{n_0} g(n_0 \lambda x) - \frac{1}{n_0} h(n_0 \lambda x) \| \leq \frac{1}{n_0}\omega\varphi(n_0 \lambda x, 0) \leq \frac{L}{n_0}\omega\varphi(\lambda x, 0), \forall x \in E,$$

that is

$$d(Jg, Jh) \leq \frac{L}{n_0} d(g, h), \forall g, h \in X.$$

Letting $x = n_0 t$ and $y = 0$ in (3.1), we obtain

$$\| n_0 f(\lambda t) - f(n_0 \lambda t) \| \leq \varphi(n_0 \lambda t, 0) \leq L\varphi(\lambda t, 0), \forall t \in E,$$

thus

$$\| f(\lambda x) - \frac{1}{n_0} f(n_0 \lambda x) \| \leq \frac{1}{n_0}\varphi(n_0 \lambda x, 0) \leq \frac{L}{n_0}\varphi(\lambda x, 0), \forall x \in E,$$

so $d(f, Jf) \leq \frac{L}{n_0} < 1$.

According to Lemma 1.1, there exists a mapping $c : X \to X$ such that

1. c is a fixed point of J, that is

$$Jc(\lambda x) = c(\lambda x) = \frac{1}{n_0} c(n_0 \lambda x), \forall x \in E. \tag{3.3}$$

The mapping c is a unique fixed point of J in the set

$$Y = \{g \in X, d(f, g) < \infty\}.$$

This shows that c is a unique mapping satisfying (3.3) and there is $\delta \in (0, \infty)$, such that

$$\| f(\lambda x) - c(\lambda x) \| \leq \delta\varphi(\lambda x, 0), \forall x \in E.$$

2. $d(J^n f, c) \to 0, n \to \infty$, which implies the equality

$$\lim_{n\to\infty} \frac{f(n_0^n \lambda x)}{n_0^n} = c(\lambda x), \forall x \in E.$$

3. $d(f, c) \leq \frac{n_0}{n_0 - L} d(f, Jf)$, which implies the inequality

$$\| f(\lambda x) - c(\lambda x) \| \leq \frac{L}{n_0 - L}\varphi(\lambda x, 0), \forall x \in E.$$

Finally, we prove that c is additive.

Letting $x = n_0^{n+1} u$ and $y = n_0^{n+1} v$ in (2.1), we get

$$\| \frac{f(n_0^n \lambda u + n_0^n \lambda v)}{n_0^n} - \frac{f(n_0^{n+1} \lambda u)}{n_0^{n+1}} - \frac{f(n_0^{n+1} \lambda v)}{n_0^{n+1}} \| \leq \frac{\varphi(n_0^{n+1} \lambda u, n_0^{n+1} \lambda v)}{n_0^{n+1}}, \forall u, v \in E,$$

taking into consideration (3.2) and letting $n \to \infty$, we obtain

$$c(\lambda u + \lambda v) = c(\lambda u) + c(\lambda v), \forall u, v \in E.$$

Consequently, the mapping $c : E \to F$ is additive. The proof is completed. □

Corollary 3.4. *Let E be a complex normed space and F be a Banach space, $0 < p < 1$ and θ be nonnegative real numbers, and suppose that the mapping $f : E \to F$ satisfies $f(0) = 0$ and the following inequality*

$$(3.4) \qquad \| n_0 f\left(\frac{\lambda x + \lambda y}{n_0}\right) - f(\lambda x) - f(\lambda y) \| \leq \theta \lambda^p (\| x \|^p + \| y \|^p), \forall x, y \in E,$$

where n_0 is a positive integer, $\lambda > 0$ is a fixed constant.Then there exists a unique additive mapping $c : E \to F$ such that

$$\| f(x) - c(x) \| \leq \frac{n_0^p \lambda^p \theta}{n_0 - n_0^p} \| x \|^p, \forall x \in E.$$

Proof. For $\forall x, y \in E$, letting $L = n_0^p$ and $\varphi(\lambda x, \lambda y) = \theta \lambda^p (\| x \|^p + \| y \|^p)$ in Theorem 3.3, and the rest of the proof is similar to Corollary 2.1. □

Theorem 3.4. *Let E be a complex normed space and F be a Banach space, and assume that the mapping $f : E \to F$ satisfies $f(0) = 0$ and (3.1), where n_0 is a positive integer, $\lambda > 0$ is a fixed constant, and $\varphi : E \times E \to [0, \infty)$ is a given function. Moreover, there exists $L \in (0, \frac{1}{n_0})$ such that*

$$\varphi(\lambda x, 0) \leq L \varphi(n_0 \lambda x, 0), \forall x \in E,$$

and the mapping φ satisfies

$$\lim_{n \to \infty} \frac{\varphi(n_0^{1-n} \lambda x, n_0^{1-n} \lambda y)}{n_0^{1-n}} = 0, \forall x, y \in E.$$

Then there exists a unique additive mapping $c : E \to F$ such that

$$\| f(\lambda x) - c(\lambda x) \| \leq \frac{1}{1 - n_0 L} \varphi(\lambda x, 0), \forall x \in E.$$

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 3.3. Now we consider the mapping

$$J : X \to X, Jg(\lambda x) = n_0 g\left(\frac{1}{n_0} \lambda x\right), \forall x \in E.$$

By letting $y = 0$ in (3.1), we get

$$\| n_0 f\left(\frac{1}{n_0} \lambda x\right) - f(\lambda x) \| \leq \varphi(\lambda x, 0) \leq L \varphi(n_0 \lambda x, 0), \forall x \in E,$$

hence $d(f, Jf) \leq 1$.

The remaining proof is similar to the proof of Theorem 3.3. □

Corollary 3.5. *Let E be a complex normed space and F be a Banach space, $p > 1$ and θ be nonnegative real numbers, and suppose that the mapping $f : E \to F$ satisfies $f(0) = 0$ and*

(3.4), where n_0 is a positive integer, $\lambda > 0$ is a fixed constant. Then there exists a unique additive mapping $c : E \to F$ such that

$$\| f(x) - c(x) \| \leq \frac{n_0^p \, \lambda^p \, \theta}{n_0^p - n_0} \| x \|^p, \forall x \in E.$$

Proof. *For $\forall x, y \in E$, letting $L = n_0^{-p}$and $\varphi(\lambda x, \lambda y) = \theta \, \lambda^p(\| x \|^p + \| y \|^p)$ in Theorem 3.4, the rest of the proof is similar to Corollary 2.1.*

4. Partial multipliers related to additive functional equation (1.1)

In this section, we will study partial multipliers related to additive functional equation (1.1) in complex Banach $*$-algebra.

Theorem 4.5. *Let G be a complex Banach $*$-algebra, and assume that the mapping $f : G \to G$ satisfies $f(0) = 0$ and the following inequality*

$$\| n_0 \, f\left(\frac{x+y}{n_0}\right) - f(x) - f(y) \| \leq \varphi(x, y) \, , \forall x, y \in G, \tag{4.1}$$

where n_0 is a positive integer, $\varphi : G \times G \to [0, \infty)$ is a given function. Moreover, there exists $L \in (0, n_0)$ such that

$$\varphi(x, y) \leq L\varphi\left(\frac{x}{n_0}, \frac{y}{n_0}\right), \forall x \in G, \tag{4.2}$$

and the mapping φ also satisfies

$$\lim_{n\to\infty} \frac{\varphi(n_0^{n+1} \, x, n_0^{n+1} \, y)}{n_0^{n+1}} = 0, \forall x, y \in G.$$

Then there exists a unique additive mapping $d : G \to G$ such that

$$\| f(x) - d(x) \| \leq \frac{L}{n_0 - L} \varphi(x, 0), \forall x \in G. \tag{4.3}$$

In addition, if the mapping $f : G \to G$ satisfies $f(n_0 \, x) = n_0 \, f(x)$ and

$$\| f \circ f(xy) - f(x) f(y) \| \leq \varphi(x, y), \forall x, y \in G, \tag{4.4}$$

$$\| f(x^*) - f(x)^* \| \leq \varphi(x, 0), \forall x \in G, \tag{4.5}$$

then the mapping f is a partial multiplier.

Proof. According to Theorem 2.1, there exists a unique additive mapping $d : X \to X$ such that (4.3) is established and

$$d(x) := \lim_{n\to\infty} \frac{1}{n_0^n} f(n_0^n \, x), \forall x \in G.$$

If $f(n_0 \, x) = n_0 \, f(x)$, then

$$d(x) := \lim_{n\to\infty} \frac{1}{n_0^n} f(n_0^n \, x) = \lim_{n\to\infty} \frac{1}{n_0^{n-1}} f(n_0^{n-1} \, x) = \lim_{n\to\infty} \frac{1}{n_0} f(n_0 \, x) = f(x), \forall x \in G.$$

On the basis of (4.2) and (4.4), we get

$$\begin{aligned}\| f\circ f(xy)-f(x)f(y)\| &= \| d\circ d(xy)-d(x)d(y)\| \\ &= \lim_{n\to\infty}\frac{1}{n_0^{2n}}\| f\circ f(n_0^n x\cdot n_0^n y)-f(n_0^n x)f(n_0^n y)\| \\ &\le \lim_{n\to\infty}\frac{1}{n_0^{2n}}\varphi(n_0^n x, n_0^n y) \\ &\le \lim_{n\to\infty}\frac{L^n}{n_0^{2n}}\varphi(x,y)=0, \forall x,y\in G.\end{aligned}$$

Therefore,

$$f\circ f(xy)=f(x)f(y), \forall x,y\in G.$$

On the basis of (4.2) and (4.5), we get

$$\begin{aligned}\| f(x^*)-f(x)^*\| &= \| d(x^*)-d(x)^*\| \\ &= \lim_{n\to\infty}\frac{1}{n_0^{n}}\| f(n_0^n x^*)-f(n_0^n x)^*\| \\ &\le \lim_{n\to\infty}\frac{1}{n_0^{n}}\varphi(n_0^n x, 0) \\ &\le \lim_{n\to\infty}\frac{L^n}{n_0^{n}}\varphi(x,0)=0, \forall x,y\in G.\end{aligned}$$

Therefore,

$$f(x^*)=f(x)^*, \forall x,y\in G.$$

Thus, the mapping $f: G\to G$ is a partial multiplier. □

Corollary 4.6. *Let G be a complex Banach $*$-algebra, $0<\mathrm{p}<1$ and θ be nonnegative real numbers, and suppose that the mapping $f: E\to F$ satisfies $f(0)=0$ and the following inequality*

$$\| n_0 f\left(\frac{x+y}{n_0}\right)-f(x)-f(y)\| \le \theta(\| x\|^p+\| y\|^p), \forall x,y\in G, \tag{4.6}$$

where n_0 is a positive integer. Then there exists a unique additive mapping $d: G\to G$ such that

$$\| f(x)-d(x)\| \le \frac{n_0^p\theta}{n_0-n_0^p}\| x\|^p, \forall x\in G.$$

In addition, if the mapping $f: G\to G$ satisfies $f(n_0 x)=n_0 f(x)$ and

$$\| f\circ f(xy)-f(x)f(y)\| \le \theta(\| x\|^p+\| y\|^p), \forall x,y\in G, \tag{4.7}$$

$$\| f(x)^* - f(x^*)\| \le \theta\| x\|^p, \forall x\in G, \tag{4.8}$$

then the mapping f is a partial multiplier.

Proof. For $x,y\in G$, letting $L=n_0^p$ and $\varphi(x,y)=\theta(\| x\|^p+\| y\|^p)$ in Theorem 4.5, we get

$$\varphi\left(\frac{x}{n_0},\frac{y}{n_0}\right)=\frac{\theta}{n_0^p}(\| x\|^p+\| y\|^p).$$

Since $\theta\| x\|^p=\frac{n_0^p\theta}{n_0^p}\| x\|^p$, we have

$$\varphi(x,y)\le L\varphi\left(\frac{x}{n_0},\frac{y}{n_0}\right), \forall x\in G.$$

Meanwhile,

$$\begin{aligned} \| f\circ f(xy)-f(x)f(y) \| &\le \lim_{n\to\infty}\frac{L^{2n}}{n_0^n}\varphi(x,y) \\ &= n_0^{(p-2)n}\,\theta(\| x \|^p+\| y \|^p)=0,\forall x,y\in G, \end{aligned}$$

$$\begin{aligned} \| f(x^*)-f(x)^* \| &\le \lim_{n\to\infty}\frac{L^{n}}{n_0^n}\varphi(x,0) \\ &= \lim_{n\to\infty} n_0^{(p-1)n}\,\theta \| x \|^p=0,\forall x\in G. \end{aligned}$$

Namely,

$$f\circ f(xy)=f(x)f(y), f(x^*)=f(x)^*,\forall x,y\in G.$$

So the conclusion is established. □

Theorem 4.6. *Let G be a complex Banach $*$-algebra, and assume that the mapping $f : G\to G$ satisfies $f(0)=0$ and (4.1), where n_0 is a positive integer, $\varphi : G\times G\to[0,\infty)$ is a given function. Moreover, there exists $L\in(0,\frac{1}{n_0})$ such that*

$$\varphi(x,0)\le L\varphi(n_0\,x,0),\forall x\in G,$$

and the mapping φ also satisfies

$$\lim_{n\to\infty}\frac{\varphi(n_0^{1-n}\,x,n_0^{1-n}\,y)}{n_0^{1-n}}=0,\forall x,y\in G.$$

Then there exists a unique additive mapping $d : G\to G$ such that

$$\| f(x)-d(x) \|\le \frac{1}{1-n_0L}\varphi(x,0),\forall x\in G.$$

In addition, if the mapping $f : G\to G$ satisfies $f(n_0\,x)=n_0\,f(x)$, (4.4) and (4.5), then the mapping f is a partial multiplier.

Proof. The proof is similar to the proof of Theorem 4.5. □

Corollary 4.7. *Let G be a complex Banach $*$-algebra, $\mathrm{p}>1$ and θ be nonnegative real numbers, and suppose that the mapping $f : E\to F$ satisfies $f(0)=0$ and (4.6), where n_0 is a positive integer. Then there exists a unique additive mapping $d : G\to G$ such that*

$$\| f(x)-d(x) \|\le \frac{n_0^p\,\theta}{n_0^p-n_0}\| x \|^p,\forall x\in G.$$

In addition, if the mapping $f : G\to G$ satisfies $f(n_0\,x)=n_0\,f(x)$, (4.7) and (4.8), then the mapping f is a partial multiplier.

Proof. For $\forall x,y\in G$, letting $L=n_0^{-p}$ and $\varphi(x,y)=\theta(\| x \|^p+\| y \|^p)$ in Theorem 4.6, and the rest of the proof is similar to Corollary 4.6. □

5. Partial multipliers related to additive functional equation (1.2)

In this section, we will study partial multipliers related to additive functional equation (1.2) in complex Banach $*$-algebra.

Theorem 5.7. *Let G be a complex Banach $*$-algebra, and assume that the mapping $f : G\to G$ satisfies $f(0)=0$ and the following inequality*

$$\| n_0\,f\left(\frac{\lambda x+\lambda y}{n_0}\right)-f(\lambda x)-f(\lambda y) \|\le\varphi(\lambda x,\lambda y),\forall x,y\in G, \tag{5.1}$$

where n_0 is a positive integer, $\lambda > 0$ is a fixed constant, and $\varphi : G \times G \to [0, \infty)$ is a given function. Moreover, there exists $L \in (0, n_0)$ such that

$$\varphi(\lambda x, \lambda y) \leq L\varphi\left(\frac{\lambda x}{n_0}, \frac{\lambda y}{n_0}\right), \forall x \in G,$$

and the mapping φ also satisfies

$$\lim_{n\to\infty} \frac{\varphi(n_0^{n+1}\,\lambda x, n_0^{n+1}\,\lambda y)}{n_0^{n+1}} = 0, \forall x, y \in G.$$

Then there exists a unique additive mapping $d : G \to G$ such that

$$\| f(\lambda x) - d(\lambda x) \| \leq \frac{L}{n_0 - L}\varphi(\lambda x, 0), \forall x \in G.$$

In addition, if the mapping $f : G \to G$ satisfies $f(n_0\,\lambda x) = n_0\, f(\lambda x)$ and

$$\| f \circ f(\lambda^2\, xy) - f(\lambda x)f(\lambda y) \| \leq \varphi(\lambda x, \lambda y), \forall x, y \in G, \tag{5.2}$$

$$\| f(\lambda x^*) - f(\lambda x)^* \| \leq \varphi(\lambda x, 0), \forall x \in G, \tag{5.3}$$

then the mapping f is a partial multiplier.

Proof. The proof is similar to the proof of Theorem 4.5. □

Corollary 5.8. *Let G be a complex Banach $*$-algebra, $0 < \mathrm{p} < 1$ and θ be nonnegative real numbers, and suppose that the mapping $f : E \to F$ satisfies $f(0) = 0$ and the following inequality*

$$\| n_0\, f\left(\frac{\lambda x + \lambda y}{n_0}\right) - f(\lambda x) - f(\lambda y) \| \leq \theta\, \lambda^p(\| x \|^p + \| y \|^p), \forall x, y \in G, \tag{5.4}$$

where n_0 is a positive integer and $\lambda > 0$ is a fixed constant. Then there exists a unique additive mapping $d : G \to G$ such that

$$\| f(x) - d(x) \| \leq \frac{n_0^p\, \lambda^p\, \theta}{n_0 - n_0^p} \| x \|^p, \forall x \in G.$$

In addition, if the mapping $f : G \to G$ satisfies $f(n_0\,\lambda x) = n_0\, f(\lambda x)$ and

$$\| f \circ f(\lambda^2\, xy) - f(\lambda x)f(\lambda y) \| \leq \theta\, \lambda^p(\| x \|^p + \| y \|^p), \forall x, y \in G, \tag{5.5}$$

$$\| f(\lambda x^*) - f(\lambda x)^* \| \leq \theta\, \lambda^p \| x \|^p, \forall x \in G, \tag{5.6}$$

then the mapping f is a partial multiplier.

Proof. For $\forall x, y \in G$, letting $L = n_0^p$ and $\varphi(\lambda x, \lambda y) = \theta\, \lambda^p(\| x \|^p + \| y \|^p)$ in Theorem 5.7, the rest of the proof is similar to Corollary 4.6. □

Theorem 5.8. *Let G be a complex Banach $*$-algebra, and assume that the mapping $f : G \to G$ satisfies $f(0) = 0$ and (5.1), where n_0 is a positive integer, $\lambda > 0$ is a fixed constant, and $\varphi : G \times G \to [0, \infty)$ is a given function. Moreover, there exists $L \in (0, \frac{1}{n_0})$ such that*

$$\varphi(\lambda x, \lambda y) \leq L\varphi(n_0\,\lambda x, n_0\,\lambda y), \forall x \in G,$$

and the mapping φ also satisfies

$$\lim_{n\to\infty} \frac{\varphi(n_0^{1-n}\,\lambda x, n_0^{1-n}\,\lambda y)}{n_0^{1-n}} = 0, \forall x, y \in G.$$

Then there exists a unique additive mapping $d : G \to G$ such that

$$\| f(\lambda x) - d(\lambda x) \| \leq \frac{1}{1 - n_0 L}\varphi(\lambda x, 0), \forall x \in G.$$

In addition, if the mapping $f : G \to G$ satisfies $f(n_0\,\lambda x) = n_0\,f(\lambda x)$, (5.2) and (5.3), then the mapping f is a partial multiplier.

Proof. The proof is similar to the proof of Theorem 4.5. □

Corollary 5.9. *Let G be a complex Banach $*$-algebra, $\mathrm{p} > 1$ and θ be nonnegative real numbers, and suppose that the mapping $f : E \to F$ satisfies $f(0) = 0$ and (5.4), where n_0 is a positive integer and $\lambda > 0$ is a fixed constant. Then there exists a unique additive mapping $d : G \to G$ such that*

$$\| f(x) - d(x) \| \leq \frac{n_0^p\,\lambda^p\,\theta}{n_0^p - n_0} \| x \|^p, \forall x \in G.$$

In addition, if the mapping $f : G \to G$ satisfies $f(n_0\,\lambda x) = n_0\,f(\lambda x)$, (5.5) and (5.6), then the mapping f is a partial multiplier.

Proof. For $\forall x, y \in G$, letting $L = n_0^{-p}$ and $\varphi(\lambda x, \lambda y) = \theta\,\lambda^p(\| x \|^p + \| y \|^p)$ in Theorem 5.8, the rest of the proof is similar to Corollary 4.6. □

Remark 5.1. *From literature, we know that equations (1.1),(1.2) are new forms and the relative results have not been seen. When $n_0 = 2$, the results are also new. In* [13]*, the author used the Hyers' method to study the stability of additive functional equation $f(x + y) = f(x) + f(y)$. In* [17]*, the author established Hyers-Ulam-Rassias stability for the Jensen functional equation, and the result is applied to the study of an asymptotic behavior of the additive mapping $f(\frac{x+y}{2}) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$, and in* [6] *the authors studied the same problem by using fixed point theorems. Evidently, our results are some generalizations of ones in* [6, 17]*. By the same method, the paper* [39] *studied a system of additive functional equations*

$$2f(x + y) - g(x) = g(y),$$
$$g(x + y) - 2f(x - y) = 4f(x).$$

We can see that our technique is working in more situations, allowing to get, in a simple manner, general stability theorems.

6. Conclusions

In this manuscript, we studied functional equations (1.1),(1.2). We used fixed point method to give some new results of the Hyers-Ulam stability and we establish some new results about partial multipliers related to additive functional equations in complex Banach $*$-algebras. Evidently, our results include the special case $n_0 = 2$ and it is also new. It should be pointed out that the ideas from our proofs can be applied to some similar generalizations of functional equations.

Acknowledgments

This paper was supported by the Shanxi Province Science Foundation (201901D111020). The authors wish to thank the anonymous referees for their valuable suggestions.

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