

Approximation by polynomials with constant coefficients and the Thresholding Greedy Algorithm

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ABSTRACT. Greedy bases are those bases where the Thresholding Greedy Algorithm (introduced by S. V. Konyagin and V. N. Temlyakov) produces the best possible approximation up to a constant. In 2017, P. M. Berná and Ó. Blasco gave a characterization of these bases using polynomials with constant coefficients. In this paper, we continue this study improving some optimization problems and extending some results to the context of quasi-Banach spaces.

1. INTRODUCTION

Since 1999, one of the most important algorithms in the field of non-linear approximation theory in quasi-Banach spaces is the Thresholding Greedy Algorithm $(G_m)_{m \in \mathbb{N}}$, where, for a given element f in the space \mathbb{X} , the algorithm selects the largest coefficients of f (in modulus) respect to a given basis $(x_j)_{j=1}^\infty$ in the space. This algorithm was introduced by S. V. Konyagin and V. N. Temlyakov in [13] and the convergence of the TGA has been extensively studied from different perspectives by various researchers, such as F. Albiac, J. L. Ansorena, D. Kutzarova, N. J. Kalton, S. J. Dilworth and P. Wojtaszczyk ([4, 10, 11, 16]), among others. Thanks to these studies, in the literature we can find the so called *greedy-like bases*, that is, bases where the TGA converges in some way.

One of the most important contributions to the theory of greedy-like bases was introduced in [13], where the authors established the notion of a greedy basis as the one for which the TGA produces the best possible approximation up to a constant, that is, for every $f \in \mathbb{X}$ and $m \in \mathbb{N}$,

$$(1.1) \quad \|f - G_m(f)\| \approx \inf \left\{ \left\| f - \sum_{n \in A} a_n x_n \right\| : |A| \leq m, a_n \in \mathbb{C} \forall n \in A \right\}.$$

In [7], P. M. Berná and Ó. Blasco proved an equivalent version of greediness showing that a basis is greedy if and only if for every $f \in \mathbb{N}$ and $m \in \mathbb{N}$,

$$(1.2) \quad \|f - G_m(f)\| \lesssim \inf \left\{ \left\| f - \sum_{j \in A} \alpha x_j \right\| : \alpha \in \mathbb{C}, |A| \leq m \right\},$$

The element $y = \sum_{j \in A} \alpha x_j$ is called a *polynomial with constant coefficients*.

In this paper, we further advance the theory initiated by P. M. Berná and Ó. Blasco. Specifically, we extend the theory of [7] to the context of quasi-Banach spaces, examining whether the characterizations of greedy-like bases using polynomials of constant coefficients are applicable in this setting. Furthermore, we aim to relax the condition (1.2) even further by proving that a basis is greedy if and only if (1.2) is satisfied for a specific value of α .

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The structure of the paper is as follows: in Section 2, we introduce the main definitions that we use and the main results that we prove. In Section 3, we prove some technical results concerning some properties about symmetry and unconditionality, where respect to the last one, we provide a new characterization in terms of elements with positive coefficients. In Sections 4 and 5, we prove Theorems 2.3 and 2.4 where we characterize greediness and almost-greediness in terms of polynomials of constant coefficients with α fixed.

2. NOTATION AND MAIN DEFINITIONS

A *quasi-norm* on a vector space \mathbb{X} over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} is a map $\|\cdot\| : \mathbb{X} \rightarrow [0, +\infty)$ verifying the following three conditions:

- N1) $\|f\| \geq 0$ for all $f \neq 0$ and $\|f\| = 0$ if and only if $f = 0$,
- N2) $\|tf\| = |t|\|f\|$ for all $t \in \mathbb{F}$ and all $f \in \mathbb{X}$,
- N3) there is $c > 0$ such that for all $f, g \in \mathbb{X}$, $\|f + g\| \leq c(\|f\| + \|g\|)$.

Then, it is well known that the collection

$$\left\{ f \in \mathbb{X} : \|f\| < \frac{1}{n} \right\}, \quad n \in \mathbb{N},$$

is a base of neighbourhoods of zero, so the quasi-norm induces a metrizable linear topology and if \mathbb{X} is complete, we say that $(\mathbb{X}, \|\cdot\|)$ is a *quasi-Banach space*.

Given $0 < p \leq 1$, we remind that a map $\|\cdot\| : \mathbb{X} \rightarrow [0, +\infty)$ is a *p-norm* if the map verifies the conditions N1), N2) and

- N4) for all $f, g \in \mathbb{X}$, $\|f + g\|^p \leq \|f\|^p + \|g\|^p$.

Of course, N4) implies N3) with $c = 2^{1/p-1}$ and every quasi-Banach space whose quasi-norm is a *p-norm* is called a *p-Banach space* and thanks to the Aoiki-Rolewicz's Theory (see [5, 15]), every quasi-Banach space becomes *p-Banach* under a suitable renorming.

In the case of *p-Banach spaces*, we define the following geometrical constants

$$(2.3) \quad A_p := (2^p - 1)^{-1/p}$$

and

$$(2.4) \quad B_p := \begin{cases} 2^{1/p} A_p & \text{if } \mathbb{F} = \mathbb{R}, \\ 4^{1/p} A_p & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

The following is a useful result from [3] that allows us to circumvent in some context the lack of convexity when $0 < p < 1$.

Lemma 2.1. [3, Corollaries 2.3, 2.4] *Let \mathbb{X} be a *p-Banach space* for some $0 < p \leq 1$. Let $(a_j)_{j \in A}$ be any collection of scalars in \mathbb{F} and $(f_j)_{j \in A}$ any collection of vectors in \mathbb{X} with $A \subset \mathbb{N}$ a finite set. Then:*

$$\begin{aligned} \left\| \sum_{j \in A} a_j f_j \right\| &\leq B_p \max_{j \in A} |a_j| \max_{B \subset A} \left\| \sum_{j \in B} f_j \right\|; \\ \left\| \sum_{j \in A} a_j f_j \right\| &\leq A_p \max_{j \in A} |a_j| \sup \left\{ \left\| \sum_{j \in A} \varepsilon_j f_j \right\| : |\varepsilon_j| = 1 \right\}. \end{aligned}$$

Now, we consider that we have a *basis* $\mathcal{X} = (\mathbf{x}_n)_{n \in \mathbb{N}}$ in a quasi-Banach space \mathbb{X} over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, that is, \mathcal{X} is a collection of elements of the space satisfying the following conditions:

- $[\mathbf{x}_n : n \in \mathbb{N}] = \mathbb{X}$, where $[E]$ denotes the closure of the linear span generated by the elements of the subspace $E \subset \mathbb{X}$;

- there is a unique sequence of biorthogonal functions $\mathcal{X}^* = (\mathbf{x}_n^*)_{n \in \mathbb{N}}$ such that $\mathbf{x}_n^*(\mathbf{x}_m) = \delta_{n,m}$;
- there are $c_1, c_2 > 0$ such that

$$\sup_{n \in \mathbb{N}} \|\mathbf{x}_n\| \leq c_1 < \infty, \sup_{n \in \mathbb{N}} \|\mathbf{x}_n^*\|_* \leq c_2 < \infty.$$

If additionally \mathcal{X}^* has the property that $\mathbf{x}_n^*(f) = 0$ for all $n \in \mathbb{N}$ entails that $f = 0$, we say that \mathcal{X}^* is *total* or, equivalently, that \mathcal{X} is a *Markushevich basis*. In the sequel, unless specified we will not assume that \mathcal{X} is a Markushevich basis - though our results will imply this property in all cases (see [3, Corollary 4.5]).

Given a basis \mathcal{X} and $f \in \mathbb{X}$, we define a *greedy ordering* as a map $\pi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{supp}(f) \subset \pi(\mathbb{N})$ (where the support of f is the collection of $n \in \mathbb{N}$ where $\mathbf{x}_n^*(f) \neq 0$) and such that if $j < k$, then either $|\mathbf{x}_{\pi(j)}^*(f)| > |\mathbf{x}_{\pi(k)}^*(f)|$ or $|\mathbf{x}_{\pi(j)}^*(f)| = |\mathbf{x}_{\pi(k)}^*(f)|$ and $\pi(j) < \pi(k)$. Using this map and $m \in \mathbb{N}$, we define a *greedy sum of f of order m* like the sum

$$G_m[\mathcal{X}, \mathbb{X}](f) = G_m(f) := \sum_{i=1}^m \mathbf{x}_{\pi(i)}^*(f) \mathbf{x}_{\pi(i)}.$$

The Thresholding Greedy Algorithm (TGA) is the collection $(G_m)_{m \in \mathbb{N}}$.

Recall that the projection operator is defined as follows: given a finite set $A \subset \mathbb{N}$ and $f \in \mathbb{X}$,

$$P_A[\mathcal{X}, \mathbb{X}](f) = P_A(f) := \sum_{n \in A} \mathbf{x}_n^*(f) \mathbf{x}_n.$$

Then, a greedy sum is a projection over a *greedy set* $A_m(f)$, where $A_m(f)$ is a set verifying the following two conditions: the cardinality is exactly m and

$$\min_{n \in A_m(f)} |\mathbf{x}_n^*(f)| \geq \max_{n \notin A_m(f)} |\mathbf{x}_n^*(f)|.$$

Of course, unconditionality guarantees the convergence of the TGA, where we say that a basis is **K-unconditional** with $\mathbf{K} > 0$ if

$$\|P_A(f)\| \leq \mathbf{K} \|f\|, \quad \forall |A| < \infty, \forall f \in \mathbb{X}.$$

However, in the literature, there are examples of conditional bases where the TGA converges. For that reason, we have the following definition to study the convergence of the algorithm.

Definition 2.1. *We say that a basis \mathcal{X} in a quasi-Banach space \mathbb{X} is quasi-greedy if there is $\mathbf{C} > 0$ such that for all $f \in \mathbb{X}$,*

$$(2.5) \quad \|f - P_A(f)\| \leq \mathbf{C} \|f\|,$$

whenever A is a finite greedy set of f . The smallest constant verifying (2.5) is denoted by \mathbf{C}_q and we say that \mathcal{X} is \mathbf{C}_q -quasi-greedy.

P. Wojtaszczyk proved in [16] (also the proof could be found in [3]) that a basis is quasi-greedy if and only if the TGA converges, that is,

$$\lim_{m \rightarrow +\infty} \left\| f - \sum_{n=1}^m \mathbf{x}_{\pi(n)}^*(f) \mathbf{x}_{\pi(n)} \right\| = 0.$$

An stronger notion than quasi-greediness is greediness, that is, the case when the TGA produces the best possible approximation up to a constant. To define it precisely, we need the *best m th approximation error of $f \in \mathbb{X}$* :

$$\sigma_m[\mathcal{X}, \mathbb{X}](f) = \sigma_m(f) := \inf \left\{ \left\| f - \sum_{j \in A} a_j \mathbf{x}_j \right\| : a_j \in \mathbb{F} \forall j \in A, |A| \leq m \right\}.$$

Definition 2.2. We say that a basis \mathcal{X} in a quasi-Banach space \mathbb{X} is greedy if there is $\mathbf{C} > 0$ such that for all $f \in \mathbb{X}$ and $m \in \mathbb{N}$,

$$(2.6) \quad \|f - P_A(f)\| \leq \mathbf{C}\sigma_m(f),$$

whenever A is a finite greedy set of f of cardinality m . The smallest constant verifying (2.6) is denoted by \mathbf{C}_g and we say that \mathcal{X} is \mathbf{C}_g -greedy.

In [13], the authors characterized greediness in terms of unconditionality and democracy. To formalize the last notion, we need the following notation:

$$\mathbb{1}_{\varepsilon, A}[\mathcal{X}, \mathbb{X}] = \mathbb{1}_{\varepsilon, A} := \sum_{j \in A} \varepsilon_j \mathbf{x}_j; \quad \varepsilon = (\varepsilon_j)_{j \in A} \in \mathbb{E}^A; \quad \mathbb{E} := \{\lambda \in \mathbb{F} : |\lambda| = 1\}.$$

In general, $\varepsilon \in \mathbb{E}^A$ is called a sign and, for $f \in \mathbb{X}$,

$$\varepsilon(f) \equiv \{\text{sign}(\mathbf{x}_n^*(f))\}, n \in \mathbb{N}.$$

Thus, a basis \mathcal{X} is \mathbf{D} -democratic with $\mathbf{D} > 0$ if

$$\|\mathbb{1}_A\| \leq \mathbf{D} \|\mathbb{1}_B\|,$$

for any pair of finite sets with $|A| \leq |B|$, where $\mathbb{1}_A = \mathbb{1}_{\varepsilon, A}$ with $\varepsilon \equiv 1$. A similar characterization could be obtained by substituting superdemocracy for democracy, where \mathcal{X} is \mathbf{D}_s -superdemocratic with $\mathbf{D}_s > 0$ if

$$\|\mathbb{1}_{\varepsilon, A}\| \leq \mathbf{D}_s \|\mathbb{1}_{\eta, B}\|,$$

for any pair of finite sets A and B with $|A| \leq |B|$, $\varepsilon \in \mathbb{E}^A$ and $\eta \in \mathbb{E}^B$.

Theorem 2.1 ([3, 13]). A basis \mathcal{X} in a quasi-Banach space \mathbb{X} is greedy if and only if \mathcal{X} is unconditional and democratic or superdemocratic.

In [7], the authors introduced a new way to study greedy bases using polynomials of constant coefficients, where a polynomial of this type is basically an element of \mathbb{X} of the form $\alpha \mathbb{1}_{\varepsilon, A} = \sum_{j \in A} \alpha \varepsilon_j \mathbf{x}_j$ where $\alpha \in \mathbb{F}$ and $\varepsilon \in \mathbb{E}^A$.

Theorem 2.2 ([7]). Let \mathcal{X} be a basis in a Banach space \mathbb{X} . The basis is greedy if and only if there is $\mathbf{C} > 0$ such that for all $f \in \mathbb{X}$,

$$(2.7) \quad \|f - P_A(f)\| \leq \mathbf{C} \inf\{\|f - \alpha \mathbb{1}_{\varepsilon, B}\| : \alpha \in \mathbb{F}, |B| = |A|, \varepsilon \in \mathbb{E}^B\},$$

whenever A is a finite greedy set of f .

The purpose of this paper is to relax the optimization problem given in (2.7) trying to fix the coefficient α . That is, in the approximation error

$$\inf\{\|f - \alpha \mathbb{1}_{\varepsilon, B}\| : \alpha \in \mathbb{F}, |B| = |A|, \varepsilon \in \mathbb{E}^B\},$$

we are taking the infimum over the sets, over the signs ε and over α . Here, we show that we can fix the value of α and take only the infimum over the sets since we can also fix the sign. For that, we introduce the following two new approximation errors:

- For each $m \in \mathbb{N}$ and each $f \in \mathbb{X}$, let

$$\rho_m[\mathcal{X}, \mathbb{X}] = \rho_m(f) := \inf \left\{ \|f - \alpha \mathbb{1}_{\varepsilon, A}\| : \alpha = \min_{n \in A_m(f)} |\mathbf{x}_n^*(f)|, \quad |A| = m, \varepsilon \in \mathbb{E}^A \right\}.$$

- For each $m \in \mathbb{N}$ and each $f \in \mathbb{X}$, let

$$\varrho_m[\mathcal{X}, \mathbb{X}] = \varrho_m(f) := \inf \left\{ \|f - \alpha \mathbb{1}_A\| : \alpha = \min_{n \in A_m(f)} |\mathbf{x}_n^*(f)|, \quad |A| = m \right\}.$$

Definition 2.3. We say that a basis \mathcal{X} is *relaxed greedy* for polynomials with constant coefficients (RGPC for short) if there is $\mathbf{C} > 0$ such that for all $m \in \mathbb{N}$ and $f \in \mathbb{X}$,

$$(2.8) \quad \|f - P_A(f)\| \leq \mathbf{C}\rho_m(f), \quad \forall A \text{ greedy set of cardinality } m.$$

The smallest constant verifying (2.8) is denoted by \mathbf{C}_{pg} and we say that \mathcal{X} is \mathbf{C}_{pg} -RGPC.

Definition 2.4. We say that a basis \mathcal{X} is *unsigned-relaxed greedy* for polynomials with constant coefficients (URGPC for short) if there is $\mathbf{C} > 0$ such that for all $m \in \mathbb{N}$ and $f \in \mathbb{X}$,

$$(2.9) \quad \|f - P_A(f)\| \leq \mathbf{C}\varrho_m(f), \quad \forall A \text{ greedy set of cardinality } m.$$

The smallest constant verifying (2.9) is denoted by \mathbf{C}_{pgu} and we say that \mathcal{X} is \mathbf{C}_{pgu} -URGPC.

Here we will show the following result:

Theorem 2.3. Let \mathcal{X} be a basis in a quasi-Banach space \mathbb{X} . The following are equivalent:

- i) \mathcal{X} is greedy.
- ii) \mathcal{X} is RGPC.
- iii) \mathcal{X} is URGPC.

Quantitatively, if \mathbb{X} is a p -Banach space with $0 < p \leq 1$, then

$$\mathbf{C}_{pgu} \leq \mathbf{C}_{pg} \leq \mathbf{C}_g.$$

Moreover, if $\mathbb{F} = \mathbb{R}$, then

$$(2.10) \quad \mathbf{C}_g \leq \min\{A_p^2 \mathbf{C}_{pg}^2, \mathbf{C}_{pgu}^p (1 + A_p^p \mathbf{C}_{pgu}^{2p} \min\{B_p^p, A_p^p \mathbf{C}_{pgu}^{2p}\})^{1/p}\}.$$

If $\mathbb{F} = \mathbb{C}$, there is a constant $\mathbf{L}_0 = \mathbf{L}_0(\mathbf{C}_{pgu}, p)$ such that

$$(2.11) \quad \mathbf{C}_g \leq \min\{A_p^2 \mathbf{C}_{pg}^2, \mathbf{L}_0\}.$$

Additionally, we prove similar results for almost-greedy bases.

Definition 2.5. We say that a basis \mathcal{X} in a quasi-Banach space \mathbb{X} is *almost-greedy* if there is $\mathbf{C} > 0$ such that for all $f \in \mathbb{X}$,

$$(2.12) \quad \|f - P_A(f)\| \leq \mathbf{C} \inf\{\|f - P_B(f)\| : |B| \leq |A|\},$$

whenever A is a finite greedy set of f . The least constant verifying (2.12) is denoted by \mathbf{C}_{ag} and we say that \mathcal{X} is \mathbf{C}_{ag} -almost-greedy.

Related to these bases and following the idea about polynomials of constant coefficients, S. J. Dilworth and D. Khurana proved in [9] that a basis in a Banach space is almost-greedy if and only if there is $\mathbf{C} > 0$ such that

$$(2.13) \quad \|f - P_A(f)\| \leq \mathbf{C} \|f - a \mathbb{1}_B\|$$

for all $f \in \mathbb{X}$, A a greedy set of f , $B \subset \mathbb{N}$, $|B| \leq |A|$, all $a \in \mathbb{F}$ and $A < B$ or $B < A$, where $A < B$ means that $\max_{j \in A} j < \min_{i \in B} i$.

Our next theorem improves the above result.

Theorem 2.4. Let \mathcal{X} be a basis for a p -Banach space \mathbb{X} . The following are equivalent:

- i) \mathcal{X} is almost greedy.
- ii) \mathcal{X} is quasi-greedy and superdemocratic.
- iii) There is $\mathbf{C} > 0$ such that

$$\|f - P_A(f)\| \leq \mathbf{C} \|f - a \mathbb{1}_{\varepsilon, B}\|$$

for all $f \in \mathbb{X}$, A a greedy set of f , $B \subset \mathbb{N}$, $|B| \leq |A|$ and $A \cap B = \emptyset$, $\varepsilon \in \mathbb{F}^B$, and all $a \in \mathbb{F}$.

iv) There is $\mathbf{C} > 0$ such that

$$\|f - P_A(f)\| \leq \mathbf{C} \|f - \min_{n \in A} |\mathbf{x}_n^*(f)| \mathbb{1}_{\varepsilon, B}\|$$

for all $f \in \mathbb{X}$, A a greedy set of f , $B \subset \mathbb{N}$, $|B| \leq |A|$ and $A < B$ or $B < A$ and all $\varepsilon \in \mathbb{E}^B$.

v) There is $\mathbf{C} > 0$ such that

$$\|f - P_A(f)\| \leq \mathbf{C} \|f - \min_{n \in A} |\mathbf{x}_n^*(f)| \mathbb{1}_B\|$$

for all $f \in \mathbb{X}$, A a greedy set of f , $B \subset \mathbb{N}$, $|B| = |A|$ and $A < B$ or $B < A$.

In Theorem 2.4, the equivalence between i and ii is well known (see [11, Theorem 3.3], [3, Theorem 6.3]), our contribution here are the other points, in particular iii and v.

3. TECHNICAL RESULTS

In this section, we focus our attention on some characterizations and relations between unconditionality and symmetry-like properties.

As we have commented, in Theorem 2.1 it was proven that a basis in a Banach space is greedy if and only if it is democratic and unconditional and the behaviour of the constants is as follows:

$$\max\{\mathbf{K}, \mathbf{D}\} \leq \mathbf{C}_g \leq \mathbf{K}(1 + \mathbf{D}).$$

In the last decade, researchers have worked on improving this estimate since there are examples of 1-democratic and 1-unconditional bases that are not 1-greedy. In 2006, F. Albiac and P. Wojtaszczyk found necessary and sufficient conditions under which $\mathbf{C}_g = 1$. Concretely, they introduced the notion of Property (A), extended and renamed in [12] by the so called symmetry for largest coefficients.

Definition 3.6. We say that a basis \mathcal{X} in a quasi-Banach space is symmetric for largest coefficients (SLC for short) if there is $\mathbf{C} > 0$ such that

$$(3.14) \quad \|f + \mathbb{1}_{\varepsilon, A}\| \leq \mathbf{C} \|f + \mathbb{1}_{\eta, B}\|,$$

for all $f \in \mathbb{X}$ such that $\text{supp}(f) \cap (A \cup B) = \emptyset$, for all finite sets A and B such that $|A| \leq |B|$ and $A \cap B = \emptyset$ and all choices of signs $\varepsilon \in \mathbb{E}^A$ and $\eta \in \mathbb{E}^B$. The least constant verifying (3.14) is denoted by Δ and we say that \mathcal{X} is Δ -SLC.

Using this concept, in [12] we can find the following.

Theorem 3.5. A basis in a Banach space is greedy if and only if the basis is symmetric for largest coefficients and unconditional. Quantitatively,

$$\max\{\Delta, \mathbf{K}\} \leq \mathbf{C}_g \leq \Delta \mathbf{K}.$$

Remark 3.1. In the case of p -Banach spaces, in [3] we find the same result with the following behaviour:

$$\max\{\Delta, \mathbf{K}\} \leq \mathbf{C}_g \leq A_p^2 \Delta \mathbf{K},$$

where A_p is defined as in (2.3). Also, changing the symmetry for largest coefficients by democracy for general p -Banach spaces, we have the bounds (see [3])

$$\mathbf{C}_g \leq \mathbf{K}(1 + A_p^p \mathbf{D}^p \min\{B_p^p, A_p^p \mathbf{K}^p\})^{1/p},$$

where B_p is defined as in (2.4).

In this line, we modify the proof of [1, Proposition 2.11] to show that in order to prove that a basis is unconditional, it is enough consider approximations by vectors with non-negative coefficients. This result will be used in the proof of Theorem 2.3.

We use the following notation: for a given basis \mathcal{X} ,

$$\begin{aligned}\mathbb{X}_{\mathbb{R}}(\mathcal{X}) &:= \{f \in \mathbb{X}: \mathbf{x}_n^*(f) \in \mathbb{R} \quad \forall n \in \mathbb{N}\}; \\ \mathbb{X}_+(\mathcal{X}) &:= \{f \in \mathbb{X}: \mathbf{x}_n^*(f) \in \mathbb{R}_{\geq 0} \quad \forall n \in \mathbb{N}\}.\end{aligned}$$

Lemma 3.2. *Let \mathcal{X} be a basis for a p -Banach space \mathbb{X} over \mathbb{F} and $\mathbf{C} > 0$. Suppose that for every $f \in \text{span}\{\mathcal{X}\}$, $g \in \text{span}\{\mathcal{X}\} \cap \mathbb{X}_+(\mathcal{X})$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$,*

$$(3.15) \quad \|f\| \leq \mathbf{C}\|f + g\|.$$

Then $\|f\| \leq \mathbf{K}\|f + g\|$ for every $f, g \in \mathbb{X}$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, with $\mathbf{K} = \mathbf{C}^2$ if $\mathbb{F} = \mathbb{R}$ and \mathbf{K} depending only on \mathbf{C} and p if $\mathbb{F} = \mathbb{C}$. Thus, \mathcal{X} is \mathbf{K} -unconditional.

Proof. By a standard density argument (or use Lemma 3.4 below), it is enough to prove that (3.15) holds with \mathbf{K} instead of \mathbf{C} for $f \in \text{span}\{\mathcal{X}\}$, $g \in \text{span}\{\mathcal{X}\}$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$. If $\mathbb{F} = \mathbb{R}$, choose disjointly supported $f, g \in \text{span}\{\mathcal{X}\}$, and write $g = g_1 - g_2$ with $g_1, g_2 \in \mathbb{X}_+(\mathcal{X}) \cap \text{span}\{\mathcal{X}\}$, and f, g_1, g_2 pairwise disjointly supported. We have

$$\|f\| \leq \mathbf{C}\|f + g_1\| = \mathbf{C}\| -f - g_1\| \leq \mathbf{C}^2\| -f - g_1 + g_2\| = \mathbf{C}^2\|f + g\|,$$

and the proof is complete.

Now suppose $\mathbb{F} = \mathbb{C}$, and let $\mathbf{K}_1 := (1 + \mathbf{C}^p)^{\frac{1}{p}}$. By p -convexity and hypothesis

$$(3.16) \quad \begin{aligned}\|g\| &\leq (\|g + f\|^p + \|f\|^p)^{\frac{1}{p}} \leq \mathbf{K}_1\|g + f\| \\ \forall g \in \text{span}\{\mathcal{X}\} \cap \mathbb{X}_+(\mathcal{X}), f \in \text{span}\{\mathcal{X}\}, \text{supp}(f) \cap \text{supp}(g) &= \emptyset.\end{aligned}$$

We claim that there is \mathbf{K}_2 depending only on p and \mathbf{C} such that, for every $A \subset \mathbb{N}$ with $0 < |A| < \infty$, $f \in \text{span}\{\mathcal{X}\}$ with $\text{supp}(f) \cap A = \emptyset$, and $(\varepsilon_n)_{n \in A} \in \mathbb{E}^A$,

$$(3.17) \quad \|\mathbb{1}_{\varepsilon, A}\| \leq \mathbf{K}_2\|\mathbb{1}_{\varepsilon, A} + f\|.$$

To prove our claim, first choose a finite nonempty set $A \subset \mathbb{N}$, $f \in \text{span}\{\mathcal{X}\}$ with $\text{supp}(f) \cap A = \emptyset$, and $(\varepsilon_n)_{n \in A} \in \mathbb{E}^A$ so that

$$|\varepsilon_n - 1| \leq \left(2^{\frac{1}{p}} \mathbf{K}_1 B_p\right)^{-1} \quad \forall n \in A.$$

Pick $B \subset A$ so that $\|\mathbb{1}_D\| \leq \|\mathbb{1}_B\|$ for all $D \subset A$. By (3.16) and Lemma 2.1,

$$\begin{aligned}\|\mathbb{1}_B\|^p &\leq \mathbf{K}_1^p \|\mathbb{1}_B + \mathbb{1}_{\varepsilon, A \setminus B} + f\|^p \\ &\leq \mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, B} + \mathbb{1}_{\varepsilon, A \setminus B} + f\|^p + \mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, B} - \mathbb{1}_B\|^p \\ &\leq \mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, A} + f\|^p + \mathbf{K}_1^p B_p^p \left(2^{\frac{1}{p}} \mathbf{K}_1 B_p\right)^{-p} \|\mathbb{1}_B\|^p \\ &\leq \mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, A} + f\|^p + \frac{1}{2} \|\mathbb{1}_B\|^p,\end{aligned}$$

so

$$\|\mathbb{1}_B\|^p \leq 2\mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, A} + f\|^p.$$

Another application of Lemma 2.1 gives

$$\|\mathbb{1}_{\varepsilon, A}\|^p \leq \|\mathbb{1}_A\|^p + \|\mathbb{1}_{\varepsilon, A} - \mathbb{1}_A\|^p \leq \|\mathbb{1}_A\|^p + B_p^p \left(2^{\frac{1}{p}} \mathbf{K}_1 B_p\right)^{-p} \|\mathbb{1}_B\|^p \leq \frac{3}{2} \|\mathbb{1}_B\|^p.$$

Hence,

$$(3.18) \quad \|\mathbb{1}_{\varepsilon, A}\|^p \leq 3\mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, A} + f\|^p.$$

Now let $\{\nu_1, \dots, \nu_{j_1}\} \subset \mathbb{E}$ be a set of minimum cardinality with the property that, for every $\varepsilon \in \mathbb{E}$ there is $1 \leq j \leq j_1$ such that $|\varepsilon - \nu_j| \leq \left(2^{\frac{1}{p}} \mathbf{K}_1 B_p\right)^{-1}$, and fix $A \subset \mathbb{N}$ a finite nonempty set, $f \in \text{span}\{\mathcal{X}\}$ with $\text{supp}(f) \cap A = \emptyset$, and $(\varepsilon_n)_{n \in A} \in \mathbb{E}^A$. Choose $I \subset \{1, \dots, j_1\}$ and $(A_j)_{j \in I}$ a partition of A such that $|\varepsilon_n - \nu_j| \leq \left(2^{\frac{1}{p}} \mathbf{K}_1 B_p\right)^{-1}$ for each $n \in A_j$ and each $j \in I$. By (3.18),

$$\begin{aligned} \|\mathbb{1}_{\varepsilon, A}\|^p &\leq \sum_{j \in I} \|\nu_j^{-1} \mathbb{1}_{\varepsilon, A_j}\|^p \leq \sum_{j \in I} 3\mathbf{K}_1^p \|\nu_j^{-1} \mathbb{1}_{\varepsilon, A_j} + \nu_j^{-1} \mathbb{1}_{\varepsilon, A \setminus A_j} + \nu_j^{-1} f\|^p \\ &= 3|I| \mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, A} + f\|^p \leq 3j_1 \mathbf{K}_1^p \|\mathbb{1}_{\varepsilon, A} + f\|^p. \end{aligned}$$

Note that \mathbf{K}_1 only depends on \mathbf{C} and p , whereas j_1 only depends on \mathbf{K}_1 and p . Thus, we have proven (3.17) with $\mathbf{K}_2 := (3j_1)^{\frac{1}{p}} \mathbf{K}_1$. To complete the proof of the lemma, fix $f, g \in \text{span}\{\mathcal{X}\} \setminus \{0\}$ with disjoint support. Let $A := \text{supp}(g)$, and define $\mathcal{Y} = (\mathbf{y}_n)_{n \in \mathbb{N}}$ by

$$\mathbf{y}_n = \begin{cases} |\mathbf{x}_n^*(g)| \mathbf{x}_n & \text{if } n \in A; \\ \mathbf{x}_n & \text{if } n \notin A. \end{cases}$$

Then \mathcal{Y} is a basis for \mathbb{X} and, since each \mathbf{y}_n is the product of \mathbf{x}_n by a positive scalar, (3.15) also holds substituting \mathcal{Y} and $\mathbb{X}_+(\mathcal{Y})$ for \mathcal{X} and $\mathbb{X}_+(\mathcal{X})$ respectively. Since \mathbf{K}_2 depends only on \mathbf{C} and p , the same argument given above shows that (3.17) holds for \mathcal{Y} as well. Therefore,

$$\|g\| = \|\mathbb{1}_{\varepsilon(g), A}[\mathcal{Y}, \mathbb{X}]\| \leq \mathbf{K}_2 \|\mathbb{1}_{\varepsilon(g), A}[\mathcal{Y}, \mathbb{X}] + f\| = \mathbf{K}_2 \|g + f\|.$$

This completes the proof of the lemma, with $\mathbf{K} = \mathbf{K}_2$. \square

Remark 3.2. While we have defined our basis to be bounded with bounded dual basis, neither condition is required in the proof of Lemma 3.2.

With respect to the symmetry for largest coefficients, in [8], the authors proved that in the case of Schauder bases in Banach spaces, in the definition of the SLC property it is enough to consider elements $f \in \mathbb{X}$ with finite support (their proofs hold for general Markushevich bases).

The same result holds in the context of general bases of p -Banach spaces. First of all, we consider the extension for Schauder bases, where the proof is immediate.

Lemma 3.3. *Let \mathcal{X} be a Schauder basis of a p -Banach space \mathbb{X} . Suppose D is a finite subset of \mathbb{N} , and $f \in \mathbb{X}$ satisfies $\text{supp}(f) \cap D = \emptyset$. Then, for any $\varepsilon > 0$ there exists a finitely supported $y \in \mathbb{X}$ such that $\|f - y\| < \varepsilon$, $\text{supp}(y) \cap D = \emptyset$, and $\max_j |\mathbf{x}_j^*(f)| = \max_j |\mathbf{x}_j^*(y)|$.*

Proof. Choose $n_0 \in \mathbb{N}$ so that $|\mathbf{x}_{n_0}^*(f)| = \|f\|_{\ell_\infty}$, and $n_1 \geq n_0$ so that $\|f - S_{n_1}(f)\| < \varepsilon$, and let $y := S_{n_1}(f)$. \square

For general bases, the result can be obtained by adapting the proofs of [6, Lemma 7.1] or [14, Lemma 2.2]. Nevertheless, we give a proof for the sake of completion.

Lemma 3.4. *Let \mathcal{X} be a basis of a p -Banach space \mathbb{X} and let $f \in \mathbb{X}$. The following hold:*

- i) *For every $D \subset \mathbb{N}$ finite such that $\text{supp}(f - P_D(f)) \neq \emptyset$ and every $\varepsilon > 0$ there exists $g \in \text{span}\{\mathcal{X}\}$ such that*

$$(3.19) \quad \|f - g\| < \varepsilon, \quad P_D(g) = P_D(f), \quad \max_{j \in D^c} |\mathbf{x}_j^*(f)| = \max_{j \in D^c} |\mathbf{x}_j^*(g)|.$$

In particular, if $D \cap \text{supp}(f) = \emptyset$, then $D \cap \text{supp}(g) = \emptyset$ and $\max_j |\mathbf{x}_j^(g)| = \max_j |\mathbf{x}_j^*(f)|$.*

- ii) If $\text{supp}(f) = \emptyset$, for every $D \subset \mathbb{N}$ finite and every $\varepsilon > 0$ there exists $g \in \text{span}\{\mathcal{X}\}$ such that $\|f - g\| < \varepsilon$, $\max_j |\mathbf{x}_j^*(g)| < \varepsilon$, and $P_D(g) = P_D(f) = 0$.

Proof. Set

$$c := 1 + \sup_{n \in \mathbb{N}} \{\|\mathbf{x}_n^*\| + \|\mathbf{x}_n\|\}.$$

To prove (3.19), first we consider the case $D \cap \text{supp}(f) = \emptyset$. We may assume that $\max_j |\mathbf{x}_j^*(f)| = 1$ (then $\|f\| \geq 1/c$ and $\varepsilon < 1/(2c)$). Set $\delta > 0$ so that

$$0 < \delta \leq \frac{\varepsilon}{3c^2\|f\|}; \quad \left(\delta^p + \left(\frac{c\delta}{1-c\delta} \right)^p (\|f\|^p + \delta^p) \right)^{\frac{1}{p}} < \varepsilon.$$

As $\text{span}\{\mathcal{X}\}$ is dense in \mathbb{X} , there exists $h \in \text{span}\{\mathcal{X}\}$ such that $\|f - h\| < \delta/\|P_D^c\|$. Let $u = P_D^c(h)$, then $\|f - u\| = \|P_D^c(f - h)\| < \delta$. For every j , $|\mathbf{x}_j^*(f - u)| < c\delta$, hence $C = \max_j |\mathbf{x}_j^*(u)| \in (1 - c\delta, 1 + c\delta)$. Now let $g = u/C$. Then $\max_j |\mathbf{x}_j^*(g)| = 1$, and

$$\|f - g\| \leq (\|f - u\|^p + |1 - C^{-1}|^p \|u\|^p)^{\frac{1}{p}} < \left(\delta^p + \left(\frac{c\delta}{1-c\delta} \right)^p (\|f\|^p + \delta^p) \right)^{\frac{1}{p}} < \varepsilon.$$

This completes the proof of (3.19) when $P_D(f) = 0$.

Now suppose $P_D(f) \neq 0$, and let $f_1 := f - P_D(f)$. By the previous case, there is $g_1 \in \text{span}\{\mathcal{X}\}$ so that (3.19) holds for f_1 , g_1 , D and ε . Now let $g := g_1 + P_D(f)$; it is clear that (3.19) holds.

Let us now prove ii: If \mathcal{B} is a Markushevich basis, then $f = 0$ so we can take $g = 0$. Otherwise, pick $n_0 \notin D$, choose $\delta > 0$ so that

$$\delta(1 + c^p)^{\frac{1}{p}} < \varepsilon,$$

and let $f_0 := f + \delta \mathbf{x}_{n_0}$. By i, there is $g \in \text{span}\{\mathcal{X}\}$ such that $\|g - f_0\| \leq \delta$, $\max_j |\mathbf{x}_j^*(g)| = \delta$ and $P_D(g) = P_D(f) = 0$. We have

$$\|f - g\| \leq (\|g - f_0\|^p + \|f_0 - f\|^p)^{\frac{1}{p}} \leq (\delta^p + c^p \delta^p)^{\frac{1}{p}} = \delta(1 + c^p)^{\frac{1}{p}} < \varepsilon.$$

□

4. PROOF OF THEOREM 2.3

Lemma 4.5. *Let \mathcal{X} be a basis in a p -Banach space \mathbb{X} . Suppose that \mathcal{X} is \mathbf{C}_{pg} -RGPCC. Then \mathcal{X} is \mathbf{K} -unconditional and Δ -symmetric for largest coefficients with*

$$\max\{\mathbf{K}, \Delta\} \leq \mathbf{C}_{pg}.$$

Proof. Fix $f, g \in \text{span}\{\mathcal{X}\}$ such that $\text{supp}(f) \cap \text{supp}(g) = \emptyset$. Let $A := \text{supp}(g)$, $B := \text{supp}(f)$ and $t_0 := 1 + \max_{n \in \mathbb{N}} (|\mathbf{x}_n^*(g)| + |\mathbf{x}_n^*(f)|)$. Pick $n_0 \in \mathbb{N} \setminus (A \cup B)$, and let $\varepsilon_n = \text{sign}(\mathbf{x}_n^*(g))$ for all $n \in A$ and $\varepsilon_{n_0} = 1$. Note that $A_0 = A \cup \{n_0\}$ is the only greedy set of $h := f + g + t_0 \mathbb{1}_{\varepsilon, A_0}$ of cardinality $|A_0| = |A| + 1$, and that $\min_{n \in A_0} |\mathbf{x}_n^*(h)| = t_0$. Thus,

$$\|f\| = \|h - P_{A_0}(h)\| \leq \mathbf{C}_{pg} \rho_{|A|+1}(h) \leq \mathbf{C}_{pg} \|h - t_0 \mathbb{1}_{\varepsilon, A_0}\| = \mathbf{C}_{pg} \|f + g\|,$$

which proves that \mathcal{X} is \mathbf{K} -unconditional with $\mathbf{K} \leq \mathbf{C}_{pg}$.

Now fix $f \in \mathbb{X}$ with finite support and $|\mathbf{x}_n^*(f)| \leq 1$ for all $n \in \mathbb{N}$ and pick $A, B \subset \mathbb{N}$ nonempty, finite, disjoint and disjoint from $\text{supp}(f)$ with $|A| = |B|$, $\varepsilon \in \mathbb{E}^A$ and $\eta \in \mathbb{E}^B$.

Given $0 < \epsilon < 1$, we have

$$\begin{aligned} \|f + \mathbb{1}_{\epsilon, A}\|^p &= \|f + \mathbb{1}_{\epsilon, A} + (1 + \epsilon)\mathbb{1}_{\eta, B} - (1 + \epsilon)\mathbb{1}_{\eta, B}\|^p \\ &\leq \mathbf{C}_{pg}^p \rho_{|B|}^p (f + \mathbb{1}_{\epsilon, A} + (1 + \epsilon)\mathbb{1}_{\eta, B}) \\ &\leq \mathbf{C}_{pg}^p \|f + \mathbb{1}_{\epsilon, A} + (1 + \epsilon)\mathbb{1}_{\eta, B} - (1 + \epsilon)\mathbb{1}_{\epsilon, A}\|^p \\ &\leq \mathbf{C}_{pg}^p (\|f + \mathbb{1}_{\eta, B}\|^p + \epsilon^p \|\mathbb{1}_{\epsilon, A}\|^p + \epsilon^p \|\mathbb{1}_{\eta, B}\|^p). \end{aligned}$$

Letting ϵ tend to zero we obtain

$$\|f + \mathbb{1}_{\epsilon, A}\|^p \leq \mathbf{C}_{pg}^p \|f + \mathbb{1}_{\eta, B}\|^p.$$

Hence, by Lemma 3.4, \mathcal{X} is Δ -SLC with $\Delta \leq \mathbf{C}_{pg}$. \square

Our next task is obtain greediness under hypotheses that are formally weaker than those of Lemma 4.5, to complete our characterization of greediness.

Lemma 4.6. *Let \mathcal{X} be a basis for a p -Banach space \mathbb{X} . Suppose that \mathcal{X} is C_{pgu} -URGPCC. Then \mathcal{X} is unconditional and democratic. Quantitatively, if $\mathbb{F} = \mathbb{R}$, then*

$$\max\{\mathbf{K}, \mathbf{D}\} \leq \mathbf{C}_{pgu}^2.$$

If $\mathbb{F} = \mathbb{C}$, there is $\mathbf{L} = \mathbf{L}(C_{pgu}, p)$ such that

$$\mathbf{K} \leq \mathbf{L} \text{ and } \mathbf{D} \leq \mathbf{C}_{pgu}^2.$$

Proof. First, we prove that it that \mathcal{X} is unconditional. Fix $f \in \text{span}\{\mathcal{X}\}$, $g \in \text{span}\{\mathcal{X}\} \cap \mathbb{X}_+$ so that $\text{supp}(f) \cap \text{supp}(g) = \emptyset$. Set $t_0 := 1 + \max_{n \in \mathbb{N}} (x_n^*(g) + |x_n^*(f)|)$, and pick $n_0 > \text{supp}(f) \cup \text{supp}(g)$. Note that $A := \text{supp}(g) \cup \{n_0\}$ is the only greedy set of $h := f + g + t_0 \mathbb{1}_A$ of cardinality $|A|$, and that $\min_{n \in A} |x_n^*(h)| = t_0$. Thus,

$$(4.20) \quad \|f\| = \|h - P_A(h)\| \leq \mathbf{C}_{pgu} \varrho_{|A|}(h) \leq \mathbf{C}_{pgu} \|h - t_0 \mathbb{1}_A\| = \mathbf{C}_{pgu} \|f + g\|.$$

By Lemma 3.2, \mathcal{X} is \mathbf{K} -unconditional with $\mathbf{K} = \mathbf{C}_{pgu}^2$ if $\mathbb{F} = \mathbb{R}$ and $\mathbf{K} \leq \mathbf{L}$ if $\mathbb{F} = \mathbb{C}$. To prove that it is democratic, pick $A, B \subset \mathbb{N}$ with $0 < |A| \leq |B| < \infty$ and $A \neq B$, and choose $B_0 \subset B$ so that $|B_0| = |A|$. Given $0 < \epsilon < 1$, we have

$$\begin{aligned} \|\mathbb{1}_A\|^p &= \|\mathbb{1}_{A \cap B_0} + \mathbb{1}_{A \setminus B_0} + (1 + \epsilon)\mathbb{1}_{B_0 \setminus A} - (1 + \epsilon)\mathbb{1}_{B_0 \setminus A}\|^p \\ &\leq \mathbf{C}_{pgu}^p \varrho_{|B_0 \setminus A|}^p (\mathbb{1}_{A \cap B_0} + \mathbb{1}_{A \setminus B_0} + (1 + \epsilon)\mathbb{1}_{B_0 \setminus A}) \\ &\leq \mathbf{C}_{pgu}^p \|\mathbb{1}_{A \cap B_0} + \mathbb{1}_{A \setminus B_0} + (1 + \epsilon)\mathbb{1}_{B_0 \setminus A} - (1 + \epsilon)\mathbb{1}_{A \setminus B_0}\|^p \\ &\leq \mathbf{C}_{pgu}^p \|\mathbb{1}_{B_0}\|^p + \mathbf{C}_{pgu}^p \epsilon^p \|\mathbb{1}_{B_0 \setminus A}\|^p + \mathbf{C}_{pgu}^p \epsilon^p \|\mathbb{1}_{A \setminus B_0}\|^p. \end{aligned}$$

Letting ϵ tend to zero we obtain $\|\mathbb{1}_A\| \leq \mathbf{C}_{pgu} \|\mathbb{1}_{B_0}\|$. If $B_0 = B$, there is nothing else to prove. Otherwise, an application of (4.20) with $f = \mathbb{1}_{B_0}$ and $g = \mathbb{1}_{B \setminus B_0}$ gives $\|\mathbb{1}_{B_0}\| \leq \mathbf{C}_{pgu} \|\mathbb{1}_B\|$. It follows that

$$\|\mathbb{1}_A\| \leq \mathbf{C}_{pgu}^2 \|\mathbb{1}_B\|,$$

so the basis is D -democratic with $\mathbf{D} \leq \mathbf{C}_{pgu}^2$. \square

Proof of Theorem 2.3. First, we show i) \Leftrightarrow ii). For that, of course, if \mathcal{X} is \mathbf{C}_g -greedy, then it is trivial that \mathcal{X} is \mathbf{C}_{pg} -RGPCC with $\mathbf{C}_{pg} \leq \mathbf{C}_g$. Now, if \mathcal{X} is \mathbf{C}_{pg} -RGPCC, then, using Lemma 4.5, the basis is \mathbf{K} -unconditional and Δ -SLC with $\max\{\mathbf{K}, \Delta\} \leq \mathbf{C}_{pg}$. Hence, using now Remark 3.1, the basis is \mathbf{C}_g -greedy with $\mathbf{C}_g \leq A_p^2 \mathbf{C}_{pg}^2$. Now, we show that i) \Leftrightarrow iii). Again, if \mathcal{X} is \mathbf{C}_g -greedy, then it is trivial that \mathcal{X} is \mathbf{C}_{pgu} -URGPCC with $\mathbf{C}_{pgu} \leq \mathbf{C}_g$. Now, using Lemma 4.6, the basis is \mathbf{D} -democratic and \mathbf{K} -unconditional with $\mathbf{D} \leq \mathbf{C}_{pgu}^2$ and \mathbf{K} depends on \mathbf{C}_{pgu} and p . Then, using now Remark 3.1, the basis is greedy and the constant is as it appears in (2.10) and (2.11). \square

5. PROOF OF THEOREM 2.4

In this section, we prove Theorem 2.4. In addition to the classical characterization of almost greedy bases as those that are quasi-greedy and democratic or superdemocratic ([11, Theorem 3.3], [3, Theorem 6.3]), we will use the following result from the literature (see [11, Lemma 2.2], [3, Theorem 4.3])

Theorem 5.6. [3, Theorem 4.13] *If \mathcal{X} is a C_q -quasi-greedy basis in a p -Banach space with $0 < p \leq 1$, then for all $f \in \mathbb{X}$ and A a greedy set of f ,*

$$(5.21) \quad \min_{n \in A} |\mathbf{x}_n^*(f)| \|\mathbb{1}_{\varepsilon(f), A}\| \leq \mathbf{C} \|f\|.$$

The minimum constant \mathbf{C} for which the above inequality holds is denoted Γ_t , and it's no greater than $\mathbf{C}_q^2 \eta_p(C_q)$, where

$$\eta_p(u) = \min_{0 < t < 1} (1 - t^p)^{-1/p} (1 - (1 + A_p^{-1} u^{-1} t)^{-p})^{-1/p} \quad \forall u > 0.$$

Bases for which (5.21) holds have been recently named 'truncation quasi-greedy' (see [2]), but have been known since the early days of the theory (see [11]) (it is known that there are bases satisfying (5.21) that are not quasi-greedy, see for example [2] or [11, Proposition 4.8]).

Proof of Theorem 2.4. i) \iff ii) follows from [11, Theorem 3.3], [3, Theorem 6.3], whereas iii) \implies iv) \implies v) is immediate.

ii) \implies iii) Let \mathbf{D}_s , \mathbf{C}_q and Γ_t be the superdemocracy, quasi-greedy and truncation quasi-greedy constants of \mathcal{X} respectively. Given f , A , B , and ε as in the statement, set $b := \min_{n \in A} |\mathbf{x}_n^*(f)|$. We may assume $b > 0$ and we consider two cases:

Case 1: If $|a| \leq 2b$, let $D \subset \mathbb{N}$ be a greedy set of $g := f - a \mathbb{1}_{\varepsilon, B}$ of minimum cardinality that contains A . Then

$$\min_{n \in D} |\mathbf{x}_n^*(g)| = b,$$

so

$$(5.22) \quad \begin{aligned} \|f - P_A(f)\|^p &\leq \mathbf{C}_q^p \|f\|^p \leq \mathbf{C}_q^p \|g\|^p + \mathbf{C}_q^p 2^p (2^{-1}|a| \|\mathbb{1}_{\varepsilon, B}\|)^p \\ &\leq \mathbf{C}_q^p \|g\|^p + \mathbf{C}_q^p \mathbf{D}_s^p 2^p (2^{-1}|a| \|\mathbb{1}_{\varepsilon(g), D}\|)^p \\ &\leq \mathbf{C}_q^p (1 + 2^p \mathbf{D}_s^p \Gamma_t^p) \|g\|^p. \end{aligned}$$

Case 2: If $|a| > 2b$, set g as before, and let D be a greedy set of g of minimum cardinality containing B . Then

$$d := \min_{n \in D} |\mathbf{x}_n^*(g)| = \min_{n \in B} |\mathbf{x}_n^*(g)| \geq \frac{|a|}{2}.$$

Thus,

$$\|a \mathbb{1}_{\varepsilon, B}\| \leq 2 \mathbf{D}_s d \|\mathbb{1}_{\varepsilon(g), D}\| \leq 2 \mathbf{D}_s \Gamma_t \|g\|.$$

Hence,

$$(5.23) \quad \begin{aligned} \|f - P_A(f)\|^p &\leq \mathbf{C}_q^p \|f\|^p \leq \mathbf{C}_q^p \|g\|^p + \mathbf{C}_q^p (\|a \mathbb{1}_{\varepsilon, B}\|)^p \\ &\leq \mathbf{C}_q^p (1 + 2^p \mathbf{D}_s^p \Gamma_t^p) \|g\|^p. \end{aligned}$$

Combining (5.22) and (5.23), we obtain iii) with

$$\mathbf{C} \leq \mathbf{C}_q^p (1 + 2^p \mathbf{D}_s^p \Gamma_t^p).$$

v) \implies ii)

Choose a finite set A , $\varepsilon \in \mathbb{E}^A$, and $f \in \mathbb{X}$ so that

$$(5.24) \quad |\{n \in \text{supp}(f) : |\mathbf{x}_n^*(f)| \geq 1\}| \geq |A|.$$

We claim that

$$(5.25) \quad \|\mathbb{1}_{\varepsilon, A}\| \leq B_p \mathbf{C}^2 2^{\frac{1}{p}-1} \|f\|.$$

To prove the claim, first assume that f has finite support and apply Lemma 2.1 to find $A_0 \subset A$ such that

$$\|\mathbb{1}_{\varepsilon, A}\| \leq B_p \|\mathbb{1}_{A_0}\|.$$

Pick B a greedy set of f with $|B| = |A_0|$. Note that $\min_{n \in B} |\mathbf{x}_n^*(f)| \geq 1$ since B is a greedy set with $|A| \geq |B|$ and (5.24) is satisfied. Let $f_1 := \frac{f}{\min_{n \in B} |\mathbf{x}_n^*(f)|}$, and $g := f_1 - P_B(f_1)$, and choose $D \supset \text{supp}(f) \cup A$ so that $|D| = |A_0|$. By hypothesis and p -convexity,

$$\|\mathbb{1}_{A_0}\|^p = \|\mathbb{1}_{A_0} + \mathbb{1}_D - \mathbb{1}_D\|^p \leq \mathbf{C}^p \|\mathbb{1}_D\|^p \leq 2^{-p} \mathbf{C}^p \|\mathbb{1}_D + g\|^p + 2^{-p} \mathbf{C}^p \|\mathbb{1}_D - g\|^p.$$

On the other hand, as B is a greedy set of both $\mathbb{1}_D + f_1$ and $\mathbb{1}_D - f_1$, we have

$$\begin{aligned} \|\mathbb{1}_D + g\| &= \|\mathbb{1}_D + f_1 - P_B(\mathbb{1}_D + f_1)\| \leq \mathbf{C} \|\mathbb{1}_D + f_1 - \mathbb{1}_D\| \leq \mathbf{C} \|f\|; \\ \|\mathbb{1}_D - g\| &= \|\mathbb{1}_D - f_1 - P_B(\mathbb{1}_D - f_1)\| \leq \mathbf{C} \|\mathbb{1}_D - f_1 - \mathbb{1}_D\| \leq \mathbf{C} \|f\|. \end{aligned}$$

Combining the above inequalities we obtain (5.25) for f with finite support, and the general case follows at once by Lemma 3.4. In particular, it follows that \mathcal{X} is superdemocratic. To prove that it is also quasi-greedy, fix $f \in \mathbb{X}$ and A a greedy set of f , and pick $B \subset \mathbb{N}$ with $B \supset A$, and $|A| = |B|$. By hypothesis and (5.25) we have

$$\begin{aligned} \|f - P_A(f)\|^p &\leq \mathbf{C}^p \|f - \min_{n \in A} |\mathbf{x}_n^*(f)| \mathbb{1}_B\|^p \leq \mathbf{C}^p \|f\|^p + \mathbf{C}^p \min_{n \in A} |\mathbf{x}_n^*(f)|^p \|\mathbb{1}_B\|^p \\ &\leq \mathbf{C}^p \|f\|^p + B_p^p \mathbf{C}^{3p} 2^{1-p} \|f\|^p, \end{aligned}$$

and the proof is complete. \square

6. CONCLUSIONS

In this article, we have focused on the characterization of two types of greedy-like bases: greedy bases and almost-greedy bases. On the one hand, for the case of Schauder bases in Banach spaces, in [7] the authors characterize greedy bases as those bases where there is $C > 0$ such that for every f in the space and $m \in \mathbb{N}$,

$$\|f - G_m(f)\| \leq C \inf_{\alpha, |A|=m} \|f - \alpha \mathbb{1}_A\|.$$

In our paper, we improve this result by proving that it is not necessary to take any element of the form $y = \alpha \mathbb{1}_A$. Concretely, for general semi-normalized Markushevich bases in quasi-Banach spaces, we prove that a basis is greedy if and only if

$$\|f - G_m(f)\| \leq C \inf_{|A|=m} \|f - \min_{n \in A_m(f)} |\mathbf{x}_n^*(f)| \mathbb{1}_A\|.$$

On the other hand, for the case of almost-greediness, in [9], the authors proved that a Schauder basis in a Banach space is almost-greedy if and only if there is $C > 0$ such that for every element f and $m \in \mathbb{N}$,

$$\|f - G_m(f)\| \leq C \inf \{ \|f - \alpha \mathbb{1}_A\|, \alpha \in \mathbb{F}, |A| = m, A < A_m(f) \text{ or } A_m(f) < A \}.$$

In our paper, as for greediness, we improve this result proving that for general semi-normalized Markushevich bases in quasi-Banach spaces, a basis is almost-greedy if and only if there is $C > 0$ such that for every element f and $m \in \mathbb{N}$,

$$\|f - G_m(f)\| \leq C \inf \left\{ \|f - \min_{n \in A_m(f)} |x_n^*(f)| \mathbb{1}_A\|, |A| = m, A < A_m(f) \text{ or } A_m(f) < A \right\}.$$

Furthermore, we show more equivalences substituting $\mathbb{1}_A$ for $\mathbb{1}_{\varepsilon, A}$ where ε is a collection of signs and for all the results we proved, we also used a new characterization of unconditionality and some extensions of density results about the symmetry for largest coefficients.

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