CARPATHIAN J. MATH.
Volume **41** (2025), No. 1,
Pages 45-55

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2025.01.04

Conjectures about wheels without one edge

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ABSTRACT. The main aim of the paper is to give the crossing number of the join product $G^* + D_n$ for the graph G^* isomorphic to 4-regular graph on six vertices except for two distinct edges with no common vertex such that two remaining vertices are still adjacent, and where D_n consists of n isolated vertices. The proofs are done with the help of well-known exact values for crossing numbers of join products of four subgraphs H_k of G^* with discrete graphs. Further, we give a conjecture concerning crossing numbers of the join products of D_n with $W_m \setminus e$ for both types edges e of wheels W_m of $m + 1$ vertices.

1. INTRODUCTION

The crossing number is an important parameter of a graph, as it provides information about the complexity of the graph and the difficulty of visualizing it. In addition, the crossing numbers are related to many other graph parameters and algorithms, such as graph coloring, graph embedding, and planarity testing. In general reducing the number of crossings on graph edges can be useful in various applications, including circuit design, network visualization, cartography or social choice theory. Simple graphs are widely used to represent complex networks such as social, communication, and transportation networks. Reducing the number of edge crossings in network visualizations helps understand the network's underlying structure and identify important nodes and connections. Note that examining number of crossings of simple graphs is an NP-complete problem by Garey and Johnson [6].

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane (for the definition of a *drawing* see Klešč [19]). It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good drawing*, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

Throughout this paper, some parts of proofs will be based on Kleitman's result [16] on crossing numbers for some complete bipartite graphs $K_{m,n}$ on $m + n$ vertices with

Received: 22.12.2023. In revised form: 15.07.2024. Accepted: 28.07.2024
2010 *Mathematics Subject Classification.* 05C10, 05C38.
Key words and phrases. *Graph, drawing, crossing number, join product, discrete graph, wheel, cycle, star.*
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a partition $V(K_{m,n}) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ containing an edge between every pair of vertices from V_1 and V_2 of sizes m and n, respectively. He showed that

$$\text{cr}(K_{m,n}) = \Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor \Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor, \quad \text{if} \quad \min\{m,n\} \leq 6. \tag{1.1}$$

For an overview of several exact values of crossing numbers for specific graphs or some families of graphs, see Clancy [4]. The main goal of this survey is to summarize all such published results for crossing numbers along with references also in an effort to give priority to the author who published the first result. Chapter 4 of such survey is devoted to the issue of crossing numbers of join product with all simple graphs of order at most six mainly due to unknown values of $\text{cr}(K_{m,n})$ for both m, n more than six in (1.1). The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let P_n and C_n be the *path* and the *cycle* on n vertices, respectively, and let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices. Besides, let W_m and S_m denote the *wheel* and the *star* of $m+1$ vertices, respectively. The exact values for crossing numbers of $G + D_n$ for all graphs G of order at most four are given by Klešč and Schrötter [25], and also for some connected graphs G of order five and six [1, 2, 3, 5, 9, 10, 11, 12, 13, 14, 17, 18, 19, 21, 22, 23, 24, 28, 29, 31, 34, 39, 40]. The aim of this paper is to extend known results concerning this topic to new connected graphs. Note also that $\text{cr}(G + D_n)$ are known only for some disconnected graphs G [26, 27, 33]. For this purpose, we present a new technique regarding the use of knowledge from the subgraphs whose values of crossing numbers are already known.

Section 2 is devoted to the graph G^* isomorphic to 4-regular graph on six vertices except for two distinct edges with no common vertex such that two remaining vertices are still adjacent. In the rest of the paper, we will use the following notation of the vertex set $V(G^*) = \{v_1, v_2, \dots, v_6\}$. Many possible drawings of G^* are partially solved using its clearly established cycle $C_5^\star$ as a subgraph whose edges do not cross each other in any optimal drawing of $G^* + D_n$. The main aim of the paper is to establish $\text{cr}(G^* + D_n)$ for all integers n. The crossing number of $G^* + D_n$ equal to $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ is determined in Theorem 2.6 with the proof that is strongly based on Lemma 2.2. This lemma includes well-known values of $\text{cr}(H_k + D_n)$ for four subgraphs H_k of G^* (with planar drawings shown in Fig. 2) presented in Theorems 2.1, 2.2, 2.3, and 2.4. The paper concludes by giving a new conjecture concerning crossing numbers of the join products of D_n with $W_m \setminus e$ obtained by removing one edge (of both possible types) from the wheel W_m of $m+1$ vertices. In the proofs of the paper, we will often use the term "region" also in nonplanar subdrawings. In this case, crossings are considered to be vertices of the "map".

2. The Crossing Number of $G^* + D_n$

The join product $G^* + D_n$ (sometimes the notation $G^* + nK_1$ used) consists of one copy of the graph G^* and n vertices $t_1, \dots, t_n$, and any vertex t_i is adjacent to every vertex of the graph G^*. We denote the subgraph induced by six edges incident with the fixed vertex t_i by T^i, which yields that

$$G^* + D_n = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.2}$$

We consider a good drawing D of $G^* + D_n$. By the *rotation* $\text{rot}_D(t_i)$ of a vertex t_i in D we understand the cyclic permutation that records the (cyclic) counterclockwise order in which edges leave t_i, as defined by Hernández-Vélez et al. [8] or Woodall [41]. We use the notation (123456) if the counter-clockwise order of edges incident with the fixed vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5 and t_iv_6. We recall that rotation is a cyclic permutation. In the given drawing D, it is highly desirable to separate n subgraphs T^i into three mutually disjoint subsets depending on how many times edges of G^* could be crossed by T^i in D. For $i = 1, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G^*, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G^*, T^i) = 1\}$. Edges of G^* are crossed by each remaining subgraph T^i at least twice in D. Note that if D is a good drawing of $G^* + D_n$ with the empty set $R_D \cup S_D$, then $\sum_{i=1}^{n} \mathrm{cr}_D(G^*, T^i) \geq 2n$ enforces at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ crossings in D provided by

$$\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D(K_{6,n}) + \mathrm{cr}_D(G^*, K_{6,n}) + \mathrm{cr}_D(G^*) \geq$$
$$\geq 6\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + 2n \geq 6\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + n + \Big\lfloor \frac{n}{2} \Big\rfloor.$$

According to the expected result of the main Theorem 2.6, this leads to a consideration of the nonempty set $R_D \cup S_D$ in all good drawings of $G^* + D_n$.

Let us discuss all possible drawings of G^* induced by D. In the rest of the paper, let v_6 be the vertex notation of one vertex of degree 4 in all considered good subdrawings $D(G^*)$. The graph G^* contains a cycle C_5 induced on the remaining five vertices of degrees 2, 3, 3, 3, and 3 as a subgraph (for brevity, we will write $C_5^\star$). As we can always redraw a crossing of two edges of $C_5^\star$ in an effort to get a new drawing of $C_5^\star$ (with vertices in a different order) with less number of edge crossings, the proof of Lemma 2.1 can be omitted.

Lemma 2.1. *For $n \geq 1$, the edges of $C_5^\star$ do not cross each other in any optimal drawing of the join product $G^* + D_n$.*

A similar idea has already been presented in the proof for $W_5 + D_n$ by Berežný and Staš [2]. Based on the arguments above, we will assume that edges of the cycle $C_5^\star$ do not cross each other in all considered subdrawings $D(G^*)$, and let v_1, v_2, v_3, v_4, and v_5 be their vertex notation in the appropriate order of $C_5^\star$. We only need to consider possibilities of crossings between subdrawings of $C_5^\star$ and four remaining edges incident with the vertex v_6. If we would like to obtain an optimal drawing D of $G^* + D_n$, then in addition the set $R_D \cup S_D$ must be nonempty. Thus, we will only consider subdrawings of the graph G^* induced by D for which there is a possibility of obtaining a subgraph $T^i \in R_D \cup S_D$. Let us first consider a good subdrawing of G^* in which the edges of $C_5^\star$ are crossed at most once. In this case, we obtain three non isomorphic drawings shown in Fig. 1(a)-(c). If we consider a good subdrawing of G^* in which two different edges of $C_5^\star$ are crossed once, then we obtain two possibilities that are shown in Fig. 1(d) and (e). Two crossings on only one edge of $C_5^\star$ can be achieved in Fig. 1(f)-(h). Finally, the drawing with three crossings on $C_5^\star$ is shown in Fig. 1(i).

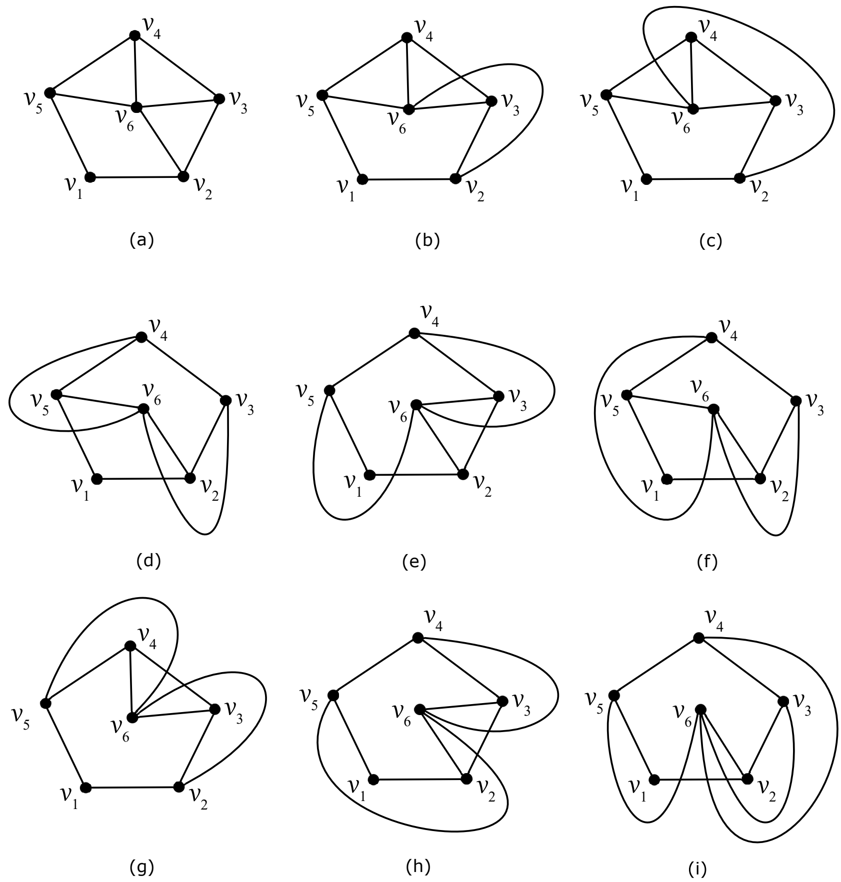

FIGURE 1. Nine possible non isomorphic drawings of the graph G^* with no crossing among edges of $C_5^{\star}$ and also with a possibility of obtaining a subgraph T^i whose edges can cross G^* at most once.

In the proof of the main Theorem 2.6 of this section, the following Lemma 2.2 related to some restricted subdrawings of $G^* + D_n$ will be also required. It includes well-known exact values for crossing numbers of join products of four subgraphs H_k of G^* with discrete graphs, and their planar drawings are shown in Fig. 2. The four mentioned results are described in Theorems 2.1, 2.2, 2.3 and 2.4.

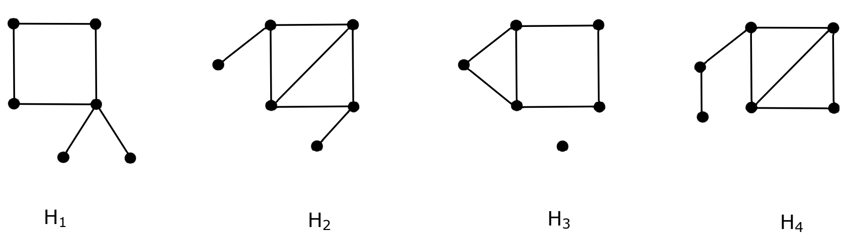

FIGURE 2. Four graphs H_k on six vertices with well-known values of $\mathrm{cr}(H_k + D_n)$.

Theorem 2.1 (see [32], Theorem 3.4). $\mathrm{cr}(H_1 + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Theorem 2.2 (see [35], Theorem 6). $\mathrm{cr}(H_2+D_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 1$.

Theorem 2.3 (see [33], Theorem 3.4). $\mathrm{cr}(H_3+D_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 1$.

Theorem 2.4 (see [34], Corollary 3). $\mathrm{cr}(H_4+D_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 1$.

For $k=1,2,3,4$, let G^*-H_k denote the graph difference of graphs G^* and H_k.

Lemma 2.2. *For* $n\geq 1$*, let* D *be a good drawing of* G^*+D_n *with the empty set* R_D*. If* $|S_D|\geq\lceil\frac{n}{2}\rceil$ *and each subgraph* $T^i\in S_D$ *can cross only some edge of* G^*-H_k*, then there are at least* $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor$ *crossings in* D*.*

*P*roof. As the graph G^* consists of two edge-disjoint subgraphs G^*-H_k and H_k, let us consider that $\mathrm{cr}_D(\bigcup_{j=1}^{n}T^j,G^*-H_k)\geq\lceil\frac{n}{2}\rceil$ is fulfilling in the good drawing D of G^*+D_n. The edges of H_k+D_n must be crossed at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ times in D according to Theorems 2.1, 2.2, 2.3, and 2.4. Consequently, we have

$$\mathrm{cr}_D(G^*+D_n)=\mathrm{cr}_D(H_k+D_n)+\mathrm{cr}_D(H_k+D_n,G^*-H_k)+\mathrm{cr}_D(G^*-H_k)\geq$$
$$\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor+\left\lceil\frac{n}{2}\right\rceil=6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

□

Theorem 2.5 (see [20], Theorem 2.3). *If* $m\geq 2$*,* $n\geq 3$ *and* $\min\{m,n\}\leq 6$*, then* $\mathrm{cr}(P_m+C_n)=\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+1$.

Lemma 2.3. $\mathrm{cr}(G^*+D_1)=1$ *and* $\mathrm{cr}(G^*+D_2)=3$.

*P*roof. Fig. 3 offers the subdrawing of G^*+D_1 with one crossing, and so $\mathrm{cr}(G^*+D_1)\leq 1$. The graph G^*+D_1 contains a subgraph that is a subdivision of the join product P_2+C_3, and therefore, $\mathrm{cr}(G^*+D_1)\geq\mathrm{cr}(P_2+C_3)=1$ by Theorem 2.5. The verification proceeds in a similar way also for the graph G^*+D_2 using a subgraph that is a subdivision of P_4+C_3. This completes the proof of Lemma 2.3. □

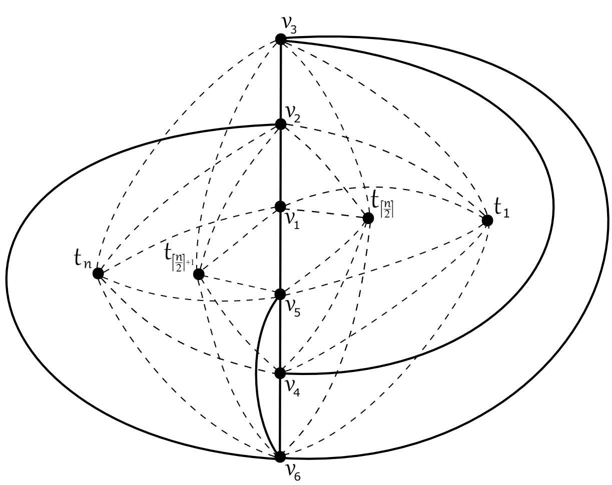

FIGURE 3. The good drawing of G^*+D_n with $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor$ crossings.

Theorem 2.6. $\mathrm{cr}(G^* + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

*P*roof. In Fig. 3, the edges of $K_{6,n}$ cross each other

$$6\binom{\lceil \frac{n}{2} \rceil}{2} + 6\binom{\lfloor \frac{n}{2} \rfloor}{2} = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

times, each subgraph T^i, $i = 1, \ldots, \lceil \frac{n}{2} \rceil$ on the right side crosses edges of G^* exactly once and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \ldots, n$ on the left side crosses edges of G^* exactly twice. Thus, $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ crossings appear among edges of the graph $G^* + D_n$ in this drawing. We prove the reverse inequality by induction on n. Lemma 2.3 confirms this result for $n = 1$ and $n = 2$. Suppose now that there is an optimal drawing D of $G^* + D_n$ with

$$\mathrm{cr}_D(G^* + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor \quad \text{for some } n \geq 3, \tag{2.3}$$

and let

$$\mathrm{cr}(G^* + D_m) = 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + m + \left\lfloor \frac{m}{2} \right\rfloor \quad \text{for any positive integer } m < n. \tag{2.4}$$

The assumption (2.3) together with $\mathrm{cr}_D(K_{6,n}) \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ thanks to (1.1) imply the following relation with respect to edge crossings of G^* in D:

$$\mathrm{cr}_D(G^*) + \sum_{T^i \in R_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \in S_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \notin R_D \cup S_D} \mathrm{cr}_D(G^*, T^i) < n + \left\lfloor \frac{n}{2} \right\rfloor.$$

In the case, if the set R_D is empty and $s = |S_D|$, then

$$\mathrm{cr}_D(G^*) + 1s + 2(n - s) < n + \left\lfloor \frac{n}{2} \right\rfloor, \tag{2.5}$$

which forces $s \geq \mathrm{cr}_D(G^*) + \lceil \frac{n}{2} \rceil + 1$. Now, we will deal with the possibilities of obtaining a subgraph $T^i \in R_D \cup S_D$ in the considered drawing D and we will show that in all cases a contradiction with the assumption (2.3) can be obtained.

Case 1: $\mathrm{cr}_D(G^*) = 0$. Without loss of generality, we can consider the planar subdrawing $D(G^*)$ induced by D with the vertex notation in such a way as shown in Fig. 1(a). Because no face is incident to all vertices in $D(G^*)$, there is no possibility to obtain a subdrawing of $G^* \cup T^i$ for a $T^i \in R_D$. As the set R_D is empty, there are at least $\lceil \frac{n}{2} \rceil + 1$ subgraphs T^i by which the edges of G^* are crossed just once. Let us denote by H_1 the subgraph of G^* with the vertex set $V(G^*)$, and the edge set $E(G^*) \setminus \{v_2v_3, v_3v_4, v_4v_5\}$. By Theorem 2.1, the edges of $H_1 + D_n$ are crossed at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ times in D. Clearly, each subgraph $T^i \in S_D$ must cross one edge of the cycle $C_5^\star$, and therefore, there is at least one subgraph $T^i \in S_D$ by which the edge v_1v_2 or v_1v_5 of $C_5^\star$ is crossed. In the rest of the proof, let the edge v_1v_2 of $C_5^\star$ be crossed by a $T^i \in S_D$. It is not difficult to verify over all possible regions of $D(G^* \cup T^i)$ that the edges of $G^* \cup T^i$ are crossed at least five and four times by each subgraph $T^j \in S_D$, $j \neq i$, and $T^k \notin S_D$, respectively. Thus, by fixing the subgraph $G^* \cup T^i$, we have

$$\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G^* \cup T^i) + \mathrm{cr}_D(G^* \cup T^i) \geq$$
$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(s-1) + 4(n-s) + 1 = 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n +$$
$$+ s - 4 \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n + \left\lceil \frac{n}{2} \right\rceil + 1 - 4 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor.$$

Case 2: $\mathrm{cr}_D(G^*) \geq 1$. At first, we can consider the nonplanar subdrawing $D(G^*)$ induced by D given in Fig. 1(b). The set R_D is also empty, and so $s \geq \lceil \frac{n}{2} \rceil + 2$. Let us denote by H_2 the subgraph of G^* with the vertex set $V(G^*)$, and the edge set $E(G^*) \setminus \{v_1v_5, v_5v_6\}$. The edges of $H_2 + D_n$ are crossed at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ times in D due to Theorem 2.2, which yields that there is at least one subgraph $T^i \in S_D$ by which the edge v_1v_2 of $C_5^\star$ must be crossed. Consequently, the same fixation of the subgraph $G^* \cup T^i$ as in Case 1 can be applied. For four drawings of G^* given in Fig. 1(c)-(f), Lemma 2.2 contradicts the assumption (2.3) using two its subgraphs H_3 and H_4. Moreover, if we consider some subdrawing $D(G^*)$ given in Fig. 1(g)-(i), then the set R_D cannot be empty also due to Lemma 2.2 for the subgraph H_3 of G^* with the edge set $E(G^*) \setminus \{v_1v_2, v_1v_5, v_4v_6\}$.

Now, let us turn to the possibility of obtaining a subdrawing $G^* \cup T^i$ for some $T^i \in R_D$, that is, the set R_D must be nonempty. Without lost of generality, let T^n be a subgraph by which the edges of G^* are not crossed. For the drawing of G^* given in Fig. 1(g), the reader can easily see that the subgraph $G^* \cup T^n$ is uniquely represented by $\mathrm{rot}_D(t_n) = (123645)$. If there is a subgraph T^j, $j \neq n$ such that $\mathrm{cr}_D(G^* \cup T^n, T^j) < 4$, then the vertex t_j must be placed in the outer region of subdrawing $D(G^*)$ with three vertices v_1, v_2 and v_5 of G^* on its boundary, and $\mathrm{cr}_D(G^* \cup T^n, T^j) = 3$ enforces $\mathrm{cr}_D(T^n, T^j) = 0$. Thus, by fixing the subgraph $T^n \cup T^j$, we have

$$\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D(G^* + D_{n-2}) + \mathrm{cr}_D(T^n \cup T^j) + \mathrm{cr}_D(K_{6,n-2}, T^n \cup T^j) +$$

$$+\mathrm{cr}_D(G^*, T^n \cup T^j) \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + n - 2 + \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 6(n-2) + 3 =$$

$$= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor,$$

where edges of $T^n \cup T^j$ are crossed by each other subgraph T^k at least six times using $\mathrm{cr}_D(K_{6,3}) \geq 6$ thanks to (1.1). In the following, let the edges of T^n be crossed by each other subgraph T^j at least once. It is not difficult to verify over possible regions of $D(G^* \cup T^n)$ that edges of $G^* \cup T^n$ are crossed by each other subgraph T^j, $j \neq n$ at least four times and just four crossings could be achieved for some subgraphs by which the edge v_3v_4 of G^* is crossed. This implies $\mathrm{cr}_D(G^* \cup T^n, \bigcup_{j=1}^{n-1} T^j) \geq 5(n-1) - \alpha$, where α is the number of such subgraphs forcing at least one crossing on the edge v_3v_4. By fixing the subgraph $G^* \cup T^n$, we have at least

$$6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(n-1) - \alpha + \mathrm{cr}_D(G^* \cup T^n) \tag{2.6}$$

crossings in D. As $\alpha < \lceil \frac{n}{2} \rceil$ using the subgraph H_1 from Case 1, both considered subcases contradict the assumption (2.3) in D. For the drawing of G^* given in Fig. 1(h), the edges of $G^* \cup T^n$ are crossed by each other subgraph T^j, $j \neq n$ at least four times and just four crossings can be achieved for some subgraphs by which the edge v_2v_3 of G^* is crossed. Thus, the same idea of the subgraph H_1 from Case 1 can also be applied. Finally, the edges of $G^* \cup T^n$ are crossed by each other subgraph T^j, $j \neq n$ at least five times if we consider the last possible drawing of G^* given in Fig. 1(i).

We have shown that there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ crossings in each good drawing D of $G^* + D_n$, and the proof of Theorem 2.6 is done. □

3. Some Consequences of the Main Result

Each wheel W_m of $m+1$ vertices consists of two edge-disjoint subgraphs $C_m^\star$ and $S_m^\star$. First, we deal with the possibility of deleting one edge e_S from the star $S_m^\star$ of W_m. Berežný and Staš [2] gave a conjecture regarding the crossing number of $W_m + D_n$ equal to $Z(m+$

$1)Z(n) + [Z(m) - 1]\lfloor \frac{n}{2} \rfloor + n$, where $Z(n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is Zarankiewicz's number, see also [4]. Now, we are able to postulate that

$$\mathrm{cr}(W_m \setminus e_S + D_n) = Z(m+1)Z(n) + [Z(m-1) - 1]\left\lfloor \frac{n}{2} \right\rfloor + n, \tag{3.7}$$

for all integers $m \geq 4, n \geq 1$.

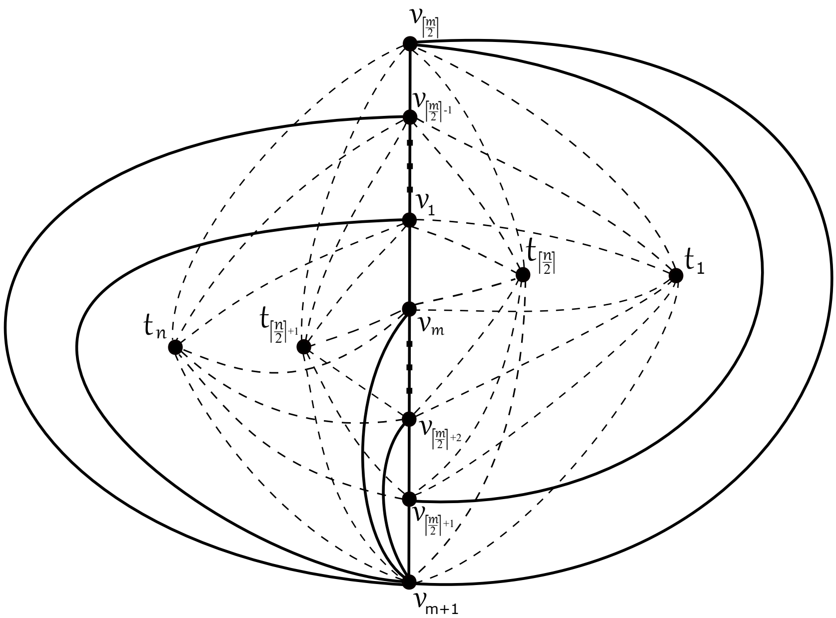

FIGURE 4. The good drawing of $W_m + D_n$ with exactly $Z(m+1)Z(n) + [Z(m) - 1]\lfloor \frac{n}{2} \rfloor + n$ crossings.

For $m \geq 4$, the upper bound for the conjecture (3.7) can be reached by removing the edge $v_1 v_{m+1}$ from the drawing in Fig. 4 because $e_S = v_1 v_{m+1}$ is crossed by each subgraph T^i on the left side exactly $\lceil \frac{m}{2} \rceil - 1$ times. Note that for $m = 3$, the optimal drawing of $W_3 \setminus e_S + D_n$ with $2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings could be obtained if we remove the edge $v_{\lceil \frac{m}{2} \rceil} v_{\lceil \frac{m}{2} \rceil +1}$. This special situation is first caused by the fact that the wheel W_3 is isomorphic to the complete graph K_4, see also Klešč and Schrötter [25].

Recently, our conjecture (3.7) was proved for the graph $W_4 \setminus e_S + D_n$ by Asano [1]. Theorem 2.6 also confirms the validity of this conjecture for $W_5 \setminus e_S + D_n$. On the other hand, the graphs $W_m \setminus e_S + D_1$ and $W_m \setminus e_S + D_2$ contain a subgraph that is a subdivision of the graph $W_{m-1} + D_1$ and $W_{m-1} + D_2$, respectively. The crossing numbers of the join products of W_m with the discrete graphs D_1 and D_2 have been well-known by Berežný and Staš [2].

Theorem 3.7 (see [2], Theorem 4.2). *$\mathrm{cr}(W_m + D_1) = 1$ and $\mathrm{cr}(W_m + D_2) = Z(m) + 1$ for $m \geq 3$.*

These facts allow us to determine another results for the join product of $W_m \setminus e_S$ with the discrete graph on one and two vertices if m is at least four.

Corollary 3.1. *$\mathrm{cr}(W_m \setminus e_S + D_1) = 1$ and $\mathrm{cr}(W_m \setminus e_S + D_2) = Z(m-1) + 1$ for $m \geq 4$, $m \in \mathbb{Z}$.*

One can easily verify that these results also confirm the validity of our conjecture for the graphs $W_m \setminus e_S + D_1$ and $W_m \setminus e_S + D_2$.

Now, let us turn to the possibility of deleting one edge e_C from the cycle $C_m^\star$ of W_m. Harboth [7] gave an upper bound on the crossing number of the complete n-partite graph $K_{x_1,\dots,x_n}$ by which

$$\text{(3.8)} \qquad \mathrm{cr}(K_{1,m,n}) \leq Z(m+1)Z(n) + Z(m)\left\lfloor \frac{n}{2} \right\rfloor \quad \text{for all integers } m, n \geq 1.$$

Assuming the validity of Zarankiewicz's conjecture that $\mathrm{cr}(K_{m,n}) = Z(m)Z(n)$, the crossing numbers of the complete tripartite $K_{1,m,n}$ have been well-known by Yang and Wang [38].

Theorem 3.8 (see [38], Corollary 2)**.** *If Zarankiewicz's conjecture is true, then*

$$\text{(3.9)} \qquad \mathrm{cr}(K_{1,m,n}) = Z(m+1)Z(n) + Z(m)\left\lfloor \frac{n}{2} \right\rfloor$$

holds for all positive integers m and n.

Based on the arguments above, we are able to postulate that

$$\text{(3.10)} \quad \mathrm{cr}(W_m \setminus e_C + D_n) = Z(m+1)Z(n) + Z(m)\left\lfloor \frac{n}{2} \right\rfloor \quad \text{for all integers } m \geq 3, n \geq 1.$$

Again for all $m \geq 3$, the upper bound for the conjecture (3.10) can be reached by removing the edge $v_{\lceil \frac{m}{2} \rceil} v_{\lceil \frac{m}{2} \rceil + 1}$ from the drawing in Fig. 4 because $e_C = v_{\lceil \frac{m}{2} \rceil} v_{\lceil \frac{m}{2} \rceil + 1}$ is crossed by each subgraph T^i on the right side exactly once. On the other hand, the complete bipartite graph $K_{1,m}$ is a subgraph of $W_m \setminus e_C$, and therefore, $\mathrm{cr}(K_{1,m} + D_n) \leq \mathrm{cr}(W_m \setminus e_C + D_n)$. This together with the exact value of $\mathrm{cr}(K_{1,m,n})$ by Theorem 3.8 imply the following result.

Corollary 3.2. *If Zarankiewicz's conjecture is true, then*

$$\text{(3.11)} \qquad \mathrm{cr}(W_m \setminus e_C + D_n) = Z(m+1)Z(n) + Z(m)\left\lfloor \frac{n}{2} \right\rfloor$$

holds for all integers $m \geq 3$, $n \geq 1$.

Note that Norin and Zwols [30] obtained the best known asymptotic lower bound for the crossing number of $K_{m,n}$ with $m \geq 9$ in the form

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor} \geq \frac{0.905m}{m-1}$$

which implies that Zarankiewicz's conjecture mentioned above is "asymptotically at least $90,5\%$ true". Some useful remarks about Zarankiewicz's conjecture were also stated by Staš and Valiska [36], where another conjecture concerning $\mathcal{CF}$-connectivity for $K_{m,n}$ follows from this one.

4. Conclusions

We expect that similar forms of discussions can be used to estimate unknown values of the crossing numbers of other graphs on six vertices with a much larger number of edges in the join products with discrete graphs, and also with paths and cycles. Especially for the graph $W_5 \setminus e$ obtained by removing one edge (of both possible types) from W_5 in the join products with paths and cycles.

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