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Stability of oscillatory solutions of impulsive differential equations with general piecewise constant arguments of mixed type

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ABSTRACT. We investigate scalar impulsive differential equations with piecewise constant generalized mixed arguments, abbreviated as the IDEPCAG of mixed type. These equations have general step functions as arguments. We propose criteria for the existence of oscillatory and non-oscillatory solutions, and obtain sufficient conditions for the stability of the zero solution. Our results are novel, and extend and improve upon previous publications. Additionally, we provide several numerical examples and simulations to demonstrate the feasibility of our findings.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of all natural, integer, and real numbers, respectively. Consider a strictly ordered sequence of real numbers denoted by $t_i, i \in \mathbb{Z}$, such that $t_i < t_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. Let $\gamma_1 : \mathbb{R} \to \mathbb{R}$ and $\gamma_2 : \mathbb{R} \to \mathbb{R}$ be step functions defined as $\gamma_1(t) = t_i$ and $\gamma_2(t) = t_{i+1}$ for $t \in I_i = [t_i, t_{i+1})$.

We aim to study the global asymptotic behavior and oscillation of solutions of impulsive differential equations with piecewise constant generalized mixed arguments (abbreviated as the IDEPCAG of mixed type):

$$
\begin{cases} y'(t) = a(t)y(t) + b(t)y(\gamma_1(t)) + c(t)y(\gamma_2(t)), \quad y(\tau) = y_0, \quad t \neq t_k, & \text{(1.1a)} \\ \Delta y|_{t=t_k} = d_k y(t_k^-), \quad k \in \mathbb{Z}, & \text{(1.1b)} \end{cases}
$$

where $a(t)$, $b(t)$, and $c(t)$ are continuous real-valued functions defined on $\mathbb{R}$, $d_k \in \mathbb{R}$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+)$ and $y(t_k^-)$ denote the right-hand and left-hand limits of $y(t)$ as t approaches t_k, respectively.

The first deviation argument, $\wp_1(t) = t - \gamma_1(t)$, is positive for $t_i < t < t_{i+1}$, and the second deviation argument, $\wp_2(t) = t - \gamma_2(t)$, is negative for $t_i < t < t_{i+1}$, where $i \in \mathbb{Z}$. Therefore, the IDEPCAG of mixed type (1.1a)-(1.1b) is of considerable interest since it contains both retarded and advanced arguments on each interval $[t_i, t_{i+1})$.

Differential equations with piecewise constant arguments (DEPCA) that have argument deviations of fixed sign were the first to be investigated [1, 31, 34]. Piecewise constant systems exist in a wide range of areas, including biomedicine, chemistry, mechanical engineering, physics, and others. Systematic studies involving mathematical models with piecewise constant arguments were initiated to solve biomedical problems. These equations have a similar structure to those found in certain sequential-continuous models of disease dynamics, as investigated in [2]. We also refer the reader to the papers

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[15, 16, 24, 25] where several dissipation problems impulsive effects are also idealized by means of external and/or drift sources.

The investigation of the DEPCA of mixed type was initiated by S. M. Shah and J. Wiener [31] in 1983, and later by G. Ladas [29] in 1988. They noted that the change of sign in the argument deviation not only led to interesting periodic properties, but also introduced complications in the asymptotic and oscillatory behavior of solutions. See also the papers [6, 8, 14, 26, 27, 30] for the periodicity, oscillation and asymptotic behavior of differential equations with deviating arguments. As a result, it was natural to attempt to study the oscillatory and stability properties of the DEPCA of mixed type with general deviation arguments.

Criteria for the existence of oscillatory solutions of the DEPCA have been studied extensively in the literature by many authors, including [1, 17, 28, 29, 32, 33]. However, the question of what additional conditions are necessary for the stability of oscillatory solutions is still an active area of research. In particular, the stability of oscillatory solutions has been studied in the area of differential equations. For example, A. R. Aftabizadeh and J. Wiener in [1] investigated the oscillatory properties of solutions of first-order linear DEPCA of retarded and advanced type:

$$x'(t) + a(t)x(t) + p(t)x([t]) = 0,$$
$$x'(t) + a(t)x(t) + q(t)x([t+1]) = 0,$$

where $a(t)$, $p(t)$, and $q(t)$ are continuous on $[0,\infty)$, and the greatest-integer function is denoted by $[\cdot]$. Sufficient conditions for the existence of oscillatory solutions of the DEPCA are presented, and the authors emphasize that these conditions are optimal in the sense that they are the "best possible". In particular, when a, p, and q are constants, the conditions can be simplified to the following necessary and sufficient expressions when a, p, and q are constants:

$$p > \frac{a}{e^a - 1} \quad \text{and} \quad q < -\frac{ae^a}{e^a - 1}.$$

In their paper [31], S. M. Shah and J. Wiener proved the following result: For any real numbers a, a_0, and a_1 satisfying $a \neq 0$ and

$$(a + a_0 + a_1)\left(a_1 - a_0 - \frac{a(e^a + 1)}{e^a - 1}\right) > 0,$$

every solution x of the differential equation with piecewise constant arguments of mixed type, given by

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t+1]), \quad x(0) = x_0, \tag{1.2}$$

tends to zero as $t \to \infty$.

In the paper [4], K.-S. Chiu and J.-C. Jeng considered the first-order linear DEPCA of generalized mixed arguments (DEPCAG of mixed type)

$$y'(t) = a(t)y(t) + b(t)y(\gamma_1(t)) + c(t)y(\gamma_2(t)), \quad y(\tau) = y_0 \tag{1.3}$$

and the authors investigated the sufficient conditions for the existence and global asymptotic stability of oscillatory and non-oscillatory solutions for certain specific cases.

In the paper [5], K.-S. Chiu and T. Li investigated the sufficient conditions for the existence and global asymptotic stability of oscillatory and periodic solutions for certain specific cases of the DEPCAG of mixed type (1.3).

It is worth noting that the results established in [31] by S. M. Shah and J. Wiener, in [4] by K.-S. Chiu and J.-C. Jeng and in [5] by K.-S. Chiu and T. Li show that under the specified conditions, every solution of a certain class of the DEPCA with mixed type and a certain class of the DEPCAG with mixed type, respectively, tends to zero as $t \to \infty$.

In 2013, F. Karakoc et al. [19] investigated first order non-linear impulsive differential equation with piecewise constant arguments (IDEPCA):

$$\begin{cases} x'(t) + a(t)x(t) + x([t-1])f(x([t])) = 0, \quad x(-1) = x_{-1}, \quad x(0) = x_0 \quad t \neq n, \\ \Delta x|_{t=n} = d_k x(n^-), \quad n \in \mathbb{N}. \end{cases}$$

In addition to establishing the existence and uniqueness of a solution, the authors have also derived sufficient conditions for the oscillation of the solution. It is important to highlight that further research regarding the oscillation and asymptotic behavior of the IDEPCA with classical piecewise constant argument can be found in the following papers: [18–23]. These papers offer additional insights into the mentioned topic.

To the best of our knowledge, several studies have been conducted on the existence and global asymptotic stability of periodic solutions associated with the IDEPCAG [7, 9–13]. However, thus far, none of these studies have provided simple criteria for determining the existence and stability of oscillatory and non-oscillatory solutions specifically for the IDEPCAG of mixed type. The main objective of this paper is to extend the classical results of the DEPCA from [1] and [31], the results of the DEPCAG from [4, 5], and the results of the IDEPCA from [3] to the IDEPCAG of mixed type, represented by equations (1.1a)-(1.1b). The paper aims to provide straightforward criteria for determining the existence and stability of both oscillatory and non-oscillatory solutions in this extended framework.

For the reader's convenience, we provide some definitions that will be necessary later on.

We define a function y as a solution of the IDEPCAG of mixed type (1.1a)-(1.1b) on $\mathbb{R}$ if the following conditions hold: i) $y : \mathbb{R} \to \mathbb{R}$ is continuous for $t \in \mathbb{R}$, except possibly at the points t_k, $k \in \mathbb{Z}$. ii) $y(t)$ is right-continuous and has left-hand limits at the points t_k, $k \in \mathbb{Z}$. iii) The derivative $y'(t)$ exists at each point $t \in \mathbb{R}$, except possibly at the points t_k, $k \in \mathbb{Z}$, where the one-sided derivatives exist. iv) $y(t)$ satisfies (1.1a), except possibly at the points t_k, $k \in \mathbb{Z}$. v) $y(t_k)$ satisfies (1.1b) for $k \in \mathbb{Z}$.

A function $y(t)$ defined on $\mathbb{R}$ is said to be oscillatory if there exist two real-valued sequences $(\nu_n)_{n\geq 0}$ and $(\nu'_n)_{n\geq 0}$ in $\mathbb{R}$ such that $\nu_n \to \infty$ and $\nu'_n \to \infty$ as $n \to \infty$, and $y(\nu_n) \leq 0 \leq y(\nu'_n)$ for $n \geq N$, where N is sufficiently large. Otherwise, the solution is called non-oscillatory.

A solution $\{x_n\}_{n\in\mathbb{Z}}$ of the difference equation is called oscillatory if $x_n \cdot x_{n+1} \leq 0$. Otherwise, $\{x_n\}_{n\in\mathbb{Z}}$ is called non-oscillatory.

Our principal contributions are:

1. The utilization of piecewise constant generalized mixed arguments with impulsive effects has not been previously incorporated into the oscillatory theory of the IDEPCAG of mixed type.
2. A global solution for the IDEPCAG of mixed type (1.1a)-(1.1b) is proven under certain conditions, considering arbitrary initial data $(\tau, y_0) \in \mathbb{R} \times \mathbb{R}$.
3. The study of the existence and global asymptotic stability of oscillatory and non-oscillatory solutions for the IDEPCAG of mixed type (1.1) is first put forward.
4. Three special cases of the main results derived in Theorems 2.3, 3.1, and 3.2 are feasible by using the simulations given in Examples 4.1, 4.2 and 4.3, which ensures the advantages of this paper.

Our paper is organized as follows: In Section 2, we establish criteria for the existence of oscillatory and non-oscillatory solutions of scalar impulsive differential equation with piecewise constant generalized mixed arguments. In Section 3, we study the stability of solutions of linear impulsive differential equation. In the last section, we provide appropriate examples with simulations.

2. Existence of the Oscillatory and Non-oscillatory solutions

In this section, we establish sufficient conditions for oscillatory and non-oscillatory solutions of scalar impulsive differential equations with piecewise constant generalized mixed arguments.

The following assumption will be used throughout the paper:

(N) For every $t \in \mathbb{R}$, let $i = i(t) \in \mathbb{Z}$ be the unique integer such that $t \in I_i = [t_i, t_{i+1})$. Suppose that

$$d_{i+1} \cdot (\beta(t_{i+1}, t_i) - 1) + \beta(t_{i+1}, t_i) \neq 0$$

holds for all $i \in \{i(\tau) + j\}_{j\in\mathbb{N}}$, where

$$\beta(t, s) := 1 - \int_s^t e^{\int_u^t a(\kappa)d\kappa} c(u)du,$$

and

$$\lambda(t, s) := e^{\int_s^t a(\kappa)d\kappa} + \int_s^t e^{\int_u^t a(\kappa)d\kappa} b(u)du.$$

The theorem below provides conditions for the existence and uniqueness of solutions on the interval $[\tau, \infty)$, where $\tau = t_{i(\tau)}$. The proof of this assertion is similar to that of Theorem 2.2 in [13], as well as with proofs presented in [4] and in [19].

Theorem 2.1. *Suppose that* **(N)** *holds. Then, the IDEPCAG of mixed type* (1.1a)-(1.1b) *has a unique solution on* $[\tau, \infty)$ *with the initial condition* $y(\tau) = y_0$*. Furthermore, for* $t \in [t_n, t_{n+1})$ *with* $n > i(\tau)$*,* y *has the form*

$$y(t) = \left(\lambda(t, t_n) + \frac{(1 + d_{n+1})\lambda(t_{n+1}, t_n)}{d_{n+1} \cdot (\beta(t_{n+1}, t_n) - 1) + \beta(t_{n+1}, t_n)}(1 - \beta(t, t_n))\right) x_n \tag{2.1}$$

where $x_n = y(t_n)$ *and the sequence* $\{x_n\}_{n\geq i(\tau)}$ *is the unique solution of the difference equation*

$$x_{n+1} = (1 + d_{n+1}) \frac{\lambda(t_{n+1}, t_n)}{d_{n+1} \cdot (\beta(t_{n+1}, t_n) - 1) + \beta(t_{n+1}, t_n)} x_n, \tag{2.2}$$

for $n > i(\tau)$ *with the initial condition* $x_{i(\tau)} = y_0$*.*

Proof. Let $y_n(t)$ be a solution of the IDEPCAG of mixed type (1.1a)-(1.1b) on the interval $t_n \leq t < t_{n+1}$. On the interval, we can express (1.1a) as follows:

$$y_n'(t) = a(t)y_n(t) + b(t)y_n(t_n) + c(t)y_n(t_{n+1}).$$

The general solution to this equation on the given interval is:

$$\begin{aligned} y_n(t) &= \left[e^{\int_{t_n}^t a(\kappa)d\kappa} + \int_{t_n}^t e^{\int_s^t a(\kappa)d\kappa} b(s)ds\right] y_n(t_n) + \left[\int_{t_n}^t e^{\int_s^t a(\kappa)d\kappa} c(s)ds\right] y_n(t_{n+1}) \\ &= \lambda(t, t_n)y_n(t_n) + \left[\int_{t_n}^t e^{\int_s^t a(\kappa)d\kappa} c(s)ds\right] y_n(t_{n+1}). \end{aligned} \tag{2.3}$$

As $t \to t_{n+1}$ in (2.3) and under the impulse condition (1.1b), we obtain

$$\begin{aligned} y_n(t_{n+1}) &= (1 + d_{n+1})y(t_{n+1}^-) \\ &= (1 + d_{n+1}) \left\{\lambda(t_{n+1}, t_n)y_n(t_n) + \left[\int_{t_n}^{t_{n+1}} e^{\int_s^{t_{n+1}} a(\kappa)d\kappa} c(s)ds\right] y_n(t_{n+1})\right\}, \end{aligned}$$

or

(2.4)

$$
\begin{aligned}
y_n(t_{n+1}) &= \frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{1-(1+d_{n+1})\left[\int_{t_n}^{t_{n+1}} e^{\int_s^{t_{n+1}} a(\kappa)d\kappa} c(s)ds\right]} y_n(t_n) \\
&= (1+d_{n+1})\,\frac{\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)} y_n(t_n), \quad \text{for all } n > i(\tau).
\end{aligned}
$$

Hence, substituting (2.4) into the previous equation yields:

(2.5)
$$
y_n(t) = \left(\lambda(t,t_n) + \frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)}(1-\beta(t,t_n))\right) y_n(t_n).
$$

Using equation (2.5), we obtain the difference equation (2.2). With the initial condition $x_{i(\tau)} = y(\tau) = y_0$, we can obtain a unique solution for (2.2). Therefore, the unique solution of the IDEPCAG of mixed type (1.1a)-(1.1b) with the initial condition $y(\tau) = y_0$ is given by (2.1). □

By the recurrence relation, we can see that the solution of the IDEPCAG of mixed type (1.1a)-(1.1b) on $[\tau, \infty)$, where $\tau \neq t_{i(\tau)}$, is uniquely determined and can be expressed as follows:

$$
y(t) = \frac{\lambda_t(t,t_{i(t)})}{\lambda_\tau(\tau,t_{i(\tau)})}\left(\prod_{j=i(\tau)}^{i(t)-1} \frac{(1+d_{j+1})\lambda(t_{j+1},t_j)}{d_{j+1}\cdot(\beta(t_{j+1},t_j)-1)+\beta(t_{j+1},t_j)}\right) y(\tau)
$$

where,

$$
\lambda_t(t,t_{i(t)}) = \lambda(t,t_{i(t)}) + \frac{(1+d_{i(t)+1})\lambda(t_{i(t)+1},t_{i(t)})(1-\beta(t,t_{i(t)}))}{d_{i(t)+1}\cdot\big(\beta(t_{i(t)+1},t_{i(t)})-1\big)+\beta(t_{i(t)+1},t_{i(t)})},
$$

and

$$
\lambda_\tau(\tau,t_{i(\tau)}) = \lambda(\tau,t_{i(\tau)}) + \frac{\big(1+d_{i(\tau)+1}\big)\lambda(t_{i(\tau)+1},t_{i(\tau)})(1-\beta(\tau,t_{i(\tau)}))}{d_{i(\tau)+1}\cdot\big(\beta(t_{i(\tau)+1},t_{i(\tau)})-1\big)+\beta(t_{i(\tau)+1},t_{i(\tau)})}.
$$

In particular, $\tau = t_{i(\tau)}$, we have

$$
y(t) = \lambda_t(t,t_{i(t)})\left(\prod_{j=i(\tau)}^{i(t)-1} \frac{(1+d_{j+1})\lambda(t_{j+1},t_j)}{d_{j+1}\cdot(\beta(t_{j+1},t_j)-1)+\beta(t_{j+1},t_j)}\right) y(\tau).
$$

Remark 2.1. *If we consider the IDEPCAG of mixed type* (1.1) *without the impulsive condition* (1.1b)*, i.e., when $d_k = 0$ for all $k \in \mathbb{Z}$, the IDEPCAG can be reduced to the following DEPCAG of mixed type* (1.1a)*:*

$$
y'(t) = a(t)y(t) + b(t)y(\gamma_1(t)) + c(t)y(\gamma_2(t)), \quad y(\tau) = y_0.
$$

It directly follows from Theorem 2.1 that the solution is given by

$$
y(t) = \frac{\lambda_t(t,t_{i(t)})}{\lambda_\tau(\tau,t_{i(\tau)})}\prod_{j=i(\tau)}^{i(t)-1} \frac{\lambda(t_{j+1},t_j)}{\beta(t_{j+1},t_j)} y(\tau),
$$

which is also proved in Theorem 2.1 of [4] and Theorem 2.2 of [5].

The following results are particular cases of Theorem 2.1.

Corollary 2.1. *Let* $\hat{\lambda}(t) = e^{at} + \frac{b}{a}(e^{at} - 1)$, $\hat{\beta}(t) = 1 - \frac{c}{a}(e^{at} - 1)$, $\vartheta_n = t_{n+1} - t_n$ *for all* $n \in \{i(\tau) + j\}_{j\in\mathbb{N}}$, *and assume that* $d_{n+1} \cdot \left(\hat{\beta}(\vartheta_n) - 1\right) + \hat{\beta}(\vartheta_n) \neq 0$ *for all* $n \in \{i(\tau) + j\}_{j\in\mathbb{N}}$. *If* $a(t) = a \neq 0$, $b(t) = b$, $c(t) = c$ *are constants, then the IDEPCAG of mixed type* (1.1a)-(1.1b) *has a unique solution* y *which is given by*

$$y(t) = \left(\hat{\lambda}(t - t_n) + \frac{(1 + d_{n+1})\hat{\lambda}(\vartheta_n)[1 - \hat{\beta}(t - t_n)]}{d_{n+1} \cdot (\hat{\beta}(\vartheta_n) - 1) + \hat{\beta}(\vartheta_n)}\right) x_n, \quad t_n \leq t < t_{n+1} \tag{2.6}$$

where $x_n = y(t_n)$. *We can express the sequence* $\{x_n\}_{n\geq i(\tau)}$ *as a solution to the difference equation*

$$x_{n+1} = \frac{(1 + d_{n+1})\hat{\lambda}(\vartheta_n)}{d_{n+1} \cdot (\hat{\beta}(\vartheta_n) - 1) + \hat{\beta}(\vartheta_n)} x_n, \tag{2.7}$$

for $n > i(\tau)$ *with the initial condition* $x_{i(\tau)} = y_0$.

Corollary 2.2. *Let* $\eta(t) := \int_{t_{i(t)}}^{t} b(u)du$, $\mu(t) := \int_{t_{i(t)}}^{t} c(u)du$, $\mu_0(t) := \int_{\tau}^{t} c(u)du$, $(1 + d_{n+1})\mu(t_{n+1}) \neq 1$ *for all* $n \in \{i(\tau) + j\}_{j\in\mathbb{N}}$ *and* $(1 + d_{i(\tau)+1})\mu_0(t_{i(\tau)+1}) \neq 1$. *Then*

$$\begin{cases} u'(t) = b(t)u(\gamma_1(t)) + c(t)u(\gamma_2(t)), \quad u(\tau) = u_0, \quad t \neq t_k, & (2.8a) \\ \Delta u|_{t=t_k} = d_k u(t_k^-), \quad k \in \mathbb{Z}, & (2.8b) \end{cases}$$

with the initial condition $u(\tau) = u_0$ *has a unique solution* u *which is given by*

$$u(t) = \left(1 + \eta(t) + \frac{(1 + d_{n+1})(1 + \eta(t_{n+1}))}{1 - (1 + d_{n+1})\mu(t_{n+1})}\mu(t)\right) u_n, \quad t_n \leq t < t_{n+1} \tag{2.9}$$

where $u_n = u(t_n)$ *and the sequence* $\{u_n\}_{n\geq i(\tau)}$ *satisfies the difference equation*

$$u_{n+1} = \frac{(1 + d_{n+1})(1 + \eta(t_{n+1}))}{1 - (1 + d_{n+1})\mu(t_{n+1})} u_n, \tag{2.10}$$

for $n > i(\tau)$ *with the initial condition* $u_{i(\tau)} = u_0$.

Remark 2.2. *Corollary 2.1 deduces the result presented in [4, Corollary 2.2], while Corollary 2.2 deduces the finding in [5, Corollary 2.3] without considering impulsive effects. Theorem 2.1 deduces [3, Theorem 1] by incorporating classical piecewise constant delays* $\gamma_1(t) = [t]$ *and* $\gamma_2(t) = [t+1]$. *Furthermore, Corollary 2.1 deduces the result presented in [31, Theorem 2.1] without impulsive effects and with classical piecewise constant delays. These deductions demonstrate the generality of our results and their complementary nature to the previously known findings.*

The following theorems provide some sufficient conditions for the existence of oscillatory and non-oscillatory solutions of the IDEPCAG of mixed type (1.1a)-(1.1b).

Theorem 2.2. *Suppose that* **(N)** *holds and let* $y : [\tau, \infty) \to \mathbb{R}$ *be a solution of the IDEPCAG of mixed type* (1.1a)-(1.1b). *If the solution* $\{x_n\}_{n\geq i(\tau)}$ *of the difference equation (2.2) is oscillatory, then the solution* $y(t)$ *of Eq.(1.1) is also oscillatory.*

Proof. By using Equation (2.5), we can express $y(t)$ for $t_n \leq t < t_{n+1}$, where $n \in \{i(\tau) + j\}_{j\in\mathbb{N}}$, as:

$$y(t) = \left(\lambda(t, t_n) + \frac{(1 + d_{n+1})\lambda(t_{n+1}, t_n)}{d_{n+1} \cdot (\beta(t_{n+1}, t_n) - 1) + \beta(t_{n+1}, t_n)}(1 - \beta(t, t_n))\right) x_n.$$

Note that $y(t) = x_n$ for $t = t_n$. According to the theory of difference equations, x_n oscillates if and only if $x_n \cdot x_{n+1} \leq 0$ for $n \geq N'$, where N' is a sufficiently large integer. Therefore, $y(t)$ is an oscillatory solution. □

Theorem 2.3. *If $b(t)$ and $c(t)$ are locally integrable on $[\tau,\infty)$ and the sequence*

$$\left\{\frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)}\right\}_{n\geq i(\tau)}$$

is not eventually positive, then every solution of the IDEPCAG of mixed type (1.1a)-(1.1b) *is oscillatory.*

Proof. Using equation (2.2), we can express the sequence $\{x_n\}_{n\geq i(\tau)}$ as

$$x_{n+1}=\frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)}x_n.$$

It is clear that $\{x_n\}_{n\geq i(\tau)}$ oscillates if the sequence $\left\{\frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)}\right\}_{n\geq i(\tau)}$ is not eventually positive. Thus, according to Theorem 2.2, we conclude that $y(t)$ is oscillatory if $\{x_n\}_{n\geq i(\tau)}$ is oscillatory. This completes the proof. □

Theorem 2.4. *Suppose that $1+d_{n+1}>0$, for $n\geq i(\tau)$. Then every solution of the IDEPCAG of mixed type* (1.1a)-(1.1b) *is oscillatory, if either of the following hypotheses is satisfied:*

i)

$$\begin{aligned}
&\lim_{n\to\infty}\inf\int_{t_n}^{t_{n+1}}e^{\int_s^{t_n}a(\kappa)d\kappa}b(s)ds<-1,\\
&\lim_{n\to\infty}\sup\left\{(1+d_{n+1})\int_{t_n}^{t_{n+1}}e^{\int_s^{t_{n+1}}a(\kappa)d\kappa}c(s)ds\right\}<1,
\end{aligned}\tag{2.11}$$

ii)

$$\begin{aligned}
&\lim_{n\to\infty}\sup\int_{t_n}^{t_{n+1}}e^{\int_s^{t_n}a(\kappa)d\kappa}b(s)ds>-1,\\
&\lim_{n\to\infty}\inf\left\{(1+d_{n+1})\int_{t_n}^{t_{n+1}}e^{\int_s^{t_{n+1}}a(\kappa)d\kappa}c(s)ds\right\}>1.
\end{aligned}\tag{2.12}$$

Proof. Suppose that y is a solution of the IDEPCAG of mixed type (1.1a)-(1.1b) such that $y(t)>0$ (or $y(t)<0$) for $t>t_j$, where $j\in\mathbb{N}$ is sufficiently large. If $t\in I_n$, $n>j$, then using (2.4), we can obtain

$$y(t_{n+1})=\frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)}y(t_n).$$

Since $y(t_{n+1})$ and $y(t_n)$ are both positive, thus

$$\frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)}>0$$

if

$$(1+d_{n+1})\lambda(t_{n+1},t_n)>0\quad\text{and}\quad d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)>0,$$

or

$$\int_{t_n}^{t_{n+1}}e^{\int_s^{t_n}a(\kappa)d\kappa}b(s)ds>-1\quad\text{and}\quad(1+d_{n+1})\cdot\int_{t_n}^{t_{n+1}}e^{\int_s^{t_{n+1}}a(\kappa)d\kappa}c(s)ds<1,$$

i.e,

$$\lim_{n\to\infty}\inf\int_{t_n}^{t_{n+1}}e^{\int_s^{t_n}a(\kappa)d\kappa}b(s)ds\geq-1$$

and

$$\lim_{n\to\infty} \sup \left\{ (1+d_{n+1}) \int_{t_n}^{t_{n+1}} e^{\int_s^{t_{n+1}} a(\kappa)d\kappa} c(s)ds \right\} \leq 1.$$

However, this contradicts the first condition stated in (2.11).
Similarly,

$$\frac{(1+d_{n+1})\lambda(t_{n+1},t_n)}{d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n)} > 0$$

if

$$(1+d_{n+1})\lambda(t_{n+1},t_n) < 0 \quad \text{and} \quad d_{n+1}\cdot(\beta(t_{n+1},t_n)-1)+\beta(t_{n+1},t_n) < 0,$$

we obtain

$$\int_{t_n}^{t_{n+1}} e^{\int_s^{t_n} a(\kappa)d\kappa} b(s)ds < -1,$$

and

$$(1+d_{n+1})\cdot \int_{t_n}^{t_{n+1}} e^{\int_s^{t_{n+1}} a(\kappa)d\kappa} c(s)ds > 1;$$

or

$$\lim_{n\to\infty} \sup \int_{t_n}^{t_{n+1}} e^{\int_s^{t_n} a(\kappa)d\kappa} b(s)ds \leq -1,$$

and

$$\lim_{n\to\infty} \inf \left\{ (1+d_{n+1}) \int_{t_n}^{t_{n+1}} e^{\int_s^{t_{n+1}} a(\kappa)d\kappa} c(s)ds \right\} \geq 1.$$

This contradiction to (2.12) implies that the IDEPCAG of mixed type (1.1a)-(1.1b) admits only oscillatory solutions. □

Remark 2.3. *Note that condition (2.11) or (2.12) represents a classical hypothesis utilized for verifying the existence of oscillatory solutions in the DEPCA and in the DEPCAG. This particular condition has been documented in previous studies, such as references [1], [4], [5], [31], and [34]. Moreover, it is worth mentioning that the first condition of Theorem 2.4 can yield the conclusion regarding the presence of oscillatory solutions, as demonstrated in [3, Theorem 7]. This result specifically considers the DEPCA of mixed type* (1.2)*, incorporating both impulsive effects and classical piecewise constant delays.*

In a similar manner to Theorem 2.4, we can obtain

Theorem 2.5. *Assume that* $1+d_{n+1} > 0$*, for* $n \geq i(\tau)$*. If either of the conditions holds true:*

i)

$$\lim_{n\to\infty} \inf \int_{t_n}^{t_{n+1}} e^{\int_s^{t_n} a(\kappa)d\kappa} b(s)ds > -1,$$

$$\lim_{n\to\infty} \sup \left\{ (1+d_{n+1}) \int_{t_n}^{t_{n+1}} e^{\int_s^{t_{n+1}} a(\kappa)d\kappa} c(s)ds \right\} < 1,$$

ii)

$$\lim_{n\to\infty} \sup \int_{t_n}^{t_{n+1}} e^{\int_s^{t_n} a(\kappa)d\kappa} b(s)ds < -1,$$

$$\lim_{n\to\infty} \inf \left\{ (1+d_{n+1}) \int_{t_n}^{t_{n+1}} e^{\int_s^{t_{n+1}} a(\kappa)d\kappa} c(s)ds \right\} > 1,$$

then the sequence $\{x_n\}_{n\geq i(\tau)}$ *of the difference equation (2.2) is non-oscillatory.*

In the following, we present some oscillation and non-oscillation results for the IDEPCAG of mixed type with constant coefficients, which are derived from the previous theorem.

Let us consider the IDEPCAG of mixed type (1.1a)-(1.1b) with constant coefficients:

$$\begin{cases} y'(t) = ay(t) + by(\gamma_1(t)) + cy(\gamma_2(t)), \quad y(\tau) = y_0, \quad t \neq t_k, & (2.13a) \\ \Delta y|_{t=t_k} = d_k y(t_k^-), \quad k \in \mathbb{Z}, & (2.13b) \end{cases}$$

where $a \neq 0$, b, c and d_k are real constants.

Analogously to Theorem 2.4, we present the following theorem for the IDEPCAG of mixed type (2.13a)-(2.13b).

Corollary 2.3. *Suppose that* $1 + d_{n+1} > 0$*, for* $n \geq i(\tau)$*. If* $a \neq 0$ *each one of the conditions*

i)

$$b < -\lim_{n\to\infty} \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}, \quad c < \lim_{n\to\infty} \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1}, \tag{2.14}$$

ii)

$$b > -\lim_{n\to\infty} \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}, \quad c > \lim_{n\to\infty} \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1}, \tag{2.15}$$

implies that every solution of the IDEPCAG of mixed type (2.13a)-(2.13b) *is oscillatory.*

Remark 2.4. *The first condition of Corollary 2.3 can indeed lead to a conclusion regarding the existence of oscillatory solutions, as demonstrated in [3, Theorem 8], and in [5, Corollary 2.8].*

The following corollary shows that either (2.14) or (2.15) is the "best possible" (sharp) condition.

Corollary 2.4. *Suppose that* $1 + d_{n+1} > 0$*, for* $n \geq i(\tau)$*. If* $a \neq 0$ *any one of the following conditions holds true,*

i)

$$b > -\lim_{n\to\infty} \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}, \quad c < \lim_{n\to\infty} \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1}, \tag{2.16}$$

ii)

$$b < -\lim_{n\to\infty} \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}, \quad c > \lim_{n\to\infty} \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1}, \tag{2.17}$$

then the IDEPCAG of mixed type (2.13a)-(2.13b) *has no oscillatory solutions.*

Proof. Condition (2.16) or (2.17) implies that for all $n \geq i(\tau)$, we have $\frac{(1+d_{n+1})\hat{\lambda}(\vartheta_n)}{d_{n+1}\cdot(\hat{\beta}(\vartheta_n)-1)+\hat{\beta}(\vartheta_n)} > 0$. Then, from (2.6), it follows that the solution $y(t)$ of the IDEPCAG of mixed type (2.13a)-(2.13b) is always of one sign on $[\tau, \infty)$. □

The following results are particular cases of Corollaries 2.3 and 2.4.

Corollary 2.5. *Suppose that* $1 + d_{n+1} > 0$*, for* $n \geq i(\tau)$ *and let* $a \neq 0$, b *be real constants. If*

$$b < -\lim_{n\to\infty} \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1},$$

holds, then the delayed impulsive differential equation

$$\begin{cases} y'(t) = ay(t) + by(\gamma_1(t)), \quad y(\tau) = y_0, \quad t \neq t_k, & (2.18a)\\ \Delta y|_{t=t_k} = d_k y(t_k^-), \quad k \in \mathbb{Z}, & (2.18b) \end{cases}$$

has oscillatory solutions only.

Corollary 2.6. *Assume that*

$$1 + d_{n+1} > 0 \quad \textit{and} \quad b > -\lim_{n\to\infty} \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1},$$

for $n \geq i(\tau)$. Then the delayed impulsive differential equation (2.18a)-(2.18b) *has no oscillatory solutions.*

Remark 2.5. *Corollary 2.5 provides an extension of Corollary 2.1 in the work of Aftabizadeh and Wiener [1], specifically addressing the scenario where there is a change of sign for a and $b = -p$, while $\gamma_1(t) = [t]$. On the other hand, Corollary 2.6 extends Theorem 2.3 from the same reference [1], utilizing a similar argument deviation.*

Corollary 2.7. *If $a \neq 0$, c are real constants and the conditions*

$$1 + d_{n+1} > 0 \quad \textit{and} \quad c > \lim_{n\to\infty} \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1},$$

hold for $n \geq i(\tau)$, then the advanced impulsive differential equation

$$\begin{cases} y'(t) = ay(t) + cy(\gamma_2(t)), \quad y(\tau) = y_0, \quad t \neq t_k, & (2.19a)\\ \Delta y|_{t=t_k} = d_k y(t_k^-), \quad k \in \mathbb{Z}, & (2.19b) \end{cases}$$

has only oscillatory solutions.

Corollary 2.8. *Assume that*

$$1 + d_{n+1} > 0 \quad \textit{and} \quad c < \lim_{n\to\infty} \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1},$$

for $n \geq i(\tau)$. Then the advanced impulsive differential equation (2.19a)-(2.19b) *has no oscillatory solutions.*

Remark 2.6. *Corollary 2.7 serves as an extension of Corollary 2.4 in the work of Aftabizadeh and Wiener [1]. In this extension, the variables a and b are replaced with $-a$ and $-q$, respectively, while the advanced argument is given as $\gamma_2(t) = [t+1]$. Similarly, Corollary 2.8 extends Theorem 2.7 from the same reference [1], incorporating the same modifications as mentioned above.*

3. Global asymptotic stability

Theorem 3.1. *Suppose that $b(t)$ and $c(t)$ are locally integrable on $[\tau, \infty)$.*

i) *The zero solution of the IDEPCAG of mixed type* (1.1a)-(1.1b) *is stable, if*

$$\left| \frac{(1 + d_{n+1})\lambda(t_{n+1}, t_n)}{d_{n+1} \cdot (\beta(t_{n+1}, t_n) - 1) + \beta(t_{n+1}, t_n)} \right| \leq 1 \tag{3.1}$$

for all $n > i(\tau)$.

ii) *The zero solution of the IDEPCAG of mixed type* (1.1a)-(1.1b) *is globally asymptotically stable, if*

$$\left| \frac{(1 + d_{n+1})\lambda(t_{n+1}, t_n)}{d_{n+1} \cdot (\beta(t_{n+1}, t_n) - 1) + \beta(t_{n+1}, t_n)} \right| \leq \ell < 1 \tag{3.2}$$

for all $n > i(\tau)$.

Proof. Since $t \in [t_{i(t)}, t_{i(t)+1})$ and $\lambda(t, t_{i(t)}) + \frac{\lambda(t_{i(t)+1}, t_{i(t)})[1-\beta(t, t_{i(t)})]}{\beta(t_{i(t)+1}, t_{i(t)})}$ is continuous, the function $\lambda(t, t_{i(t)}) + \frac{\lambda(t_{i(t)+1}, t_{i(t)})[1-\beta(t, t_{i(t)})]}{\beta(t_{i(t)+1}, t_{i(t)})}$ is bounded for all t. The proof can be easily derived from (2.1). □

Remark 3.1. *The global asymptotic stability result in [3] exclusively addresses differential equations with constant coefficients. Consequently, their result in Theorem 9 cannot be directly applied to Theorem 3.1, as the latter deals with variable coefficients within the framework of piecewise constant delay.*

The following theorem provides necessary and sufficient conditions for the global asymptotic stability of the zero solution of the IDEPCAG of mixed type (2.13a)-(2.13b) with constant coefficients.

Theorem 3.2. *Suppose that $a \neq 0$, b, c are real constants and $\vartheta_n = t_{n+1} - t_n$, $n \in \mathbb{Z}$.*

i) *The zero solution of the IDEPCAG of mixed type* (2.13a)-(2.13b) *with constant coefficients is stable, if*

$$(1 + d_{n+1})(a + b + c)\left\{(1 + d_{n+1})\left(c - b - \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}\right) - \frac{a}{e^{a\vartheta_n} - 1}\right\} \geq 0, \tag{3.3}$$

for all $n > i(\tau)$.

ii) *The zero solution of the IDEPCAG of mixed type* (2.13a)-(2.13b) *with constant coefficients is globally asymptotically stable, if*

$$(1 + d_{n+1})(a + b + c)\left\{(1 + d_{n+1})\left(c - b - \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}\right) - \frac{a}{e^{a\vartheta_n} - 1}\right\} > 0, \tag{3.4}$$

for all $n > i(\tau)$.

Proof. i) The condition (3.1) can be written

$$-1 \leq \frac{(1 + d_{n+1})\left(e^{a\vartheta_n} + \frac{b}{a}\left(e^{a\vartheta_n} - 1\right)\right)}{1 - (1 + d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right)} \leq 1.$$

If $1 - (1 + d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right) > 0$, then

$$-1 + (1 + d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right) \leq (1 + d_{n+1})\left(e^{a\vartheta_n} + \frac{b}{a}\left(e^{a\vartheta_n} - 1\right)\right),$$

$$(1 + d_{n+1})\left(e^{a\vartheta_n} + \frac{b}{a}\left(e^{a\vartheta_n} - 1\right)\right) \leq 1 - (1 + d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right),$$

and

$$\begin{aligned} &(1 + d_{n+1})\left(e^{a\vartheta_n} + \frac{b}{a}\left(e^{a\vartheta_n} - 1\right) + \frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right) \leq 1, \\ &(1 + d_{n+1})\left(-e^{a\vartheta_n} - \frac{b}{a}\left(e^{a\vartheta_n} - 1\right) + \frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right) \leq 1. \end{aligned} \tag{3.5}$$

Since (3.5) implies $(1 + d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right) \leq 1$, we only need to analyze inequality (3.5). From (3.5), we have

$$(1 + d_{n+1})\left(e^{a\vartheta_n} + \frac{b}{a}\left(e^{a\vartheta_n} - 1\right) + \frac{c}{a}\left(e^{a\vartheta_n} - 1\right)\right) \leq 1,$$

that is,

$$(1 + d_{n+1})(a + b + c) \leq 0, \tag{3.6}$$

and

$$(1+d_{n+1})\left\{-e^{a\vartheta_n}-\frac{b}{a}\left(e^{a\vartheta_n}-1\right)+\frac{c}{a}\left(e^{a\vartheta_n}-1\right)\right\}\le 1,$$

which is equal to

$$(1+d_{n+1})\left(c-b-\frac{ae^{a\vartheta_n}}{e^{a\vartheta_n}-1}\right)-\frac{a}{e^{a\vartheta_n}-1}\le 0. \tag{3.7}$$

If $1-(1+d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n}-1\right)\right)<0$, then

$$-1+(1+d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n}-1\right)\right)\ge(1+d_{n+1})\left(e^{a\vartheta_n}+\frac{b}{a}\left(e^{a\vartheta_n}-1\right)\right),$$

and

$$(1+d_{n+1})\left(e^{a\vartheta_n}+\frac{b}{a}\left(e^{a\vartheta_n}-1\right)\right)\ge 1-(1+d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n}-1\right)\right).$$

These inequalities imply $(1+d_{n+1})\left(\frac{c}{a}\left(e^{a\vartheta_n}-1\right)\right)\ge 1$. Therefore, we focus only on the inequalities:

$$(1+d_{n+1})\left(e^{a\vartheta_n}+\frac{b}{a}\left(e^{a\vartheta_n}-1\right)+\frac{c}{a}\left(e^{a\vartheta_n}-1\right)\right)\ge 1,$$

and

$$(1+d_{n+1})\left(-e^{a\vartheta_n}-\frac{b}{a}\left(e^{a\vartheta_n}-1\right)+\frac{c}{a}\left(e^{a\vartheta_n}-1\right)\right)\ge 1.$$

These inequalities lead to opposite inequalities to those in (3.6) and (3.7), which proves the theorem.

ii) The proof follows a similar technique as in part i). □

Remark 3.2. *Theorem 3.2 serves as a generalization of several previously established results. It extends Theorem 1.28 from the work of S. M. Shah and J. Wiener [31], incorporating classical piecewise constant delays $\gamma_1(t)=[t]$ and $\gamma_2(t)=[t+1]$.*

Furthermore, Theorem 3.2 extends the findings presented in Theorem 10 by H. Bereketoglu et al. [3] by incorporating classical piecewise constant delays. Notably, Theorem 3.2 in our context does not impose the restriction of the condition $1-d>0$, where $d_k\equiv d$. Consequently, the result presented in Theorem 10 of [3] cannot be directly applied to our result in Theorem 3.2.

Moreover, Theorem 3.2 also generalizes Theorem 3.2 presented in K.-S. Chiu and J.-C. Jeng [4], which focuses on the DEPCAG of mixed type without impulsive effects.

By applying Theorem 3.2, several particular cases of results can be derived.

Corollary 3.1. *Suppose that $a\neq 0$, b are real constants. The zero solution of the delayed impulsive differential equation* (2.18a)-(2.18b)

$$\begin{cases} y'(t)=ay(t)+by(\gamma_1(t)), \quad y(\tau)=y_0, \quad t\neq t_k,\\ \Delta y|_{t=t_k}=d_k y(t_k^-), \quad k\in\mathbb{Z}, \end{cases}$$

is globally asymptotically stable, if

$$-a\left(\frac{e^{a\vartheta_n}+1+d_{n+1}e^{a\vartheta_n}}{e^{a\vartheta_n}-1}\right)\le(1+d_{n+1})\,b\le -a\left(1+d_{n+1}\frac{e^{a\vartheta_n}}{e^{a\vartheta_n}-1}\right), \tag{3.9}$$

for all $n>i(\tau)$, where $\vartheta_n=t_{n+1}-t_n$, $n\in\mathbb{Z}$.

Corollary 3.1 can be seen as an extension of Corollary of Theorem 2.5 in [31, pp. 678], where $\gamma_1(t)=[t]$ is considered.

Corollary 3.2. *Suppose that $a \neq 0$, c are real constants. The zero solution of the advanced impulsive differential equation* (2.19a)-(2.19b)

$$\begin{cases} y'(t) = ay(t) + cy(\gamma_2(t)), \quad y(\tau) = y_0, \quad t \neq t_k, \\ \Delta y|_{t=t_k} = d_k y(t_k^-), \quad k \in \mathbb{Z}, \end{cases}$$

is globally asymptotically stable, if

$$(1 + d_{n+1})\, c \leq a \frac{(1 + d_{n+1})\, e^{a\vartheta_n} + 1}{e^{a\vartheta_n} - 1} \quad \text{and} \quad a + \frac{a d_{n+1} e^{a\vartheta_n}}{e^{a\vartheta_n} - 1} + (1 + d_{n+1})\, c \leq 0, \tag{3.11}$$

for all $n > i(\tau)$, where $\vartheta_n = t_{n+1} - t_n$, $n \in \mathbb{Z}$.

Based on Theorem 3.2 and Corollary 2.3, we can conclude that:

Corollary 3.3. *Suppose that $1 + d_{n+1} > 0$, for $n \geq i(\tau)$. Then, every oscillatory solution of the IDEPCAG of mixed type* (2.13a)-(2.13b) *with constant coefficients tends to zero if at least one of the following hypotheses is fulfilled:*

i)

$$\begin{aligned} &a + b + c > 0, \quad c > \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1} \\ &c\,(1 + d_{n+1}) - \frac{a}{e^{a\vartheta_n} - 1} \left(e^{a\vartheta_n}\,(1 + d_{n+1}) + 1\right) > b\,(1 + d_{n+1}) > -\frac{a e^{a\vartheta_n}}{e^{a\vartheta_n} - 1}, \end{aligned} \tag{3.12}$$

ii)

$$\begin{aligned} &a + b + c < 0, \quad c < \frac{1}{1 + d_{n+1}} \frac{a}{e^{a\vartheta_n} - 1}, \\ &c\,(1 + d_{n+1}) - \frac{a}{e^{a\vartheta_n} - 1} \left(e^{a\vartheta_n}\,(1 + d_{n+1}) + 1\right) < b\,(1 + d_{n+1}) < -\frac{a e^{a\vartheta_n}}{e^{a\vartheta_n} - 1}. \end{aligned} \tag{3.13}$$

Remark 3.3. *Corollary 3.3 extends the findings presented in Theorem 10 by H. Bereketoglu et al. [3] by incorporating classical piecewise constant delays.*

4. Illustrative examples and simulations

In this section, we will present relevant examples that serve to demonstrate the practical applicability and usefulness of our results.

Example 4.1. Let us consider the IDEPCAG of mixed type

$$\begin{cases} y'(t) = 1.3\, y(t) - 2.2 y(\gamma_1(t)) - 3 \ln 3 y(\gamma_2(t)), \quad y(0) = 4, \quad t \neq t_k, & \text{(4.1a)} \\ \Delta y|_{t=t_k} = 1.5 y(t_k^-), \quad k \in \mathbb{Z}, & \text{(4.1b)} \end{cases}$$

where $t_n = 2n - 2$, for all $n \in \mathbb{N}$. The IDEPCAG of mixed type (4.1a)-(4.1b) is a particular case of the IDEPCAG of mixed type (2.13a)-(2.13b) with $a = 1.3$, $b = -2.2$, $c = -3\ln 3$ and $d_k = 1.5$.
It is evident that

$$d_{n+1} \cdot (\hat{\beta}(\vartheta_n) - 1) + \hat{\beta}(\vartheta_n) = 1 + 2.5 \frac{3 \ln 3}{1.3} (e^{2.6} - 1) \approx 79.99701 \neq 0 \quad \text{and} \quad \vartheta_n = t_{n+1} - t_n = 2,$$

for all $n \in \mathbb{N}$.
Through simple computation, we have

$$a + b + c \approx -4.1958,$$

$$c\left(1+d_{n+1}\right)-\frac{a}{e^{a\vartheta_n}-1}\left(e^{a\vartheta_n}\left(1+d_{n+1}\right)+1\right)$$
$$=-3\ln 3\cdot 2.5-\frac{1.3}{e^{a\vartheta_n}-1}\left(2.5e^{2.6}+1\right)\approx -11.8546,$$

$$b\left(1+d_{n+1}\right)=-5.5,$$

$$-\frac{ae^{a\vartheta_n}}{e^{a\vartheta_n}-1}=-\frac{1.3e^{2.6}}{e^{2.6}-1}\approx -1.4043$$

and

$$\frac{1}{1+d_{n+1}}\frac{a}{e^{a\vartheta_n}-1}=\frac{1}{2.5}\frac{1.3}{e^{2.6}-1}\approx 0.0417.$$

In this case, the second hypothesis (3.13) of Corollary 3.3 is satisfied, which implies that every solution of the IDEPCAG of mixed type (4.1a)-(4.1b) is oscillatory. Moreover, the hypothesis ii) of Theorem 3.2 is also satisfied, and therefore, we can conclude that every solution of the IDEPCAG of mixed type (4.1a)-(4.1b) approaches zero as $t\to\infty$ by oscillating.

Figures 1 and 2 illustrate the simulation results showcasing the global asymptotic stability of the oscillatory solution for the IDEPCAG of mixed type (4.1a)-(4.1b) with and without impulses.

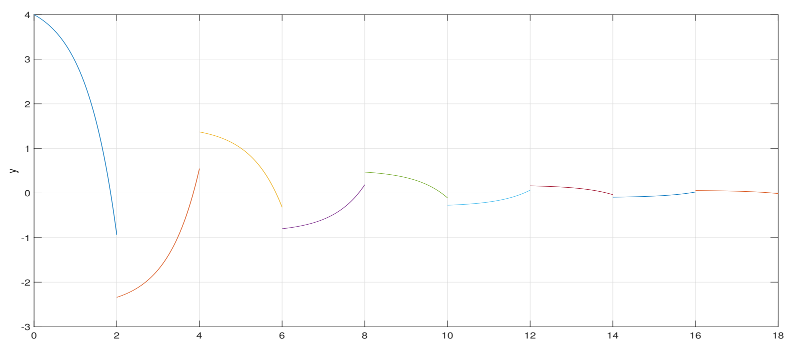

Fig. 1. The global asymptotic stability of the oscillatory solution for the IDEPCAG of mixed type (4.1a)-(4.1b).

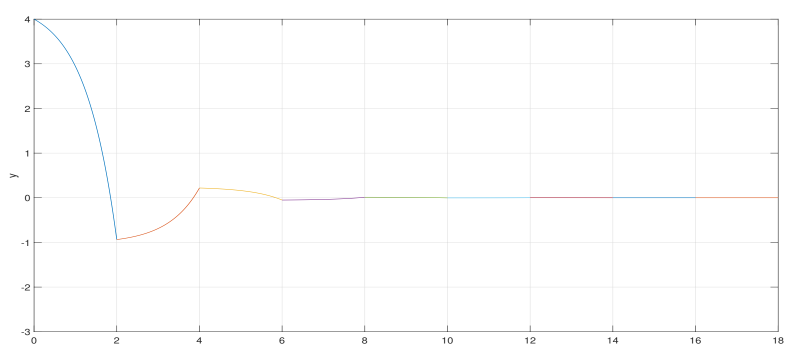

Fig. 2. The global asymptotic stability of the oscillatory solution for the DEPCAG of mixed type (4.1a).

Example 4.2. Let us consider the IDEPCA of mixed type

$$\begin{cases} y'(t) = -\dfrac{2}{e-1}y(t) - \dfrac{\sqrt{3}}{5}y([t]) + \dfrac{6}{5}y([t+1]), \quad y(0)=1, \quad t \neq k, & (4.2a) \\ \Delta y|_{t=k} = 0.25y(k^-), \quad k \in \mathbb{Z}, & (4.2b) \end{cases}$$

where $[\cdot]$ signifies the greatest integer function.
It is evident that $\gamma_1(t) = [t]$, $\gamma_2(t) = [t+1]$, $t_n = n$, $\vartheta_n = t_{n+1} - t_n = 1$, for all $n \in \mathbb{N}$, and

$$d_{n+1} \cdot (\hat{\beta}(\vartheta_n) - 1) + \hat{\beta}(\vartheta_n) = 1 + \frac{3\,(e-1)}{4}(e^{-\frac{2}{e-1}} - 1) \approx 0.1136 \neq 0.$$

In this case,

$$1 + d_{n+1} = 1.25 > 0, \quad b = \frac{\sqrt{3}}{5} \approx -0.3464 > -\frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1} = \frac{2}{e-1}\frac{e^{-\frac{2}{e-1}}}{e^{-\frac{2}{e-1}} - 1} \approx -0.5284,$$

and

$$c = 1.2 < \frac{1}{1+d_{n+1}}\frac{a}{e^{a\vartheta_n} - 1} = -\frac{8}{5\,(e-1)\left(e^{-\frac{2}{e-1}} - 1\right)} \approx 1.3539,$$

for $n \in \mathbb{N}$.
By verifying the hypotheses (2.16) of Corollary 2.4, we can conclude that every solution of the IDEPCA of mixed type (4.2a)-(4.2b) is nonoscillatory.

The second hypothesis of Theorem 3.2 is satisfied because

$$(1 + d_{n+1})(a + b + c) \approx -0.3879 < 0,$$

and

$$(1 + d_{n+1})\left(c - b - \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}\right) - \frac{a}{e^{a\vartheta_n} - 1} \approx -0.4199 < 0.$$

Then,

$$(1 + d_{n+1})(a + b + c)\left((1 + d_{n+1})\left(c - b - \frac{ae^{a\vartheta_n}}{e^{a\vartheta_n} - 1}\right) - \frac{a}{e^{a\vartheta_n} - 1}\right) > 0$$

for all $n \in \mathbb{N}$. Therefore, by Theorem 3.2, the zero solution of the IDEPCA of mixed type (4.2a)-(4.2b) is globally asymptotically stable.

Figures 3 and 4 illustrate the simulation results showcasing the global asymptotic stability of the non-oscillatory solution for the IDEPCAG of mixed type (4.2a)-(4.2b) with and without impulses.

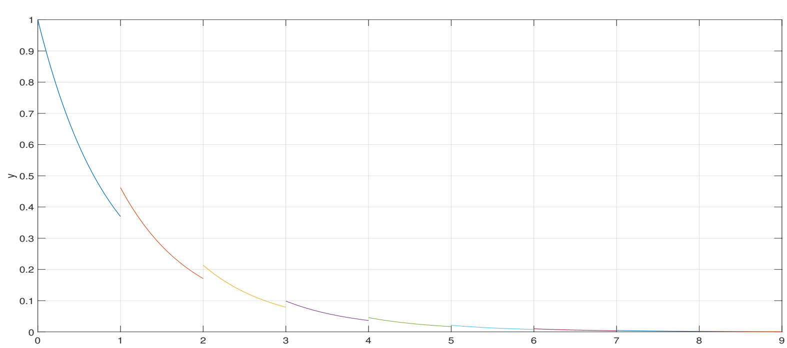

Fig. 3. The global asymptotic stability of the non-oscillatory solution for the IDEPCA of mixed type (4.2a)-(4.2b).

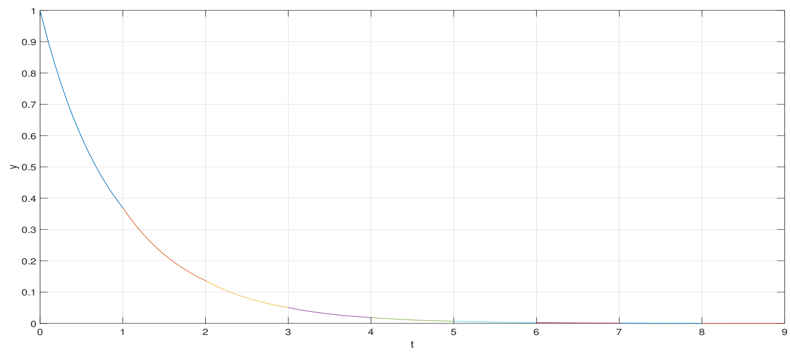

Fig. 4. The global asymptotic stability of the non-oscillatory solution for the DEPCA of mixed type (4.2a).

Example 4.3. Let us consider the IDEPCAG of mixed type

$$\begin{cases} y'(t) = \dfrac{\sin t}{8} y(\gamma_1(t)) + \left(\dfrac{\sqrt{17}}{3} \cos t + \dfrac{1}{2e^2 - 1} \right) y(\gamma_2(t)), \; y(0) = 10, \quad t \neq t_k, & \text{(4.3a)} \\ \Delta y|_{t=t_k} = -1.5 y(t_k^-), \quad k \in \mathbb{Z}, & \text{(4.3b)} \end{cases}$$

where $t_n = 2\pi n$, for all $n \in \mathbb{N} \cup \{0\}$. The IDEPCAG of mixed type (4.3a)-(4.3b) is a particular case of the IDEPCAG of mixed type (2.8a)-(2.8b). It is evident that $b(t) = \frac{\sin t}{8}$ and $c(t) = \frac{\sqrt{17}}{3} \cos t + \frac{1}{2e^2-1}$ in (2.9), we have

$$\begin{aligned} \frac{(1 + d_{n+1})(1 + \eta(t_{n+1}))}{1 - (1 + d_{n+1})\mu(t_{n+1})} &= \frac{-0.5 \left(1 + \int_{2\pi n}^{2\pi(n+1)} \frac{\sin u}{8} du \right)}{1 + 0.5 \int_{2\pi n}^{2\pi(n+1)} \left(\frac{\sqrt{17}}{3} \cos u + \frac{1}{2e^2-1} \right) du} \\ &= -\frac{2e^2 - 1}{2\left(2e^2 - 1\right) + 2\pi} \approx -0.40716. \end{aligned}$$

Then, $\left\{ \frac{(1+d_{n+1})(1+\eta(t_{n+1}))}{1-(1+d_{n+1})\mu(t_{n+1})} \right\}_{n \geq i(0)}$ is not eventually positive. All assumptions of Theorem 2.3 are satisfied, then every solution of the IDEPCAG of mixed type (4.3a)-(4.3b) is oscillatory. Moreover, the condition (3.2) is fulfilled, so, due to Theorem 3.1, the zero solution of the IDEPCAG of mixed type (4.3a)-(4.3b) is globally asymptotically stable.
However, it is important to note that in our case,

$$\frac{1 + \eta(t_{n+1})}{1 - \mu(t_{n+1})} = \frac{1 + \int_{2\pi n}^{2\pi(n+1)} \frac{\sin u}{8} du}{1 - \int_{2\pi n}^{2\pi(n+1)} \left(\frac{\sqrt{17}}{3} \cos u + \frac{1}{2e^2-1} \right) du} = \frac{2e^2 - 1}{2e^2 - 1 - 2\pi} \approx 1.8383 > 1,$$

the condition (3.2) from the paper [4] is not satisfied. As a result, according to Theorem 3.1 in [4], the zero solution of the DEPCAG of mixed type represented by equation (4.3a) is not globally asymptotically stable.

Figure 5 illustrates the simulation results demonstrating the global asymptotic stability of the oscillatory solution for the IDEPCAG of mixed type (4.3a)-(4.3b). On the other hand, Figure 6 depicts the simulation results demonstrating the presence of an oscillatory solution for the DEPCAG of mixed type without impulses. It is worth noting that in this particular case, the DEPCAG of mixed type (4.3a) is not globally asymptotically stable.

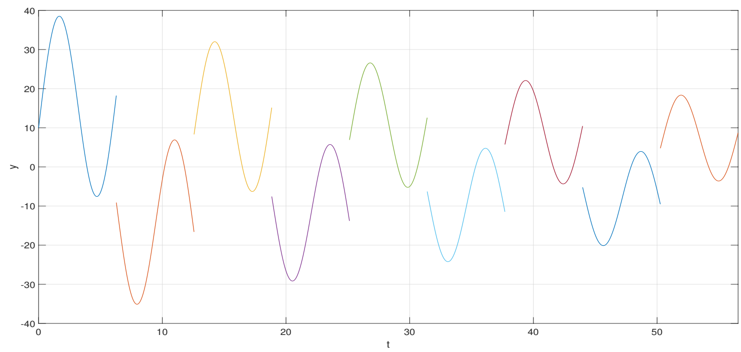

Fig. 5. The oscillatory solution of the IDEPCAG of mixed type (4.3a)-(4.3b) is globally asymptotically stable.

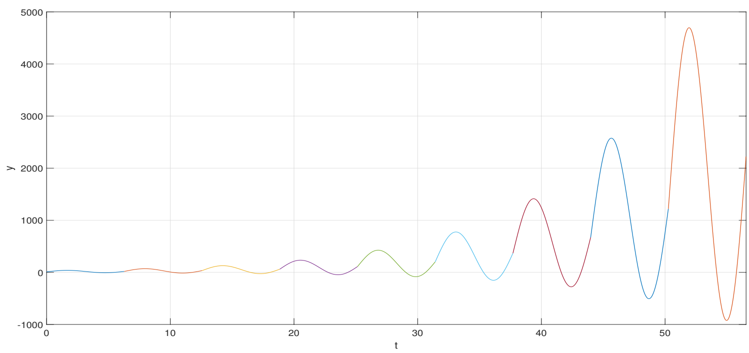

Fig. 6. The oscillatory solution of the DEPCAG of mixed type (4.3a) is not globally asymptotically stable.

Indeed, the presence or absence of impulsive effects plays a significant role in the stability of the system. As observed from the simulation results, the inclusion of impulsive effects in the IDEPCAG of mixed type leads to global asymptotic stability, while their absence in the DEPCAG of mixed type results in the lack of global asymptotic stability. This highlights the importance and influence of impulsive effects on the stability behavior of the system.

5. Conclusions and perspectives

The article explores the existence of oscillatory and non-oscillatory solutions in a scalar impulsive differential equation with piecewise constant generalized mixed arguments. We establish theorems confirming the existence and uniqueness of oscillatory and non-oscillatory solutions for the IDEPCAG of mixed type. Employing techniques based on differential inequalities, we derive several sufficient conditions ensuring global asymptotic stability under appropriate assumptions. Notably, our results extend classical findings from [3,4,31]. Furthermore, we provide numerous numerical examples and simulations to showcase the applicability of our findings.

In the following, a few open theses can be considered further in order.

1. Discussions on the existence of periodic solutions for the IDEPCAG of mixed type.
2. In-depth exploration of the oscillatory and asymptotic behavior of second-order DEPCAG of mixed type.

3. Comparative discussions on the asymptotic and oscillatory properties of first-order neutral differential equations with arguments of mixed type, including a comparison with the properties of the same equations with linearly transformed arguments.

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