CARPATHIAN J. MATH.
Volume **41** (2025), No. 1,
Pages 89 - 93

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2025.01.08

A quaternionic product of simple ratios

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ABSTRACT. This note introduces a quaternionic inspired product of triples of collinear points and consequently a product of their simple ratio. Some technical conditions are necessary for the existence of these products and some general examples (squares, the arithmetic and the geometric means) are discussed. Moreover, some concrete computations are performed and then some periodic ratios are obtained.

INTRODUCTION

Together with the real algebra $\mathbb{R}$ and the complex algebra $\mathbb{C}$ the quaternionic algebra $\mathbb{H}$ is a well-known setting of the modern mathematics. From the very beginning this algebraic structure has a geometric aim being an useful tool in modeling the rotations of the three-dimensional Euclidean space.

The purpose of the present work is to use the product of $\mathbb{H}$ into another framework namely the projective line. Since its invariant, namely the cross-ratio ([1, p. 29]), is based on a combination of simple ratio we introduce a quaternionic product on a special class of simple ratio. The choice of the identification of a triple of special collinear points with a quaternion is based on some previous papers of the first author. We point out also that in order to perform a suitable quaternionic product we introduce a technical condition in our definition 1.3.

This new product is discussed especially from the point of view of examples. In addition to squares we study some concrete examples by giving also numerical details.

1. THE QUATERNIONIC PRODUCT OF SIMPLE RATIOS

Fix the set $\mathbb{R}^2_< := \{P(a,b) \in \mathbb{R}^2; a < b\}$. The well-known notion of *simple ratio of three collinear points* is the function parameterized by $P \in \mathbb{R}^2_<$:

$$\rho_P : \mathbb{R} \setminus \{b\} \to \mathbb{R} \setminus \{-1\}, \quad \rho_P(c) := \frac{c-a}{b-c}, \quad \lim_{c\to\infty} \rho_P(c) = -1. \tag{1.1}$$

Our choice is motivated by the vectorial relation inspired by the *baricenter coordinate* of c with respect to the ordered pair (a,b):

$$\overrightarrow{ac} = \rho_P(c)\overrightarrow{cb} \tag{1.2}$$

giving the value 1 for the arithmetic mean $c = m_{arith}(a,b) := \frac{a+b}{2}$. With the natural projections:

$$\pi_1, \pi_2 : \mathbb{R}^2_< \to \mathbb{R}, \quad \pi_1(P) := a, \quad \pi_2(P) := b \tag{1.3}$$

the definition domain of ρ_P can be expressed as $\mathbb{R} \setminus \{\pi_2(P)\}$. Obviously, an affine map $T : x \in \mathbb{R} \to \mathbb{R}$, $T(x) = ux + v$ for $u \neq 0$ preserves the simple ratio:

$$\rho_{T(P)=(T(a),T(b))}(T(c)) = \rho_P(c).$$

Received: 20.02.2024. In revised form: 04.07.2024. Accepted: 18.07.2024
1991 *Mathematics Subject Classification.* 51N10; 51N15; 11R52; 51N20.
Key words and phrases. *triple of collinear points; quaternion; product; projective geometry; simple ratio.*
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Example 1.1 Following the remark 1.1 of the paper [3] *the mediant* of two strictly positive ratios $a := \frac{p}{q} < b := \frac{p'}{q'} < 1$ is the ratio c expressed as:

$$a < c = a \oplus b := \frac{p+p'}{q+q'} < b.$$

Then a straightforward computation yields the simple ratio:

$$\rho_P(c) = \frac{q}{q'} > \frac{p}{p'}.$$

□

Remark 1.2 The map ρ_P of (1.1) can be interpreted as a Möbius (or rational) transformation with the associated matrix:

$$\rho_P := \begin{pmatrix} 1 & -a \\ b & -1 \end{pmatrix}$$

and then we can perform the matrix product of two such matrices:

$$\rho_P \cdot \rho_{\tilde{P}} = \begin{pmatrix} 1 & -a \\ b & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\tilde{a} \\ \tilde{b} & -1 \end{pmatrix} = \begin{pmatrix} 1 - a\tilde{b} & a - \tilde{a} \\ b - \tilde{b} & 1 - \tilde{a}b \end{pmatrix}.$$

Thinking in a projective manner the last matrix is of ρ-type above if and only if the following two conditions hold:

$$1 \neq 1 - \tilde{a}b = -1 + a\tilde{b}$$

which means the pair of conditions:

$$a\tilde{b} + \tilde{a}b = 2, \quad a\tilde{b} \neq 0.$$

In the 4-dimensional linear space $\mathbb{R}^4$ of the variables $(a, b, \tilde{a}, \tilde{b})$ the first equation above represents a quadric. □

The motivation of this work is to introduce another product involving also the point c and hence the starting point of this paper is the identification of the given pair $(P, c) \in \mathbb{R}^2_< \times \mathbb{R} \setminus \{b\}$ (which we will call *the set of special collinear points*) with the quaternion:

$$q(P, c) := \bar{k} + a\bar{i} + b\bar{j} + c = (c, a, b, 1) \in \mathbb{R}^4. \tag{1.4}$$

The quaternion $q(P, c)$ is pure imaginary if and only if the $c = 0$ and then $b \neq 0$. We point out that although there are several ways to associate a quaternion to a given triple of points we choose the expression (1.4) according to our two previous studies, namely (in the chronological order) [4] and [2].

From the real algebra structure of the quaternions ([5, p. 89]) it follows a product of two triples of special collinear points:

$$(P_1, c_1) \odot_q (P_2, c_2) := q^{-1}(q(P_1, c_1) \cdot q(P_2, c_2)). \tag{1.5}$$

For our special collinear points $(a_i, b_i, c_i), i = 1, 2$ we derive immediately:

$$q(P_1, c_1) \cdot q(P_2, c_2) = (a_1 b_2 - a_2 b_1 + c_1 + c_2)\bar{k} + (b_1 - b_2 + a_1 c_2 + a_2 c_1)\bar{i} +$$

$$+ (a_2 - a_1 + b_1 c_2 + b_2 c_1)\bar{j} + (c_1 c_2 - 1 - a_1 a_2 - b_1 b_2) =: D\bar{k} + A\bar{i} + B\bar{j} + C \tag{1.6}$$

and due to the expression of the coefficient of $\bar{k}$ we need a more definition:

Definition 1.3 The given pair of special collinear points is called *q-distinguished* if the following two conditions hold:

$$\begin{cases} A := b_1 - b_2 + a_1 c_2 + a_2 c_1 < B := a_2 - a_1 + b_1 c_2 + b_2 c_1 \\ D := a_1 b_2 - a_2 b_1 + c_1 + c_2 > 0. \end{cases} \tag{1.7}$$

Example 1.4 For a triple of special collinear points (P, c) we have the square:

$$q(P,c) \cdot q(P,c) = 2c\bar{k} + 2ac\bar{i} + 2bc\bar{j} + (c^2 - 1 - a^2 - b^2). \tag{1.8}$$

If $c > 0$ then both conditions from (1.7) are satisfied and then the pair $(P, c), (P, c)$ is q-distinguished; also $q(P, c) \in \mathbb{R}^4$ is a purely imaginary quaternion only for $c = \sqrt{a^2 + b^2 + 1} > 1$. $\square$

Let now $(\mathbb{R}^2_< \times \mathbb{R} \setminus \{\pi_2(\cdot)\})^2_q$ be the set of q-distinguished pair of triples of special collinear points. Working in a projective manner it follows a *quaternionic product* of their simple ratio:

$$\rho_{P_1}(c_1) \odot_q \rho_{P_2}(c_2) := \rho_P(c), \quad P := \left(\frac{A}{D}, \frac{B}{D}\right), \quad c := \frac{C}{D} \tag{1.9}$$

and the required inequality $\frac{A}{D} < \frac{B}{D}$ motivates the second condition from (1.7). An important tool of the quaternionic theory is that of *conjugate*, which for our quaternion (1.4) means:

$$\overline{q(P,c)} := -\bar{k} - a\bar{i} - b\bar{j} + c = (c, -a, -b, -1) = -q(P, -c) \tag{1.10}$$

and our projective way of thinking allows the identification: $\overline{q(P,c)} = q(P, -c)$. Let us remark that if the bi-point P is a symmetric one i.e. $P = (-b, b)$ for $b > 0$ and $c \in \mathbb{R}\setminus\{-b, b\}$ then:

$$\rho_P(c) = \frac{c+b}{b-c}, \quad \rho_P(-c) = \frac{-c+b}{b+c} \to \rho_P(c) \cdot \rho_P(-c) = 1. \tag{1.11}$$

2. Concrete examples

In the following we study this new product introduced in (1.9) through two large examples.

Example 2.1 Revisiting the example 1.4 (recall that $c > 0$) we have immediately the square of a simple ratio:

$$(\rho_P(c))^2_{\odot_q} := \frac{c^2 - a^2 - b^2 - 1 - 2ac}{1 + a^2 + b^2 + 2bc - c^2}. \tag{2.1}$$

This ratio is undefined on the points of the two-sheeted hyperboloid $H_2 \subset \mathbb{R}^3$ which is obtained via the substitution $(a, b, c) \to (x, y, z)$:

$$H_2 : x^2 + y^2 + 2yz - z^2 + 1 = 0 \tag{2.2}$$

and having the parametrization:

$$H_2 : x(u,v) = \sinh u \cos v, y(u,v) = \sinh u \sin v - \frac{1}{\sqrt{2}} \cosh u, z(u,v) = \frac{1}{\sqrt{2}} \cosh u, (u,v) \in \mathbb{R}^2.$$

Hence for $(P, c) \in (\mathbb{R}^2_< \times \mathbb{R} \setminus \{\pi_2(\cdot)\}) \setminus H_2$ the ratio (2.1) and its expression suggests as remarkable example the case of right triangle, $\triangle$: $c^2 = a^2 + b^2$, which gives:

$$\begin{cases} \rho(\triangle) = \frac{c-a}{b-c} = \frac{c - c\sin A}{c\cos A - c} = \frac{1-\sin A}{\cos A - 1} = \frac{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)^2}{-2\sin^2\frac{A}{2}} = -\frac{1}{2}\left(\cot\frac{A}{2} - 1\right)^2 < 0, \\ (\rho(\triangle))^2_{\odot_q} = \frac{-2ac-1}{2bc+1} > -1. \end{cases} \tag{2.3}$$

A particular case of the right triangle $\triangle$ is provided by the case when (a, b, c) is a Pythagorean triple and hence we know its parametrization ([3]):

$$a := \beta^2 - \alpha^2, \quad b := 2\alpha\beta, \quad c := \alpha^2 + \beta^2, \quad 0 < \alpha < \beta \in \mathbb{N}^*. \tag{2.4}$$

It follows the ratios:

$$\begin{cases} \rho(Pythagorean) = \frac{2\alpha^2}{-(\beta-\alpha)^2} = -2\left(\frac{\alpha}{\beta-\alpha}\right)^2 < 0, \\ (\rho(Pythagorean))^2_{\odot_q} = -\frac{2(\beta^4-\alpha^4)+1}{4\alpha\beta(\alpha^2+\beta^2)+1}. \end{cases} \tag{2.5}$$

For a concrete example we choose the minimal pair $\alpha = 1 < \beta = 2$ giving *the minimal Pythagorean triple* $(a = 3, b = 4, c = 5)$ with the associated ratios:

$$\rho(minimal) = -2 < (\rho(minimal))^2_{\odot_q} = -\frac{31}{41} = -0.\overline{75609} \in (-1, 0). \tag{2.6}$$

The positive ratio $31/41$ has the following expression as continued fraction and the Egyptian fraction expansion:

$$\frac{31}{41} = [0; 1, 3, 10] = \frac{1}{2} + \frac{1}{4} + \frac{1}{164}. \tag{2.7}$$

□

Example 2.2 In this example we will perform the quaternionic product of two different simple ratio starting from the initial $P(a, b) \in \mathbb{R}^2_<$ with $0 < a$ and considering as points $c_1 = m_{arith}(P)$ and $c_2 = m_{geom}(P) := \sqrt{ab}$. Then:

$$\rho_P(c_1) = 1 > \rho_P(c_2) = \sqrt{\frac{a}{b}} = m_{geom}\left(a, \frac{1}{b}\right) > 0. \tag{2.8}$$

The pair $((P, c_1), (P, c_2))$ is q-distinguished since the expressions A, B, C of the definition 1.3 are:

$$A = a\left(1 + \sqrt{\frac{a}{b}}\right) < B = b\left(1 + \sqrt{\frac{a}{b}}\right), \quad D = 1 + \sqrt{\frac{a}{b}} \in (1, 2) \tag{2.9}$$

and then we can compute the product (1.9):

$$\rho_P(c_1) \odot_q \rho_P(c_2) = \frac{c_1c_2 - 1 - a^2 - b^2 - a(c_1 + c_2)}{b(c_1 + c_2) + 1 + a^2 + b^2 - c_1c_2} = \rho_P(c_2) \odot_q \rho_P(c_1). \tag{2.10}$$

More precisely, the last ratio is:

$$\rho_P(c_1) \odot_q \rho_P(c_2) = \frac{\sqrt{\frac{a}{b}} - 1 - a^2 - b^2 - a(1 + \sqrt{\frac{a}{b}})}{b + \sqrt{ab} + 1 + a^2 + b^2 - \sqrt{\frac{a}{b}}}. \tag{2.11}$$

For a concrete example we choose $0 < a = 1 < b = 4$ for which $c_1 = \frac{5}{2} = a \oplus b > c_2 = 2$ and the quaternionic product of their simple ratio is:

$$\rho_P(c_1) \odot_q \rho_P(c_2) = \frac{-19}{24 - \frac{1}{2}} = -\frac{38}{47} \in (-1, 0). \tag{2.12}$$

We note that the positive ratio $38/47$ is a decimal number of period 46 with:

$$\frac{38}{47} = [0; 1, 4, 4, 2] = \frac{1}{2} + \frac{1}{4} + \frac{1}{18} + \frac{1}{339} + \frac{1}{191196}. \tag{2.13}$$

□

3. Discussions and conclusions

The product of the algebra $\mathbb{H}$ may be used in several geometric frameworks. The present one is that of the line as first step to a product of cross-ratios, in order to enrich the projective geometry. We point out that our main interest was to produce geometric examples with remarkable flavours. Interestingly, the interplay between algebra, geometry and arithmetic still works in our setting.

References

[1] Berger, M., *Geometry revealed. A Jacob's ladder to modern higher geometry*, Berlin: Springer, 2010

[2] Crasmareanu, M., *Quaternionic product of circles and cycles and octonionic product for pairs of circles*, Iran. J. Math. Sci. Inform. **17** (2022), no. 1, 227-237

[3] Crasmareanu, M., *The Farey sum of Pythagorean and Eisenstein triples*, Mathematical Sciences and Applications E-Notes **12** (2024), no. 1, 28-36

[4] Crasmareanu, M. and Popescu, M., *Quaternionic product of equilateral hyperbolas and some extensions*, Mathematics **8** (2020), no. 10, paper 1686

[5] Cushman, R. H. and Bates, L. M., *Global aspects of classical integrable systems*, 2nd edition, Basel: Birkhäuser/Springer, 2015.

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