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On the existence and uniqueness of fixed points in Banach spaces using the Krasnoselskij iterative method

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ABSTRACT. We analyze the global convergence for the Krasnoselskij method. More specifically, we obtain domains of global convergence in which we locate and separate fixed points of a given operator. To do that, we use auxiliary points instead of imposing conditions on the solution, that is generally unknown. Then, we apply our study to obtain fixed point type results. We finish our study by applying the results to Fredholm integral equations.

1. INTRODUCTION

In this work we focus our study on locating a fixed point of an operator defined in a Banach space and separating it from other possible fixed points. Banach contraction principle is a very important tool in the theory of metric spaces. The Banach contraction principle says [7]:

Theorem 1.1. *Let X be a Banach space. Let $T : X \to X$ be a contraction mapping, with Lipschitz constant $k < 1$. Then, T has a unique fixed point $x^* \in X$. Moreover, for every $x_0 \in X$, the method of successive approximations, $x_{n+1} = T(x_n)$, $n \in \mathbb{N}$, converges to x^*.*

Obviously, this result can be applied only if the operator T has a unique fixed point x^* in X. In addition, it is not necessary to separate it from other possible fixed points. Note that, in this case, the method of successive approximations converges globally in X, that is, this method converges starting at any point, $x_0 \in X$ ([1], [8]).

If the considered operator T has more than one fixed point, obviously, the previous result is not applicable. Therefore, we have to restrict to some domain of the space X where there is only one fixed point of T. In this situation we consider the following theorem. Notice that this is a fixed point theorem but restricted to $\mathcal{K}$ that is a non-empty closed subset of a Banach space X.

Theorem 1.2. ([13]) *If $\mathcal{K}$ is a non-empty closed subset of a Banach space X and the operator $T : \mathcal{K} \to \mathcal{K}$ is a contraction, then T has a unique fixed point x^* in $\mathcal{K}$. Moreover, for every $x_0 \in \mathcal{K}$, the method of successive approximations, $x_{n+1} = T(x_n)$, $n \in \mathbb{N}$, converges to x^*.*

In practice, to apply Theorem 1.2 it is necessary to have certain information about a fixed point of the operator, otherwise the location of an non-empty closed subset, $\mathcal{K}$, of the Banach space X, is complicated. Another major difficulty to apply Theorem 1.2 is the condition $T : \mathcal{K} \to \mathcal{K}$, which usually turns out to be quite restrictive.

The main aim of this work is to offer an alternative to Theorem 1.2. Hence, we provide a procedure that allows us to locate a fixed point of T and separate it from other possible fixed points. To do this, we study global, semilocal and local convergence of an iterative

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scheme associated to operator T which has always special interest. From the analysis of global convergence we obtain closed ball where the existence and uniqueness of fixed point for T is guaranteed. We will also get semilocal convergence by assuming conditions on initial approximations belonging to the ball in which the fixed point of T is located and is unique. The local study is based on demanding conditions to a solution, from certain conditions on the operator T, and provides the convergence ball, that shows the accessibility to the solution from the initial approximations belonging to the ball.

In this paper, we consider the Krasnoselskij operator ([8]):

$$T_\alpha = (1-\alpha)I_\Omega + \alpha T, \tag{1.1}$$

where I_Ω is the identity in Ω being Ω a non-empty open convex domain in the Banach space X and $\alpha \in (0,1]$. Moreover, we focus our study on the qualitative properties of fixed point. In particular, in this study we locate and separate the fixed points of T. Note that operator (1.1) has the same fixed points as T, and has a better asymptotic behavior than T itself [3]. Therefore, operator (1.1) can be used as an iteration function to approximate fixed points of T. In our development we consider the operator $T : \Omega \subseteq X \to X$, a Fréchet differentiable operator, where Ω as aforesaid.

Thus, we study the global convergence of the Krasnoselskij method ([2, 3, 10]):

$$x_{n+1} = T_\alpha(x_n) = (1-\alpha)x_n + \alpha T(x_n), \; n \in \mathbb{N}, \tag{1.2}$$

with $x_0 \in \Omega \subset X$ given, and $\alpha \in (0,1]$. Notice that, the Successive Approximations method is obtained for $\alpha = 1$.

To study the global convergence of Krasnoselskij method (1.2), we consider convergence conditions on the operator (1.1) not usually considered. In addition, we obtain domains of global convergence for Krasnoselskij method (1.2) from auxiliary points $\widetilde{x} \in X$, see [6]. We obtain closed ball where the existence and uniqueness of fixed point for T_α is guaranteed and consequently also for T. Moreover, we obtain conditions on $\widetilde{x}$ that makes it easier to find these balls which we be able to isolate a single fixed point of T.

Throughout the paper, we denote by $(X, \|\cdot\|)$ a Banach space and $\overline{B(\widetilde{x}, R)} := \{x \in X; \|x-\widetilde{x}\| \le R\}$, with $\widetilde{x} \in X$ and $R > 0$.

The paper is organized as follows: In Section 2, we motivate the study considering different Fredholm integral equations. In Section 3, for $\alpha \in (0,1]$, we obtain a global convergence result for Krasnoselskij method (1.2) from auxiliary points. Moreover, we obtain both local and semi-local convergence results for method (1.2), see [11], [14]. And finally, in Section 4, we apply the results obtained to locate and separate fixed points to integral equations [9], in several situations.

2. Motivation

An example to motivate our study, we consider a simple Fredholm linear integral equation [5]:

$$x(s) = 2\,s + \lambda \int_0^1 s\,\mathrm{e}^{-t} x(t)\,dt, \quad s \in [0,1], \quad \lambda \in \mathbb{R}. \tag{2.3}$$

So, we take the integral operator

$$[T_1(x)](s) = 2\,s + \lambda\, s \int_0^1 \mathrm{e}^{-t} x(t)\,dt, \quad s \in [0,1], \quad \lambda \in \mathbb{R}.$$

And, for $u, v \in \mathcal{C}[0,1]$, we have

$$\|T_1(u) - T_1(v)\| \le |\lambda| \left(\int_0^1 \mathrm{e}^{-t}\,dt \right) \|u-v\| \le \frac{\mathrm{e}-1}{\mathrm{e}} |\lambda| \|u-v\|.$$

Hence, T_1 is a contraction in $\mathcal{C}[0,1]$ if and only if $|\lambda| < \frac{\mathrm{e}}{\mathrm{e}-1} = 1.58198\ldots$ Therefore, in this situation using Theorem 1.1, we can say that integral equation (2.3) with $|\lambda| < 1.58198\ldots$ has a unique solution x^* in $\mathcal{C}[0,1]$. Moreover, the method of successive approximations $x_{n+1} = T_1(x_n)$, $n \in \mathbb{N}$, converges to $x^* \in \mathcal{C}[0,1]$ for all $x_0 \in \mathcal{C}[0,1]$.

Next, we modify the linearity of the integral equation (2.3) and lose the uniqueness of solution in $\mathcal{C}[0,1]$. So, we choose $\lambda = \frac{1}{5}$ and consider

$$x(s) = s + \frac{s}{5}\int_0^1 \mathrm{e}^{-t} x(t)^2\, dt, \quad x \in [0,1], \tag{2.4}$$

which has solutions: $x_1(s) = 1.03437\ldots s$ and $x_2(s) = 30.0983\ldots s$. Analogously to the previous case, we take the operator

$$[T_2(x)](s) = s + \frac{s}{5}\int_0^1 \mathrm{e}^{-t} x(t)^2\, dt, \quad s \in [0,1],$$

and, for $u, v \in \mathcal{C}[0,1]$, it is easy to obtain

$$\|T_2(u) - T_2(v)\| \leq \frac{1}{5}\left(\int_0^1 \mathrm{e}^{-t}\, dt\right)(\|u\| + \|v\|)\|u - v\|.$$

Obviously, the operator T_2 is not a contraction in $\mathcal{C}[0,1]$. Therefore, to be able to locate a solution of the integral equation (2.4), we need to apply Theorem 1.2. Thus, we need to find a non-empty closed domain where we can apply this result. So we have to get some information about possible solutions. For this, we try to pre-locate them. Hence, if x^* is a possible fixed point of the operator T_2, we have from (2.4) the following condition

$$\|x^*\| \leq 1 + \frac{\mathrm{e}-1}{5\mathrm{e}}\|x^*\|^2,$$

which is satisfied if $\|x^*\| \leq (5\mathrm{e} - \sqrt{5\mathrm{e}(\mathrm{e}+4)})/(2\mathrm{e}-2) = 1.17435\ldots$ or $\|x^*\| \geq (5\mathrm{e} + \sqrt{5\mathrm{e}(\mathrm{e}+4)})/(2\mathrm{e}-2) = 6.73553\ldots$. Therefore, if we consider $\mathcal{K} = \overline{B(0,2)}$ as the non-empty closed subset of the space $\mathcal{C}[0,1]$, then

$$\|T_2(u) - T_2(v)\| \leq \frac{4(\mathrm{e}-1)}{5\mathrm{e}}\|u - v\|,$$

with $\frac{4(\mathrm{e}-1)}{5\mathrm{e}} = 0.5056\ldots < 1$, and T_2 is a contractive operator in $\mathcal{K}$. Also, as

$$\|T_2(x)\| \leq 1 + \frac{\mathrm{e}-1}{5\mathrm{e}}2^2 = 1.5057\ldots < 2,$$

it follows that $T_2 : \mathcal{K} \to \mathcal{K}$. Hence, from Theorem 1.2, integral equation (2.4) has a unique solution x^* in $\overline{B(0,2)}$. Moreover, the method of successive approximations, $x_{n+1} = T_2(x_n)$, $n \geq 0$, converges to $x^* \in \overline{B(0,2)}$ for all $x_0 \in \overline{B(0,2)}$.

Next, we consider a small modification in (2.4) by substituting $\lambda = \frac{1}{5}$ for $\lambda = \frac{1}{2}$. In this case, we take the following integral operator

$$[T_3(x)](s) = s + \frac{1}{2}\int_0^1 s\,\mathrm{e}^{-t} x(t)^2\, dt, \quad s \in [0,1].$$

Obviously, the operator T_3 is not a contraction in $\mathcal{C}[0,1]$.

Based on the pre-location of a fixed point, we can think about looking for a non-empty closed domain of the form $\overline{B(0,r)} \subset \mathcal{C}[0,1]$, so that the operator $T_3 : \overline{B(0,r)} \to \overline{B(0,r)}$ is a contraction on that ball. But, it is easy to check that this is not possible. Moreover, for this integral equation it is not possible to locate previously a fixed point. Therefore, we are not able to find domains such as $\overline{B(0,r)}$ where we can apply Theorem 1.2. In these cases, we consider auxiliary points, $\widetilde{x} \in \Omega$, to be able to locate a fixed point in closed ball

$\overline{B(\widetilde{x}, R)}$. Hence, as we show in section 4, we obtain a precise location of a fixed point and we separate it from other possible ones with greater accuracy.

3. THE KRASNOSELSKIJ METHOD

Next, we obtain a global convergence result restricted to a certain ball, for Krasnoselskij method (1.2) and a given $\alpha_0 \in (0,1]$. For this study, we use auxiliary points that allow us to obtain both local and semi-local convergence results as a consequence. From now on, we consider $T : \Omega \subseteq X \to X$ a continuously Fréchet differentiable operator defined on a non-empty open convex domain Ω of a Banach space $(X, \|\cdot\|)$. Fixed $\alpha_0 \in (0,1]$ and $\widetilde{x} \in \Omega$ such that $\|T(\widetilde{x}) - \widetilde{x}\| \leq \widetilde{\eta}$, with $\widetilde{\eta} \in \mathbb{R}^+$, we suppose that the following conditions are true:

$(K1)$ There exists $q_{\alpha_0} : \mathbb{R}_+ \to \mathbb{R}_+$, a continuous and non-decreasing real function, such that

$$\|(1-\alpha_0)I_\Omega + \alpha_0 T'(x)\| \leq q_{\alpha_0}(\|x - \widetilde{x}\|), \quad \text{for each } x \in \Omega.$$

$(K2)$ Let be

$$p_{\alpha_0}(t) := \int_0^1 q_{\alpha_0}(\tau t)\, d\tau. \tag{3.5}$$

There exists at least a positive real root for the scalar equation

$$t(1 - p_{\alpha_0}(t)) - \alpha_0 \widetilde{\eta} = 0, \tag{3.6}$$

being r_0 the smallest positive real root of equation (3.6).

3.1. **Main convergence results.**

Theorem 3.3. *Suppose that the conditions $(K1)$ and $(K2)$ are verified. If $q_{\alpha_0}(r_0) < 1$ and $\overline{B(\widetilde{x}, r_0)} \subseteq \Omega$, then for all $x_0 \in \overline{B(\widetilde{x}, r_0)}$ Krasnoselskij method* (1.2) *converges to x^*, the only fixed point of T in $\overline{B(\widetilde{x}, r_0)}$, and $x_n \in \overline{B(\widetilde{x}, r_0)}$, $n \in \mathbb{N}$. Moreover,*

$$\|x_n - x^*\| \leq \frac{q_{\alpha_0}(r_0)^n}{1 - q_{\alpha_0}(r_0)} \|x_1 - x_0\|, \quad n \in \mathbb{N}. \tag{3.7}$$

Proof. If $x_0 \in \overline{B(\widetilde{x}, r_0)}$, as $\widetilde{x} + \tau(x_0 - \widetilde{x}) \in \overline{B(\widetilde{x}, r_0)} \subseteq \Omega$, it follows:

$$\begin{aligned} x_1 - \widetilde{x} &= (1-\alpha_0)(x_0 - \widetilde{x}) + \alpha_0(T(x_0) - T(\widetilde{x})) + \alpha_0(T(\widetilde{x}) - \widetilde{x}) \\ &= \int_0^1 \left[(1-\alpha_0)I_\Omega + \alpha_0 T'(\widetilde{x} + \tau(x_0 - \widetilde{x}))\right](x_0 - \widetilde{x})\, d\tau + \alpha_0(T(\widetilde{x}) - \widetilde{x}). \end{aligned}$$

Taking into account, $(K1)$ and (3.6), we have

$$\begin{aligned} \|x_1 - \widetilde{x}\| &\leq \int_0^1 q_{\alpha_0}(\tau\|x_0 - \widetilde{x}\|)\, d\tau \|x_0 - \widetilde{x}\| + \alpha_0 \widetilde{\eta} \\ &\leq p_{\alpha_0}(r_0) r_0 + \alpha_0 \widetilde{\eta} = r_0. \end{aligned}$$

Therefore, $x_1 \in \overline{B(\widetilde{x}, r_0)}$.

Now, as T is a differentiable operator we have

$$\begin{aligned} x_2 - x_1 &= (1-\alpha_0)(x_1 - x_0) + \alpha_0(T(x_1) - T(x_0)) \\ &= \int_0^1 \left[(1-\alpha_0)I_\Omega + \alpha_0 T'(x_0 + \tau(x_1 - x_0))\right](x_1 - x_0)\, d\tau. \end{aligned}$$

Taking into account that, $x_0 + \tau(x_1 - x_0) \in \overline{B(\widetilde{x}, r_0)} \subseteq \Omega$, it follows

$$\|x_2 - x_1\| \le q_{\alpha_0}(r_0)\|x_1 - x_0\| < \|x_1 - x_0\|.$$

To continue, we suppose that $x_1, x_2, \ldots, x_{n-1} \in \overline{B(\widetilde{x}, r_0)}$, and

$$\|x_{j+1} - x_j\| \le q_{\omega_0}(r_0)\|x_j - x_{j-1}\| \le q_{\omega_0}(r_0)^j\|x_1 - x_0\|, \text{ for } j = 1, 2, \ldots, n-1.$$

By applying an inductive procedure, it follows that

$$x_n - \widetilde{x} = \int_0^1 \left[(1-\alpha_0)I_\Omega + \alpha_0 T'(\widetilde{x} + \tau(x_{n-1} - \widetilde{x}))\right](x_{n-1} - \widetilde{x})\, d\tau - \alpha_0(\widetilde{x} - T(\widetilde{x})).$$

Therefore, as $x_{n-1} \in \overline{B(\widetilde{x}, r_0)}$, from $(K1)$ and (3.6), we obtain

$$\|x_n - \widetilde{x}\| \le p_{\alpha_0}(\|x_{n-1} - \widetilde{x}\|)\|x_{n-1} - \widetilde{x}\| + \alpha_0\widetilde{\eta} \le p_{\alpha_0}(r_0)r_0 + \alpha_0\widetilde{\eta} = r_0,$$

and then, $x_n \in \overline{B(\widetilde{x}, r_0)}$.

On the other hand, it is easy to check that

$$x_{n+1} - x_n = \int_0^1 \left[(1-\alpha_0)I_\Omega + \alpha_0 T'(x_{n-1} + \tau(x_n - x_{n-1}))\right](x_n - x_{n-1})\, d\tau,$$

and taking into account that $x_{n-1} + \tau(x_n - x_{n-1}) \in \overline{B(\widetilde{x}, r_0)} \subseteq \Omega$, from $(K1)$, we have

$$\|x_{n+1} - x_n\| \le q_{\alpha_0}(r_0)\|x_n - x_{n-1}\|.$$

Consequently, it follows that

$$\|x_{n+1} - x_n\| \le q_{\alpha_0}(r_0)\|x_n - x_{n-1}\| \le q_{\alpha_0}(r_0)^n\|x_1 - x_0\|, \quad \text{for} \quad n \in \mathbb{N}.$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence and therefore a convergent sequence. So, for $n, m \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \sum_{k=1}^{m} \|x_{n+k} - x_{n+k-1}\| \\
&\le \left(\sum_{k=1}^{m} q_{\alpha_0}(r_0)^{k-1}\right)\|x_{n+1} - x_n\| \\
&\le \frac{1 - q_{\alpha_0}(r_0)^m}{1 - q_{\alpha_0}(r_0)} q_{\alpha_0}(r_0)^n\|x_1 - x_0\|,
\end{aligned}$$

and therefore, $\{x_n\}$ is a Cauchy sequence and, there exists x^* such that $\lim x_n = x^*$. Moreover, when $m \to \infty$, (3.7) is verified. Applying the continuity of the operator T:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T_{\alpha_0}(x_n) = \lim_{n\to\infty} \left((1-\alpha_0)x_n + \alpha_0 T(x_n)\right),$$

that is,

$$x^* = (1-\alpha_0)x^* + \alpha_0 T(x^*)$$

and then, it follows that $T(x^*) = x^*$.

To prove the uniqueness of x^* we suppose that y^* is another fixed point of T in $\overline{B(\widetilde{x}, r_0)}$. Obviously, y^* is another fixed point of $T\alpha_0$. Hence,

$$x^* - y^* = T_{\alpha_0}(x^*) - T_{\alpha_0}(y^*) = \int_0^1 T'_{\alpha_0}(y^* + \tau(x^* - y^*))(x^* - y^*)d\tau,$$

so $(I_\Omega - \mathcal{G})(x^* - y^*) = 0$ with

$$\mathcal{G} = I_\Omega - \int_0^1 \left[(1-\alpha_0)I_\Omega + \alpha_0 T'(y^* + \tau(x^* - y^*))\right] d\tau.$$

Thus, if $\mathcal{G}$ is invertible, then $x^* = y^*$. Therefore, we will prove that G is invertible. By the Banach lemma [7], we only have to prove that $\|I_\Omega - \mathcal{G}\| < 1$. Indeed, from Theorem 3.3, as $y^* + \tau(x^* - y^*) \in \overline{B(\widetilde{x}, r_0)} \subseteq \Omega$, then

$$\begin{aligned} \|I_\Omega - \mathcal{G}\| &\leq \int_0^1 \|(1-\alpha_0)I_\Omega + \alpha_0 T'(y^* + \tau(x^* - y^*))\| \\ &\leq \int_0^1 q_{\alpha_0}(\|y^* + \tau(x^* - y^*) - \widetilde{x}\|)d\tau \leq q_{\alpha_0}(r_0) < 1. \end{aligned}$$

Then, the uniqueness of x^* is proved. □

Notice that, the previous result proves the existence and the uniqueness of $x^* \in \overline{B(\widetilde{x}, r_0)}$. Moreover, for all $x_0 \in \overline{B(\widetilde{x}, r_0)}$ the Krasnoselskij method converges to x^*.

So that, from the appropriate choice of the auxiliary point $\widetilde{x}$, we can obtain both local and semilocal convergence results. Thus, if we consider that there exists $\widetilde{x} = x^*$ a fixed point of T, we obtain the following local convergence result for the Krasnoselskij method.

Corollary 3.1. *Let us suppose that there exists $x^* \in \Omega$, a fixed point of T, and $(K1)$ is verified for $\widetilde{x} = x^*$. If there exists $r > 0$, such that $q_{\alpha_0}(r) < 1$ and $\overline{B(x^*, r)} \subset \Omega$, then for each $x_0 \in \overline{B(x^*, r)}$ Krasnoselskij method* (1.2) *converges to x^*, with $x_n \in \overline{B(x^*, r)}$ for all $n \in \mathbb{N}$. Moreover,*

$$\|x_n - x^*\| < p_{\alpha_0}(r)^n r,$$

and x^* is the only fixed point of T in $\overline{B(x^*, r)}$.

Proof. We consider x_0 such that $x_0 \in \overline{B(x^*, r)}$, thus

$$\begin{aligned} x_1 - x^* &= (1-\alpha_0)x_0 + \alpha_0 T(x_0) - (1-\alpha_0)x^* - \alpha_0 T(x^*) \\ &= \int_0^1 \left[(1-\alpha_0)I_\Omega + \alpha_0 T'(x^* + \tau(x_0 - x^*))\right](x_0 - x^*)\, d\tau. \end{aligned}$$

Therefore, from $(K1)$ for $\widetilde{x} = x^*$, we obtain

$$\|x_1 - x^*\| \leq p_{\alpha_0}(\|x_0 - x^*\|)\|x_0 - x^*\|,$$

where p_{α_0} is given in (3.5). Furthermore,

$$p_{\alpha_0}(\|x_0 - x^*\|) = \int_0^1 q_{\alpha_0}(\tau\|x_0 - x^*\|)\, d\tau \leq q_{\alpha_0}(\|x_0 - x^*\|) \leq q_{\alpha_0}(r) < 1.$$

Consequently, $\|x_1 - x^*\| \leq p_{\alpha_0}(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r$, so $x_1 \in \overline{B(x^*, r)}$.

By mathematical induction, it is easy to check that $x_n \in \overline{B(x^*, r)}$ and $\|x_n - x^*\| \leq q_{\alpha_0}(r)^n\|x_0 - x^*\|$, for $n \in \mathbb{N}$. Therefore, $\{x_n\}$ converges to x^*.

Moreover, it follows $\|x_n - x^*\| < p_{\alpha_0}(r)^n r$.

To finish, as in Theorem 3.3, the uniqueness of fixed point x^* is proved. □

Next, taking $\widetilde{x} = x_0$, we obtain the following semilocal convergence result.

Corollary 3.2. *Let us suppose that conditions $(K1)-(K2)$ are verified for $\widetilde{x} = x_0$. If $q_{\alpha_0}(r_0) < 1$ and $\overline{B(x_0, r_0)} \subset \Omega$, then Krasnoselskij method* (1.2) *converges to x^*, a fixed point of T, with $x_n, x^* \in \overline{B(x_0, r_0)}$, $n \in \mathbb{N}$.*

Moreover,

$$\|x_n - x^*\| \leq \frac{q_{\alpha_0}(r_0)^n}{1 - q_{\alpha_0}(r_0)}\|x_1 - x_0\|,\ \ n \in \mathbb{N},$$

and x^* is the only fixed point of T in $\overline{B(x_0, r_0)}$.

Proof. Analogously to the proof of Theorem 3.3, since $r_0(1 - p_{\alpha_0}(r_0)) = \alpha_0\eta_0$ it follows that $\alpha_0\eta_0 < r_0$. Therefore, as $\|x_1 - x_0\| \leq \alpha_0\eta_0$, $x_1 \in \overline{B(x_0, r_0)}$. The rest of the proof is analogous to that of Theorem 3.3. $\square$

Next, we introduce a small modification to condition $(K2)$.

Theorem 3.4. *Let us suppose that $T(\widetilde{x}) \neq \widetilde{x}$ and condition $(K1)$ is verified. If there exists $R \in \mathbb{R}_+$, such that*

$$\widetilde{\eta} \leq \frac{(1 - q_{\alpha_0}(R))R}{\alpha_0}, \tag{3.8}$$

with $\overline{B(\widetilde{x}, R)} \subset \Omega$, then for all $x_0 \in \overline{B(\widetilde{x}, R)}$ Krasnoselskij method (1.2) *converges to x^*, the only fixed point of T in $\overline{B(\widetilde{x}, R)}$, and $x_n \in \overline{B(\widetilde{x}, R)}$, $n \in \mathbb{N}$.*

Moreover,

$$\|x_n - x^*\| \leq \frac{q_{\alpha_0}(R)^n}{1 - q_{\alpha_0}(R)}\|x_1 - x_0\|, \ n \in \mathbb{N}. \tag{3.9}$$

Proof. Notice that, $\widetilde{\eta} > 0$ and, from (3.8), $q_{\alpha_0}(R) < 1$. If $x_0 \in \overline{B(\widetilde{x}, R)}$, we have

$$x_1 - \widetilde{x} = \int_0^1 \left[(1 - \alpha_0)I_\Omega + \alpha_0 T'(\widetilde{x} + \tau(x_0 - \widetilde{x}))\right](x_0 - \widetilde{x})\, d\tau + \omega_0(T(\widetilde{x}) - \widetilde{x}).$$

Now, from $(K1)$, taking into account that q is non-decreasing and (3.8), it follows

$$\|x_1 - \widetilde{x}\| \leq q_{\alpha_0}(R)R + (1 - q_{\alpha_0}(R))R = R,$$

therefore $x_1 \in \overline{B(\widetilde{x}, R)}$.

On the one hand, as

$$x_2 - x_1 = \int_0^1 \left[(1 - \alpha_0)I_\Omega + \alpha_0 T'(x_0 + \tau(x_1 - x_0))\right](x_1 - x_0)\, d\tau,$$

we have

$$\|x_2 - x_1\| \leq \int_0^1 q_{\alpha_0}(\|x_0 + \tau(x_1 - x_0) - \widetilde{x}\|)\, d\tau \|x_1 - x_0\| \leq q_{\alpha_0}(R)\|x_1 - x_0\| < \|x_1 - x_0\|,$$

since $x_0 + \tau(x_1 - x_0) \in \overline{B(\widetilde{x}, R)}$ and $q_{\omega_0}(R) < 1$.

Next, by mathematical induction it is easy to prove that

$$x_n \in \overline{B(\widetilde{x}, R)} \ \text{ and } \ \|x_{n+1} - x_n\| \leq q_{\omega_0}(R)^n\|x_1 - x_0\|, \ \text{ for } \ n \in \mathbb{N}.$$

On the other hand, for $n, m \in \mathbb{N} \cup \{0\}$, we have

$$\|x_{n+m} - x_n\| \leq \frac{1 - q_{\alpha_0}(R)^m}{1 - q_{\alpha_0}(R)} q_{\alpha_0}(R)^n\|x_1 - x_0\|,$$

and therefore $\{x_n\}$ is a Cauchy sequence in a Banach space X. Hence, there exists $x^* \in \overline{B(x_0, r_0)}$ such that $\lim x_n = x^*$. By the continuity of T, it follows that $T(x^*) = x^*$. Besides, when $m \to \infty$, (3.9) is verified.

The uniqueness of x^* in $\overline{B(x_0, r_0)}$ it follows as Theorem 3.3. So, the result is proved. $\square$

Notice that, in this situation, the condition imposed on the parameter $\widetilde{\eta}$ in (3.8) leads us to look for the value of R which, obviously, is a similar process to the one considered in condition $(K2)$ of Theorem 3.3. Observe that if the condition (3.8) does not hold for all $\alpha \in (0,1]$, we can find a value α_0 such that we can ensure the convergence of Krasnoselskij method (1.2).

If condition (3.8) holds for $R \in (a,b)$, then we consider the ball $\overline{B(\tilde{x},\tilde{a})}$, with $\tilde{a}$ a value close to a, to obtain a better domain of existence for the fixed point, while we take the ball $\overline{B(\tilde{x},\tilde{b})}$, with $\tilde{b}$ a value close to b, to obtain a better uniqueness domain. The last situation also give us the best global convergence domain for the Krasnoselskij method.

Finally, from this result, the corresponding local and semilocal results for the Krasnoselskij method can be obtained, as the Corollaries 3.1 and 3.2.

3.2. **Other types of conditions.** Now we focus on Theorem 3.3 condition (i), analyzing its level of restriction for the operator T. As we have already indicated in the introduction, this condition (i) arises from requiring a certain condition to the iteration function T_α. More specifically, the Fréchet derivative operator of T_α is given by $[T'_\alpha(x)]y = (1-\alpha)I_\alpha(y) + \alpha[T'(x)]y$.

Next, we consider the convergence conditions usually considered in the study of the convergence of iterative processes [12]. Obviously, all of them given on the operator T. Now, we consider a contraction condition for the Fréchet differentiable operator T.

Theorem 3.5. *Let us suppose that $\|T'(x)\| \leq M < 1$ for $x \in \Omega$. If there exists $\widetilde{x} \in \Omega$ with $\|T(\widetilde{x}) - \widetilde{x}\| \leq \widetilde{\eta}$, such that $\overline{B(\widetilde{x}, \frac{\widetilde{\eta}}{1-M})} \subset \Omega$, then, there exists x^* the only fixed point of T in $\overline{B(\widetilde{x}, \frac{\widetilde{\eta}}{1-M})}$, such that for each $x_0 \in \overline{B(\widetilde{x}, \frac{\widetilde{\eta}}{1-M})}$ the sequence $\{x_n\}$ given by the Krasnoselskij method* (1.2) *for any $\alpha \in (0,1]$, converges to x^*. Moreover,*

$$\|x^* - x_n\| \leq \frac{M^n}{1-M}\|x_1 - x_0\|, \ n \geqslant 1. \tag{3.10}$$

Proof. Consider any $\alpha \in (0,1]$. It is easy to check that

$$\|(1-\alpha)I_\Omega + \alpha T'(x)\| \leq 1 - \alpha(1-M),$$

so, in this case $q_\alpha(t)$ is a constant real function with $q_\alpha(t) = 1 - \alpha(1-M) < 1$.

Now, there exists $r_0 = \dfrac{\widetilde{\eta}}{1-M}$ the smallest positive real root of equation $t(1-p_\alpha(t)) - \alpha\widetilde{\eta} = 0$ with $p_\alpha(r_0) < 1$ and $\overline{B(\widetilde{x}, r_0)} \subset \Omega$, and by applying Theorem 3.3 the result is proved. □

As we have just seen, we can obtain a fixed point result restricted to a closed ball. Obviously, this result gives us a more precise location of the fixed point than if we apply the Restricted Fixed Point Theorem to the domain $\mathcal{K}$ and without requiring $T : \mathcal{K} \to \mathcal{K}$.

Next, we consider a Lipschitz condition for the operator T'.

Theorem 3.6. *Let us suppose that there exists a $K > 0$ such that*

$$\|T'(x) - T'(y)\| \leq K\|x - y\| \textit{ for all } x, y \in \Omega.$$

If there exists $\widetilde{x} \in \Omega$ such that $\|T(\widetilde{x}) - \widetilde{x}\| \leq \widetilde{\eta}$ and $\|T'(\widetilde{x})\| \leq L$ with $L \leq 1 - \sqrt{2K\widetilde{\eta}}$, such that $\overline{B(\widetilde{x}, R)} \subset \Omega$ where

$$R = \frac{1 - L - \sqrt{(1-L)^2 - 2K\widetilde{\eta}}}{K},$$

then, there exists x^* *the only fixed point of* T *in* $\overline{B(\widetilde{x}, R)}$, *such that Krasnoselskij method* (1.2), *for* $\alpha \in (0,1]$, *converges to* x^*, *for all* $x_0 \in \overline{B(\widetilde{x}, R)}$.

Moreover,

$$\|x_n - x^*\| \leq \frac{q_\alpha(R)^n}{1 - q_\alpha(R)}\|x_1 - x_0\|, \; n \in \mathbb{N},$$

where $q_\alpha(t) = 1 - \alpha(1 - L) + \alpha K t$.

Proof. Given $\widetilde{x} \in \Omega$ and $\alpha \in (0,1]$, from Lipschitz condition on T', we have

$$\|(1-\alpha)I_\Omega + \alpha T'(x)\| \leq 1 - \alpha + \alpha(\|T'(\widetilde{x}\| + K\|x - \widetilde{x}\|) \leq 1 - \alpha(1-L) + \alpha K\|x - \widetilde{x}\|,$$

therefore $q_\alpha(t) = 1 - \alpha(1-L) + \alpha K t$ and $p_\alpha(t) = 1 - \alpha(1-L) + \dfrac{\alpha K}{2} t$, and applying Theorem 3.3, we obtain that $r_0 = R$ with $q_\alpha(R) < 1$ and the result is proved. □

Notice that, if $K = 0$ then T is a linear operator. Hence, T has an unique fixed point.

Finally, we consider T an operator twice continuously differentiable Fréchet with second derivative ϕ-bounded.

Theorem 3.7. *Fixed,* $\alpha_0 \in (0,1]$, *let us suppose the following conditions:*

(I) $\|T''(x)\| \leq \phi(\|x\|)$ *for* $x \in \Omega$, *where* ϕ *is a non-decreasing real function with* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$.

(II) There exists $\widetilde{x} \in \Omega$, *with* $\|T(\widetilde{x}) - \widetilde{x}\| \leq \widetilde{\eta}$, *such that* $\|T'(\widetilde{x})\| \leq L < 1$.

(III) There exists at least a positive real root of scalar equation

$$t(1 - p_{\alpha_0}(t)) - \alpha_0 \widetilde{\eta} = 0,$$

where $p_{\alpha_0}(t) = 1 - \alpha_0(1-L) + \alpha_0 \int_0^1 \phi(\|\widetilde{x}\| + \tau t)\tau t \, d\tau$, *and we denote by* R *the smallest positive real root.*

If $\overline{B(\widetilde{x}, R)} \subset \Omega$ *and* $q_{\alpha_0}(R) < 1$, *for* $q_{\alpha_0}(t) = 1 - \alpha_0(1-L) + \alpha_0 \phi(\|\widetilde{x}\| + t)t$, *then, there exists* x^* *the only fixed point of* T *in* $\overline{B(\widetilde{x}, R)}$, *such that Krasnoselskij method* (1.2), *for* $\alpha_0 \in (0,1]$, *converges to* x^* *for all* $x_0 \in \overline{B(\widetilde{x}, R)}$.

Moreover,

$$\|x^* - x_n\| \leq \frac{q_{\omega_0}(R)^n}{1 - q_{\omega_0}(R)}\|x_1 - x_0\|, \; n \in \mathbb{N}.$$

Proof. By Taylor's series, we have

$$[T'_{\alpha_0}(x)](y) = (1-\alpha_0)y + [T'(x)](y) = (1-\alpha_0)y + [T'(\widetilde{x}) + T''(\widetilde{x} + \tau(x - \widetilde{x}))(x - \widetilde{x})](y),$$

for all $x, y \in \Omega$, $\tau \in [0,1]$. Then,

$$\begin{aligned}
\|(1-\alpha_0)I_\Omega + \alpha_0 T'(x)\| &\leq 1 - \alpha_0(1 - \|T'(\widetilde{x})\|) + \alpha_0\|T''(\widetilde{x} + \tau(x - \widetilde{x}))\|\|x - \widetilde{x}\| \\
&\leq 1 - \alpha_0(1 - \|T'(\widetilde{x})\|) + \alpha_0\phi(\|\widetilde{x} + \tau(x - \widetilde{x})\|)\|x - \widetilde{x}\| \\
&\leq 1 - \alpha_0(1 - \|T'(\widetilde{x})\|) + \alpha_0\phi(\|\widetilde{x}\| + \|x - \widetilde{x}\|)\|x - \widetilde{x}\|.
\end{aligned}$$

Therefore, $\|(1-\alpha_0)I_\Omega + \alpha_0 T'(x)\| \leq q_{\alpha_0}(\|x - \widetilde{x}\|)$ with $q_{\alpha_0}(t) = 1 - \alpha_0(1-L) + \alpha_0\phi(\|\widetilde{x}\| + t)t$, and applying Theorem 3.3, the result is proved. □

4. NUMERICAL EXAMPLES

In this section, we check the good applicability of the fixed point type results that we have obtained with three numerical examples. These results will allow us to locate and separate fixed points in situations where Theorem 1.2 is not applicable.

We consider the operator T_2 given in Section 2:

$$[T_2(x)](s) = s + \frac{1}{5}s\int_0^1 \mathrm{e}^{-t}x(t)^2\,dt, \quad s \in [0,1].$$

As we have already seen in Section 2, operator T_2 is not contractive in $\mathcal{C}[0,1]$, however from Theorem 1.2, has a unique fixed point x^* in $\overline{B(0,2)}$.

Now, applying Corollary 3.2, we obtain a more precise location of a fixed point of T_2. We consider $x_0(s) = s$, and taking into account the pre-location of fixed points of T_2 we take $\Omega = B(0,2)$. Therefore, we obtain $\|T(x_0) - x_0\| \leq \eta_0 = \dfrac{2\mathrm{e}-5}{5\mathrm{e}}$ and

$$\|(1-\alpha_0)I_\Omega + \alpha_0 T'(x)\| \leq q_{\alpha_0}(\|x-x_0\|) = 1 + \frac{(2(\mathrm{e}-1)\|x-x_0\| - (4+3\mathrm{e}))\alpha_0}{5\mathrm{e}}, \quad \text{for each } x \in \Omega,$$

being q_{α_0} a continuous and non-decreasing real function. Hence,

$$p_{\alpha_0}(t) = 1 + \frac{(\mathrm{e}(t-3) - (t+4))\alpha_0}{5\mathrm{e}}$$

and $r_0 = 0.0361011$ is the smallest positive real root of

$$t(1 - p_{\alpha_0}(t)) - \alpha_0\eta_0 = 0, \quad \alpha_0 \in (0,1].$$

Moreover, $q_{\alpha_0}(r_0) = 1 - 0.885175\alpha_0 < 1$ and $\overline{B(s, 0.0361011)} \subset B(0,2)$. Thus, Krasnoselskij method (1.2) for $\alpha_0 \in (0,1]$, converges to $x^* = 1.03437\ldots s$ a fixed point of T_2, with $x_n, x^* \in \overline{B(s, 0.0361011)}$, for $n \in \mathbb{N}$. Notice that, there is no restriction for the parameter α_0. Furthermore,

$$\|x^* - x_n\| \leq \frac{q_{\alpha_0}(r_0)^n}{1 - q_{\alpha_0}(r_0)}\|x_1 - x_0\|, \; n \in \mathbb{N} \tag{4.11}$$

and x^* is the only fixed point of T_2 in $\overline{B(s, 0.0361011)}$.

And from here, the numerical results that appear in Table 1 are followed, where we show the number of iterations, indicated by n, necessary to converge to x^* with the Krasnoselskij method applied to T_2. We consider stopping criterion $\|x_n - x^*\| < 10^{-3}$ and $\alpha_0 = 0.5, 0.9, 1$. Moreover, we show the error for the last iteration in each case as well as the a priori estimates (4.11).

ω_0	n	$\|x_n - x^*\|$	a priori estimates (4.11)
0.5	8	$2.24107\ldots \times 10^{-4}$	$6.76424\ldots \times 10^{-4}$
0.9	3	$1.39216\ldots \times 10^{-4}$	$3.39013\ldots \times 10^{-4}$
1	2	$1.4906\ldots \times 10^{-4}$	$4.78461\ldots \times 10^{-4}$

TABLE 1. Numerical results for the Krasnoselskij method applied to T_2, from $x_0(s) = s$.

Notice that, in this case, when α_0 is close to 1 the Krasnoselskij method applied to operator T_2 behaves better than the other cases.

Next, we study the location of a fixed point for the operator T_3, considered in Section 2, and given by

$$[T_3(x)](s) = s + \frac{s}{2}\int_0^1 \mathrm{e}^{-t}x(t)^2\,dt, \quad s \in [0,1].$$

As already indicated in Section 2, this operator is not contractive in $\mathcal{C}[0,1]$. T_3 does not admit the possible pre-location of its fixed points, nor is it possible to locate domains of the form $\overline{B(0,r)}$ in which it is contractive. Therefore, neither the application of Theorem 1.2 seems simple. However, applying Theorem 3.4 we will be able to locate a fixed point of this operator and separate it from other possible fixed points. In addition, we approximate a fixed point using the Krasnoselskij method, obtaining global convergence. For this, we consider the auxiliary point $\widetilde{x}(s) = s$ in $\Omega = \mathcal{C}[0,1]$. Hence, it is easy to check that

$$\|\widetilde{x} - T_3(\widetilde{x})\| \leq \frac{2\mathrm{e}-5}{2\mathrm{e}} = \widetilde{\eta}.$$

On the other hand, fixed $\alpha_0 \in (0,1]$, $T_3'(x) = T_3'(\widetilde{x}) + T_3''(\widetilde{x})(x-\widetilde{x})$ for each $x \in \Omega$. And, it follows

$$\|(1-\alpha_0)I_\Omega + \alpha_0 T_3'(x)\| \leq 1 - \alpha_0 + \frac{\mathrm{e}-2}{\mathrm{e}}\alpha_0 + \alpha_0\frac{\mathrm{e}-1}{\mathrm{e}}\|x-\widetilde{x}\|.$$

Thus, $q_{\alpha_0}(t) = 1 + \alpha_0\left(\frac{-2}{\mathrm{e}} + \frac{\mathrm{e}-1}{\mathrm{e}}t\right)$. Next, taking into account (3.8), we obtain that there exists $R \in (0.121909\ldots 1.04204\ldots)$ satisfying this condition. Therefore, from Theorem 3.4, we obtain that T_3 has a fixed point in $\overline{B(s,0.13)}$ and this is the only in $\overline{B(s,1.04)}$.

In the previous development, we have seen that there is no restriction for the parameter α_0, so we can apply the Krasnoselskij method for any value $\alpha_0 \in (0,1]$. Proceeding as in the previous example, in Table 2, we show the number of iterations, indicated by n, that the Krasnoselskij method needs to converges to $x^* = 1.09655\ldots s$ with $T_3(x^*) = x^*$ and stopping criterion $\|x_n - x^*\| < 10^{-5}$ and $\alpha_0 = 0.3, 0.9, 1$. Moreover, we show the error for the last iteration in each case as well as the a priori estimates (4.11) of the error provided by Theorem 3.3. Notice that, as above, the Krasnoselskij method applied to T_3 behaves better as long as the value of α_0 is close to 1.

ω_0	n	$\|x_n - x^*\|$	a priori estimates (4.11)
0.3	35	$4.61142\ldots \times 10^{-6}$	$2.36606\ldots \times 10^{-5}$
0.9	9	$4.79792\ldots \times 10^{-7}$	$1.67272\ldots \times 10^{-5}$
1	9	$1.49066\ldots \times 10^{-8}$	$3.5308\ldots \times 10^{-9}$

TABLE 2. Numerical results for the Krasnoselskij method applied to T_3, from $x_0(s) = s$.

Next, we consider another type of condition for our study. So, we consider a non-linear integral equation of Fredholm type considered in [4],

$$x(s) = \sin(\pi s) + \cos(\pi s)\lambda\int_0^1 \sin(\pi t)x(t)^3\,dt, \quad s \in [0,1], \quad \lambda \in \mathbb{R}\setminus\{0\}.$$

In this case, we take the operator $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ given by

$$[T(x)](s) = \sin(\pi s) + \cos(\pi s)\lambda\int_0^1 \sin(\pi t)x(t)^3\,dt, \quad s \in [0,1], \quad \lambda \in \mathbb{R}\setminus\{0\}.$$

Then, we have

$$\|T'(x)\| \leq \frac{3|\lambda|}{\pi}\|x\|^2$$

Hereafter, we consider $M = 0.9$. Thus, we have $\|T'(x)\| \leq M < 1$ in $\Omega = B\left(0, \sqrt{\dfrac{M\pi}{3|\lambda|}}\right)$.
Moreover, we take $\widetilde{x}(s) = \sin(\pi s)$ and obtain

$$\|T(\widetilde{x}) - \widetilde{x}\| \leq \frac{|\lambda|}{\pi}\|\widetilde{x}\|^3 \leq \frac{|\lambda|}{\pi} = \widetilde{\eta}.$$

Now, it is easy to check that $\overline{B\left(\sin(\pi s), \dfrac{10|\lambda|}{\pi}\right)} \subset \Omega$ if $|\lambda| \in (0, 0.271341)$. Therefore, by Theorem 3.5, for each $|\lambda| \in (0, 0.271341)$, there exists only one fixed point of T in $\overline{B\left(\sin(\pi s), \dfrac{10|\lambda|}{\pi}\right)}$.

5. Conclusions

In this study we have addressed two issues related to the Fixed Point Theorem. On the one hand, we obtain a more precise location of fixed points than Theorem 1.2 by means of closed balls. To do that, we consider the Krasnoselskij method to approximate the fixed point, instead of the method of successive approximations. Moreover, by using auxiliary points, we obtain procedures to obtain domains of global convergence in which we locate and separate fixed points.

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