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From Modular Spaces to Boundary Value Problems: A Survey of Recent Advances

M. A. KHAMSI[1], P. KUMAM[2,*] AND O. MÉNDEZ[3]

ABSTRACT. This survey explores the recent discovery of the connection between the geometry of modular spaces and partial differential equations with non-standard growth. Specifically, we demonstrate the solvability of the non-homogeneous Dirichlet problem for the variable exponent $p(\cdot)$-Laplacian. We place special emphasis on the case when the variable exponent $p(\cdot)$ is unbounded and the boundary data is in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$.
Tracing the historical development from Riesz's introduction of the $L^{p(\cdot)}$-spaces through the contributions by Orlicz and Nakano to further advances at the end of the twentieth century, we examine the properties and geometric characteristics of modular spaces. We discuss the uniform convexity of modulars on $L^{p(\cdot)}$, $\ell^{p(\cdot)}$, and $W^{1,p(\cdot)}$ and discuss its essential role in the analysis of the Dirichlet problem for the $p(\cdot)$-Laplacian when the variable exponent p is unbounded. It will be evident from our analysis that the Banach space structure is inadequate in treating this case.

1. INTRODUCTION

The purpose of this survey is twofold: it is intended to report recent advances in the theory of partial differential equations with non-standard growth and, on the other side, to bring attention to the overlooked connection between the classical modular space theory and boundary value problems involving variable exponent spaces.
More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $p : \Omega \to (1, \infty)$ be measurable. We will be concerned with the variable exponent $p(\cdot)$-Laplacian

$$\Delta_{p(\cdot)} u = \operatorname{div}\left(|\nabla u|^{p(\cdot)-2} \nabla u\right). \tag{1.1}$$

Our main focus is the recently obtained result (Theorem 5.5) on existence and uniqueness in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ of the solution of the non-homogeneous Dirichlet problem

$$\begin{cases} \Delta_{p(\cdot)}(w) = \operatorname{div}\left(|\nabla w|^{p(\cdot)-2} \nabla w\right) = 0 & \text{in } \Omega, \\ w|_{\partial\Omega} = \varphi, \end{cases} \tag{1.2}$$

with $n < p_- = \inf_{x \in \Omega} p(x)$ and $p(x) < \infty$ a.e. and suitable boundary datum. Notice that we include the case $p_+ = \sup_{x \in \Omega} p(x) = \infty$.

The point to be brought to the foreground here is that the standard Banach space techniques traditionally employed for treating boundary value problems are not fully adequate for handling (1.2).
The reason behind this statement is that the differential operator $\Delta_{p(\cdot)}$ acts on $C_0^\infty(\Omega)$ (in the sequel

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Corresponding author: poom.kum@kmutt.ac.th

this will symbolize the vector space of infinitely differentiable, compactly supported functions defined on Ω) as the derivative of the functional

$$u \to \mathscr{D}(u) = \int_{\Omega} p(x)^{-1} |\nabla u(x)|^{p(x)} dx,$$

which is modular in nature (in fact, it is a pseudo-modular in the terminology of [30]). Therefore, all topological issues involved when dealing with $\mathscr{D}$ are modular in principle. As long as the exponent $p(x)$ is bounded, that is, if $p_+ < \infty$, modular convergence in the sense of $\mathscr{D}$ is equivalent to norm convergence; this is the reason behind the essential difference between the modular topology and the normal topology being invisible in this case. For $p_+ < \infty$, the problem can be dealt with the well-known Banach space techniques.
However, when $p_+ = \infty$, the modular topology, and the norm topology part ways, the former being much weaker than the latter. It is at this point that a departure from the traditional approach is in order, and a deep understanding of the modular topology is indispensable.

To smoothly expound our ideas, some historical background on the theory of modular spaces will be presented in Section 2; Section 3 will be devoted to the basic properties of such spaces. Section 4 will deal with the geometrical properties of modular spaces and their implications. In Section 5, the background material will be applied to study boundary value problems, and the proof of the main theorems alluded to in the introduction will be sketched.

2. Some historical context

In his groundbreaking work, [36], F. Riesz introduced the $L^{p(\cdot)}$ class (for a constant $p : 1 \leq p < \infty$) as the collection of all Lebesgue-measurable functions f on $[a,b]$ for which $\int_a^b |f(x)|^p dx < \infty$ [36, p.457] and defined convergence $f_j \to f$ in the class $L^{p(\cdot)}$ through the equality [36, p.464]

$$\lim_{j\to\infty} \int_a^b |f_j(x) - f_(x)|^p dx = 0.$$

Not surprisingly, thus, rather than working with norms, which were not introduced until the 1930s, Riesz was dealing with the functional

$$\rho_p(f) = \int_a^b |f(x)|^p dx.$$

In 1931, W. Orlicz [33, p.207] ventured one step further and defined the $L^{p(\cdot)}([0,1])$ class for a *variable* exponent $p(x) > 1$, itself a measurable function on $[0,1]$. In the spirit of Riesz, Orlicz defined convergence in this class using the functional

$$\rho_{p(\cdot)}(f) = \int_a^b |f(x)|^{p(x)} dx,$$

namely by declaring $f_j \to f$ if and only if

$$\rho_{p(\cdot)}(f - f_j) = \int_a^b |f(x) - f_j(x)|^{p(x)} dx \to 0 \text{ as } j \to \infty.$$

It is worthwhile to point out a fundamental difference between the functionals ρ_p and $\rho_{p(\cdot)}$, namely, $(\rho_p(f))^{\frac{1}{p}}$ is a norm on the class $L^{p(\cdot)}([0,1])$, whereas no power of $\rho_{p(\cdot)}$ is a norm on $L^{p(\cdot)}([0,1])$.
Another two decades passed before the structure implied by the properties of $\rho_{p(\cdot)}$ was studied abstractly in [31], thus originating the concept of modular space. The significant weakness of modulars relative to norms represented a disadvantage. This was because modular structures remained underappreciated for a long period, and their study was mostly directed toward the underlying normed-space structure. We refer the reader to [30] and the references therein for a detailed account of the theory of modular spaces up to the early 1980's. In 1979, Sharapudinov [37] studied

some topological properties of the space $L^{p(\cdot)}([0,1])$.
The next stage in the evolution of the subject was the emergence of an interest in the geometric aspect of the theory of modular spaces, impulsed by the desire to determine the existence of fixed points for mappings that are nonexpansive in the modular sense. This question was answered in [14, Theorem 3.5]. The modular uniform convexity and the applications discussed in [14] were further developed in [15,16] for the particular case of Orlicz spaces.
Simultaneously, advances made in material sciences led to the need for suitable mathematical modeling for the hydrodynamics of non-Newtonian fluids. As it turns out, the behavior of such fluids cannot be described using the classical equations of hydrodynamics; their mathematical description led naturally to partial differential equations with non-standard growth, whose analysis requires variable exponent Lebesgue spaces [9,34,35]. The seminal paper [19] summarizing the functional analytic properties of variable exponent Lebesgue and Sobolev appeared in 1991.
A period of singularly intense research activity on variable exponent function spaces and boundary value problems involving operators with non-standard growth followed, and to an extent continues until today, see for example [1,2,6,7,9–11,19,24,25,27,28,39–41] and their included references. Yet, without exception, only the case $p_+ < \infty$ has been considered, and the unbounded case remained elusive.

We set out to briefly survey recent advances in overcoming this restriction.

3. Preliminaries

Today, it is widely acknowledged that the normed space framework is overly rigid and may not fully capture certain intricate mathematical nuances that become apparent with a more flexible approach. One significant example, central to the focus of this work, is the modular nature of the variable exponent p-Laplacian. Keeping this in perspective, we aim to provide a concise overview of definitions and established findings. For a deeper exploration of the topics in this section, interested readers are encouraged to consult [9,19–21,26,29,30].

Definition 3.1. *[30,31] A convex modular on a real vector space X is a function $\rho : X \to [0,\infty]$ that satisfies the following conditions:*

(1) $\rho(x) = 0$ if and only if $x = 0$;
(2) $\rho(\alpha x) = \rho(x)$, if $|\alpha| = 1$;
(3) $\rho(\alpha x + (1-\alpha)y) \le \alpha\rho(x) + (1-\alpha)\rho(y)$, for any $\alpha \in [0,1]$ and any $x, y \in X$.

Moreover, ρ is considered to exhibit left-continuity if, for all $x \in X$,

$$\lim_{r\to 1^-} \rho(rx) = \rho(x).$$

Remark 3.1. If the condition (1) is replaced with

$$\rho(0) = 0,$$

then ρ is said to be a pseudo-modular [30].

A modular function defined on a vector space X naturally induces a modular space.

Definition 3.2. *Given a convex modular function ρ on the vector space X, its associated modular space is defined by:*

$$X_\rho = \{x \in X;\ \lim_{\alpha\to 0} \rho(\alpha x) = 0\} = \{x \in X;\ \rho(\alpha x) < \infty \ \text{for some}\ \alpha > 0\}.$$

The Luxemburg norm, represented as $\|\cdot\|_\rho$ and defined on the vector space X_ρ, is expressed as follows:

$$\|x\|_\rho := \inf\left\{\alpha > 0;\ \rho\left(\frac{x}{\alpha}\right) \le 1\right\}.$$

The primary examples addressed in this study are listed below.

Example 3.1. *[33]* For $p : \mathbb{N} \to [1,\infty)$, consider the linear space $\ell^{p(\cdot)}$ defined as

$$\ell^{p(\cdot)} = \left\{ (x_n) \subset \mathbb{R}^{\mathbb{N}};\ \text{there exists } \beta > 0 \text{ for which } \sum_{n=0}^{\infty} \frac{1}{p(n)} \left| \frac{x_n}{\beta} \right|^{p(n)} < +\infty \right\}.$$

The functional $\rho_p : \ell^{p(\cdot)} \to [0,\infty]$ given by

$$\rho_p(x) = \rho_p((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)} \, |x_n|^{p(n)}$$

is a convex modular functional. Likewise, the functional

$$\rho_p^*(x) = \rho_p^*((x_n)) = \sum_{n=0}^{\infty} |x_n|^{p(n)}$$

defines a convex modular on the modular vector space (which coincides with $\ell^{p(\cdot)}$):

$$\ell^{p(\cdot)} = \left\{ (x_n) \subset \mathbb{R}^{\mathbb{N}};\ \text{there exists } \alpha > 0 \text{ for which } \sum_{n=0}^{\infty} \left| \frac{x_n}{\alpha} \right|^{p(n)} < +\infty \right\}.$$

Notably, the Luxemburg norms associated with ρ_p and ρ_p^* are equivalent. □

The $\ell^{p(\cdot)}$ spaces have a continuous counterpart, as illustrated in the following example.

Example 3.2. Let $\Omega \subset \mathbb{R}^n$ be a nonempty measurable subset. The notation $\mathscr{M}(\Omega)$ will stand for the collection of all extended real-valued Borel-measurable functions defined on Ω. Let $\mathscr{P}(\Omega)$ represent the subset of $\mathscr{M}(\Omega)$ consisting of functions $p : \Omega \longrightarrow [1,\infty]$ such that $p(x) < \infty$ a.e. The functional $\rho_p : \mathscr{M}(\Omega) \longrightarrow [0,\infty]$, defined by

$$\rho_p(u) = \int_\Omega \frac{1}{p(x)} \, |u(x)|^{p(x)} \, dx,$$

is a convex modular on $\mathscr{M}(\Omega)$. The corresponding modular vector space is denoted by $L^{p(\cdot)}(\Omega)$. The functional $\rho_p^* : \mathscr{M}(\Omega) \longrightarrow [0,\infty]$, defined by

$$\rho_p^*(u) = \int_\Omega |u(x)|^{p(x)} \, dx$$

is also a convex modular and, in addition, it holds that

$$L^{p(\cdot)}(\Omega) = \left\{ u \in \mathscr{M}(\Omega);\ \text{there exists } \beta > 0 \text{ for which } \rho_p^*\left(\frac{u}{\beta}\right) < +\infty \right\},$$

i.e., the associated modular vector space for ρ_p^* is $L^{p(\cdot)}(\Omega)$. As in the discrete case, the Luxemburg norms associated with ρ_p and ρ_p^* are equivalent.
Yet, one can easily construct a sequence $(u_k) \subset L^{p(\cdot)}(\Omega)$ where $\rho_p(u_k) \to 0$ but $\rho_p^*(u_k) \not\to 0$ as $k \to \infty$. Indeed, consider $\Omega = (0,1/2)$, $p(x) = 1/x$, define $u_k(x) = k^{\frac{2}{k}} \mathbb{1}_{\left(\frac{1}{k+1},\frac{1}{k}\right)}(x)$, for each $k \geq 1$. □

Example 3.3. Let $\Omega \subset \mathbb{R}^n$ be a nonempty measurable subset. The spaces $\mathscr{M}(\Omega)$ and $\mathscr{P}(\Omega)$ are defined as in the Example 3.2. On the set $\mathscr{V}(\Omega) \subset \mathscr{M}(\Omega)$ comprising functions whose distributional derivatives belong to $\mathscr{M}(\Omega)$, we have the convex modular $\rho_{1,p} : \mathscr{V}(\Omega) \to [0,\infty]$ defined by

$$\rho_{1,p}(u) := \rho_p(u) + \rho_p(|\nabla u|) = \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)} dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx,$$

where $|\nabla u|$ denotes the Euclidean norm of the gradient of u. The space $W^{1,p(\cdot)}(\Omega)$ is defined as the class of those functions $u \in \mathscr{V}(\Omega)$, such that there exists $\lambda > 0$ satisfying $\rho_{1,p}(\lambda u) < \infty$. The

Luxemburg norm $\|\cdot\|_{1,p}$, which corresponds to the modular $\rho_{1,p}$ is equivalent to the Luxemburg norm resulting from replacing ρ_p with ρ_p^*, defined as

$$\rho_{1,p}^*(u) := \rho_p^*(u) + \rho_p^*(|\nabla u|) = \int_\Omega |u(x)|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

□

Remark 3.2. It is well known (and easy to verify) that if p is constant on Ω, the variable exponent function spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ coincide with their classical counterparts.

It is at this point plain that given a modular vector space X_ρ, two modes of convergence are automatically coexisting. Definition 3.3 clarifies this point.

Definition 3.3. *[13] Let ρ be a convex modular function defined on the vector space X_ρ. The following notations will be used for convergence:*

(i) $x_k \xrightarrow{\rho} x$ iff $\rho(x_k - x) \to 0$ as $k \to \infty$ (modular convergence);

(ii) $x_k \xrightarrow{\|\cdot\|_\rho} x$ iff $\|x_k - x\|_\rho \to 0$ as $k \to \infty$ (norm convergence);

for any sequence (x_k) in X_ρ.

The two convergence concepts are equivalent if and only if ρ satisfies the following condition (known as the Δ_2-condition): For any sequence $(x_k) \subset X_\rho$, the implication

$$\lim_{k\to\infty} \rho(x_k) = 0 \implies \lim_{k\to\infty} \rho(2x_k) = 0$$

holds.

For further discussions on the Δ_2-condition, its significance, and related variants, refer to [13, 30]. In the particular case of the modular spaces $\ell^{p(\cdot)}$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, the Δ_2-condition holds if and only if there exists a positive constant K such that

$$\rho(2x) \le K\rho(x), \quad \text{for all } x \in X_\rho.$$

The latter is equivalent to the boundedness of exponent $p(\cdot)$ on Ω [9, 19]. In this work, the following notation will consistently be used:

$$p_- := \inf_{x\in\Omega} p(x) \text{ and } p_+ := \sup_{x\in\Omega} p(x).$$

Thus, the Δ_2 conditions holds if and only if $p_+ < \infty$.

The modular itself establishes a topology on X_ρ (referred to as the modular topology, whose open sets will be detailed shortly) via the modular convergence discussed earlier. Consequently, if $p(\cdot)$ is unbounded, the modular topology within any modular vector space such as $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ differs from, and is strictly weaker than, the norm topology. To introduce the solution spaces for the variable exponent $p(\cdot)$-Laplacian (Definition 3.8), a succinct description of the modular topology and a brief discussion of its key properties will be presented.

Definition 3.4. *[13] The open modular ball $B_\rho^o(x,r)$, centered at $x \in X_\rho$ with radius $r > 0$, is defined as*

$$B_\rho^o(x,r) = \{y \in X_\rho;\ \rho(x-y) < r\}.$$

Analogously, the set $B_\rho(x,r)$, defined by

$$B_\rho(x,r) = \{y \in X_\rho;\ \rho(x-y) \le r\}$$

will be referred to as the closed modular ball centered at $x \in X_\rho$ with radius $r > 0$.

As shall be apparent in the sequel, the modular topology can be entirely described by modular balls. The concept of modularly closed subsets in a modular space has been introduced in the literature, specifically:

Definition 3.5. *[13, 30] A subset C of X_ρ is said to be ρ-closed if, for any sequence (x_k) in C that ρ-converges to x, it holds that $x \in C$.*

The terminology is deliberate. Indeed, the family of all complements of ρ-closed subsets of X_ρ forms a topology on X_ρ.

Definition 3.6. *A subset $B \subseteq X_\rho$ is defined to be ρ-open if and only if $X_\rho \setminus B$ is ρ-closed.*

The following result characterizes the ρ-open subsets:

Proposition 3.1. *Let A be a subset of X_ρ. The following statements are equivalent*

(1) A is ρ-closed;
(2) for any x in $A^c = X_\rho \setminus A$, there exists $\varepsilon > 0$ such that

$$B^o_\rho(x,\varepsilon) \subset A^c = X_\rho \setminus A,$$

i.e., $B^o_\rho(x,\varepsilon) \cap A = \emptyset$.

The proof is straightforward. Consequently, the ρ-open subsets of X_ρ can be characterized through the following proposition; its proof is elementary and will be omitted.

Proposition 3.2. *A subset B of X_ρ is ρ-open if and only if for any $x \in B$, there exists $\varepsilon > 0$ such that $B^o_\rho(x,\varepsilon) \subset B$.*

It is simple to confirm that the set of modularly open subsets of X_ρ constitutes a topology, which in the sequel will be denoted by τ_ρ and referred to as the modular topology. It is well established that modularly closed ρ-balls are ρ-closed, provided the modular ρ satisfies the Fatou property, a condition satisfied by modulars ρ_p and $\rho_{1,p}$. In precise terms, a modular ρ satisfies the Fatou property if, whenever (y_n) ρ-converges to $y \in X_\rho$, the inequality

$$\rho(x-y) \le \liminf_{n\to\infty} \rho(x-y_n)$$

holds for any $x \in X_\rho$.

In what follows, the modular topologies associated with ρ_p and $\rho_{1,p}$ will be denoted by τ_p and $\tau_{1,p}$, respectively.

Definition 3.7. *[13, 30] Let C be a subset of X_ρ. The modular closure of C, denoted $\overline{C}^\rho$, is defined as the intersection of all ρ-closed subsets of X_ρ that contain C.*

Remark 3.3. As established in the literature (see [13, 30]), it is well known that

$$A \subset \overline{A} \subset \overline{A}^\rho,$$

for any subset A of X_ρ, where $\overline{A}$ represents the closure of A under the Luxemburg-norm topology. As previously discussed, these closures coincide if the modular satisfies the Δ_2-condition. However, they typically differ, as is the case of τ_p and $\tau_{1,p}$, when $p(\cdot)$ is unbounded.

Proposition 3.3. *[17] The following properties are valid:*

(1) If U is a ρ-open subset of X_ρ, then $U + x = \{u + x; u \in U\}$ is also ρ-open, for any $x \in X_\rho$. Thus, $U + V$ is ρ-open whenever U or V is ρ-open.
(2) θU is ρ-open whenever U is ρ-open and $\theta \ge 1$.
(3) For any $x \in \overline{A}^\rho$ and any ρ-open subset U such that $x \in U$, then $U \cap A \neq \emptyset$.
(4) $\overline{A}^\rho$ is convex whenever A is convex.

These properties facilitate the demonstration of the following elegant result:

Proposition 3.4. *[17] $\overline{A}^{\rho}$ is a ρ-closed vector subspace of X_ρ provided A is a vector subspace of X_ρ.*

Recall that $C_0^\infty(\Omega)$ denotes the vector space of infinitely differentiable functions on Ω with compact support.

Definition 3.8. *We denote the $\rho_{1,p}$-closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ by $V_0^{1,p(\cdot)}(\Omega)$ and the $\rho_{1,p}$-closure of the subspace of compactly supported functions in $W^{1,p(\cdot)}(\Omega)$ by $U_0^{1,p(\cdot)}(\Omega)$. Hereafter, $V^{1,p(\cdot)}(\Omega)$ will stand for the set of all $u \in W^{1,p(\cdot)}(\Omega)$ for which $\rho_{1,p}(u) < \infty$.*

The Luxemburg-norm closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ is as usual denoted by $W_0^{1,p(\cdot)}(\Omega)$.

Remark 3.4. It is clear from the preceding definition that $W_0^{1,p(\cdot)}(\Omega) \subseteq V_0^{1,p(\cdot)}(\Omega) \subseteq U_0^{1,p(\cdot)}(\Omega)$. According to Theorem 3.11 in [12], the strict inclusion $V_0^{1,p(\cdot)}(\Omega) \subsetneq U_0^{1,p(\cdot)}(\Omega)$ can occur even when $p_+ < \infty$.

The following result directly follows from Proposition 3.4.

Theorem 3.1. *$V_0^{1,p(\cdot)}(\Omega)$ and $U_0^{1,p(\cdot)}(\Omega)$ are $\rho_{1,p}$-closed real vector subspaces of $W^{1,p(\cdot)}(\Omega)$.*

Next, additional results are presented for the case of Lebesgue variable exponent spaces. As discussed earlier, a subset $C \subset W^{1,p(\cdot)}(\Omega)$ ($C \subset L^{p(\cdot)}(\Omega)$) is $\rho_{1,p}$-closed (ρ_p-closed) if and only if whenever $(x_k) \subseteq C$ and $\rho_{1,p}(x_k - x) \to 0$ ($\rho_p(x_k - x) \to 0$) as $k \to \infty$, it holds that $x \in C$. If $(x_k) \subset L^{p(\cdot)}(\Omega)$ ($W^{1,p(\cdot)}(\Omega)$) and $\lim_{k\to\infty} \rho_p(x_k - x) = 0$ ($\lim_{k\to\infty} \rho_{1,p}(x_k - x)) = 0$), (x_k) is said to ρ_p-converge ($\rho_{1,p}$-converge) to x. This is denoted as $x_k \overset{\rho_p}{\to} x$ ($x_k \overset{\rho_{1,p}}{\to} x$) as $k \to \infty$.

Lemma 3.1. *[19] If $(x_k) \subseteq L^{p(\cdot)}(\Omega)$ ρ_p-converges to $x \in L^{p(\cdot)}(\Omega)$, then there exists a subsequence (x_{k_j}) that converges to x a.e. in Ω.*

Consequently,

Corollary 3.1. *[17] If $(x_k) \subseteq W^{1,p(\cdot)}(\Omega)$ $\rho_{1,p}$-converges to $x \in W^{1,p(\cdot)}(\Omega)$, then there exists a subsequence (x_{k_j}) that converges to x a.e. in Ω and such that $\left(\nabla x_{k_j}\right)$ converges to ∇x a.e. in Ω.*

4. Modular uniform convexity

Modular uniform convexity was initially introduced by Nakano [32] (see also [30]). The original definition closely followed the classical notion of uniform convexity in Banach spaces introduced by Clarkson [8]. However, this definition failed to encompass many interesting modulars in modular vector spaces, as noted in the original work [15].

Definition 4.9. *[13] Let ρ be a convex modular on a vector space X. Fix $r > 0$ and $\varepsilon > 0$. Define*

$$D(r,\varepsilon) := \left\{ (x,y) \in X_\rho^2 \mid \rho(x) \le r,\ \rho(y) \le r,\ \rho\left(\frac{x-y}{2}\right) \ge \varepsilon r \right\}$$

and

$$\delta_2(r,\varepsilon) = \inf\left\{ 1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) \mid (x,y) \in D(r,\varepsilon) \right\}.$$

ρ is said to be uniformly convex $(UC2)$ if for every $r > 0$ and $\varepsilon > 0$, $\delta_2(r,\varepsilon) > 0$.

Note that for any $r > 0$, ε can be selected sufficiently small such that $D(r,\varepsilon)$ is non-empty.

Remark 4.5. *[30, 32]* The original definition of modular uniform convexity differs slightly from that in Definition 4.9. Specifically, we define ρ to be $(UC1)$ if for every $r > 0$ and $\varepsilon > 0$, the following condition holds:

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\left(\frac{x+y}{2}\right);\ \rho(x) \le r,\ \rho(y) \le r,\ \rho\,(x-y) \ge \varepsilon r\right\} > 0.$$

It is noteworthy that if ρ satisfies $(UC1)$, then it also satisfies $(UC2)$. The converse holds if ρ satisfies the Δ_2-condition.

Another form of modular uniform convexity that will be instrumental throughout is defined as follows:

Definition 4.10. *[9] A convex modular ρ on a vector space X is classified as (UC^*) if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $x, y \in X_\rho$:*

$$\rho\left(\frac{x-y}{2}\right) > \varepsilon\,\frac{\rho(x)+\rho(y)}{2} \implies \rho\left(\frac{x+y}{2}\right) \le \Big(1-\delta(\varepsilon)\Big)\,\frac{\rho(x)+\rho(y)}{2}.$$

It can be readily verified that if ρ satisfies (UC^*), then ρ also satisfies $(UC2)$. Next the modular uniform convexity in $\ell^{p(\cdot)}$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ will be investigated.

4.1. Modular uniform convexity of $\ell^{p(\cdot)}$ and $L^{p(\cdot)}(\Omega)$.

To examine the uniform convexity of the modular in $\ell^{p(\cdot)}$ and $L^{p(\cdot)}(\Omega)$, the classical inequalities established by Clarkson [8] and Sundaresan [38] will be employed:

Lemma 4.2. *The following inequalities are valid:*

(a) [38] If $1 \le p \le 2$ and $a, b \in \mathbb{R}$, $|a|+|b| \neq 0$, it holds that

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2^{p+1}}\,\frac{|a-b|^2}{(|a|+|b|)^{2-p}} \le \frac{1}{2}(|a|^p+|b|^p).$$

(b) [8] If $p \ge 2$ and $a, b \in \mathbb{R}$, it holds that

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \frac{1}{2}(|a|^p+|b|^p).$$

An early result concerning modular uniform convexity in variable exponent spaces was established in [4]. The proof of this result will illustrate the main ideas in this section. For completeness, it will be included below.

Theorem 4.2. *[4] For $p : \mathbb{N} \to [1,\infty]$ such that $p^- = \inf_{n\in\mathbb{N}} p(n) > 1$, the modular function $\rho_p : \ell^{p(\cdot)} \to [0,\infty]$ defined by*

$$\rho_p(x) = \rho_p((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)}\,|x_n|^{p(n)}$$

possesses the (UC^*) property.

Proof. Assume that $p^- = \inf_{n\in\mathbb{N}} p(n) > 1$. Fix $\varepsilon > 0$. Pick $x, y \in \ell^{p(\cdot)}$ in such a way that

$$\rho_p\left(\frac{x-y}{2}\right) \ge \varepsilon\,\frac{\rho_p(x)+\rho_p(y)}{2} = \varepsilon\, r,$$

where $r = (\rho_p(x)+\rho_p(y))/2$. Without loss of generality, assume $r > 0$. Since ρ_p is convex, one has

$$\varepsilon\,\frac{\rho_p(x)+\rho_p(y)}{2} \le \rho_p\left(\frac{x-y}{2}\right) \le \frac{\rho_p(x)+\rho_p(y)}{2}.$$

It follows that $\varepsilon \leq 1$. Now let $I = \{n \in \mathbb{N}; p(n) \geq 2\}$ and $J = \mathbb{N} \setminus I$. For any subset K of $\mathbb{N}$, set

$$\rho_{p,K}(x) = \rho_{p,K}((x_n)) = \sum_{n \in K} \frac{1}{p(n)} \, |x_n|^{p(n)}.$$

If $K = \emptyset$, write $\rho_{p,K}(x) = 0$. Note that $\rho_p(z) = \rho_{p,I}(z) + \rho_{p,J}(z)$ for any $z \in \ell^{p(\cdot)}$. It is clear from the assumptions that either $\rho_{p,I}((x-y)/2) \geq r\varepsilon/2$ or $\rho_{p,J}((x-y)/2) \geq r\varepsilon/2$. Suppose that $\rho_{p,I}((x-y)/2) \geq r\varepsilon/2$. Then Lemma 4.2 yields

$$\rho_{p,I}\left(\frac{x+y}{2}\right) + \rho_{p,I}\left(\frac{x-y}{2}\right) \leq \frac{\rho_{p,I}(x) + \rho_{p,I}(y)}{2},$$

which implies that

$$\rho_{p,I}\left(\frac{x+y}{2}\right) \leq \frac{\rho_{p,I}(x) + \rho_{p,I}(y)}{2} - \frac{r\varepsilon}{2}.$$

The inequality

$$\rho_{p,J}\left(\frac{x+y}{2}\right) \leq \frac{\rho_{p,J}(x) + \rho_{p,J}(y)}{2},$$

yields

$$\rho_p\left(\frac{x+y}{2}\right) \leq \frac{\rho_p(x) + \rho_p(y)}{2} - \frac{r\varepsilon}{2} \leq \left(1 - \frac{\varepsilon}{2}\right) \, \frac{\rho_p(x) + \rho_p(y)}{2}.$$

On the other hand, if one assumes $\rho_{p,J}((x-y)/2) \geq r\varepsilon/2$, then setting $C = \varepsilon/4$,

$$J_1 = \Big\{n \in J;\ |x_n - y_n| \leq C(|x_n| + |y_n|)\Big\} \ \ and \ \ J_2 = J \setminus J_1,$$

it follows that

$$\rho_{p,J_1}\left(\frac{x-y}{2}\right) \leq \sum_{n \in J_1} \frac{C^{p(n)}}{p(n)} \, \left|\frac{|x_n| + |y_n|}{2}\right|^{p(n)} \leq \frac{C}{2} \sum_{n \in J_1} \frac{|x_n|^{p(n)} + |y_n|^{p(n)}}{p(n)},$$

because the power function is convex and $C \leq 1$. Hence

$$\rho_{p,J_1}\left(\frac{x-y}{2}\right) \leq \frac{C}{2}\Big(\rho_{p,J_1}(x) + \rho_{p,J_1}(y)\Big) \leq \frac{C}{2}\Big(\rho_p(x) + \rho_p(y)\Big) = Cr.$$

Since $\rho_{p,J}((x-y)/2) \geq r\varepsilon/2$ it is readily seen that

$$\rho_{p,J_2}\left(\frac{x-y}{2}\right) = \rho_{p,J}\left(\frac{x-y}{2}\right) - \rho_{p,J_1}\left(\frac{x-y}{2}\right) \geq \frac{r\varepsilon}{2} - Cr.$$

For any $n \in J_2$, it holds that

$$p^- - 1 \leq p(n)(p(n) - 1) \ \ and \ \ C \leq C^{2-p(n)} \leq \left|\frac{x_n - y_n}{|x_n| + |y_n|}\right|^{2-p(n)},$$

which implies by Lemma 4.2 that

$$\left|\frac{x_n + y_n}{2}\right|^{p(n)} + \frac{(p^- - 1)C}{2} \, \left|\frac{x_n - y_n}{2}\right|^{p(n)} \leq \frac{1}{2}\Big(|x_n|^{p(n)} + |y_n|^{p(n)}\Big).$$

Hence

$$\rho_{p,J_2}\left(\frac{x+y}{2}\right) + \frac{(p^- - 1)C}{2} \, \rho_{p,J_2}\left(\frac{x-y}{2}\right) \leq \frac{\rho_{p,J_2}(x) + \rho_{p,J_2}(y)}{2}$$

and this yields

$$\rho_{p,J_2}\left(\frac{x+y}{2}\right) \leq \frac{\rho_{p,J_2}(x) + \rho_{p,J_2}(y)}{2} - r \, \frac{(p^- - 1)\varepsilon^2}{8}$$

because $C = \varepsilon/4$. Thus,

$$\rho_p\left(\frac{x+y}{2}\right) \leq r - r\,\frac{(p^- - 1)\varepsilon^2}{8} = \left(1 - \frac{(p^- - 1)\varepsilon^2}{8}\right)\,\frac{\rho_p(x)+\rho_p(y)}{2}.$$

Finally, writing

$$\delta(\varepsilon) = \min\left(\frac{\varepsilon}{2}, (p^- - 1)\frac{\varepsilon^2}{8}\right) > 0,$$

it is clear that

$$\rho_p\left(\frac{x+y}{2}\right) \leq \Big(1 - \delta(\varepsilon)\Big)\,\frac{\rho_p(x)+\rho_p(y)}{2}.$$

Therefore ρ_p is (UC^*) which completes the proof of Theorem 4.2. □

An analogous argument yields the following theorem:

Theorem 4.3. *[3] Let $\Omega \subset \mathbb{R}^n$ be a nonempty measurable subset. Let $p : \Omega \longrightarrow [1,\infty]$ measurable such that $p(x) < \infty$ a.e. and $p^- = \inf_{x\in\Omega} p(x) > 1$. The modular function $\rho_p : L^{p(\cdot)}(\Omega) \longrightarrow [0,\infty]$, defined by*

$$\rho_p(u) = \int_\Omega \frac{1}{p(x)}\,|u(x)|^{p(x)}\,dx,$$

satisfies the (UC^*) property.

Remark 4.6. Theorem 4.3 yields a radical improvement over what is known for the Banach space structure of $L^{p(\cdot)}(\Omega)$. The uniform convexity of the Luxemburg norm (Definition 3.2) is known to be *equivalent* to the condition $1 < p_- \leq p_+ < \infty$, see [23].

4.2. **Modular uniform convexity of $W^{1,p(\cdot)}(\Omega)$.**

The case for the uniform convexity of variable exponent Sobolev spaces is much more subtle than that of the Lebesgue spaces and necessitates more delicate considerations. Let $\Omega \subset \mathbb{R}^n$, a nonempty measurable subset. The spaces $\mathscr{V}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are introduced in Example 3.3. Recall the definition of the modular function $\rho_{1,p}$:

$$\rho_{1,p}(u) = \rho_p(u) + \rho_p(|\nabla u|) = \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx,$$

where $|\nabla u|$ stands for the Euclidean norm of the gradient of u.
To demonstrate the (UC^*) property of $\rho_{1,p}$, a variant of Lemma 4.2 where the parameters a and b are replaced by vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ has to be established. The proof of this vector form of Lemma 4.2 is rather involved and we refer the reader to [5, 18] for the details it involves. For $\mathbf{a} \in \mathbb{R}^n$, the notation $|\mathbf{a}|$ represents the Euclidean norm of the vector $\mathbf{a}$.

Lemma 4.3. *[5, 18] The subsequent inequalities are valid:*

(a) If $1 \leq p \leq 2$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $|\mathbf{a}| + |\mathbf{b}| \neq 0$, it holds that

$$\left|\frac{\mathbf{a}+\mathbf{b}}{2}\right|^p + \frac{p(p-1)}{2^{p+1}}\frac{|\mathbf{a}-\mathbf{b}|^2}{(|\mathbf{a}|+|\mathbf{b}|)^{2-p}} \leq \frac{1}{2}(|\mathbf{a}|^p + |\mathbf{b}|^p).$$

(b) If $p \geq 2$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, it holds that

$$\left|\frac{\mathbf{a}+\mathbf{b}}{2}\right|^p + \left|\frac{\mathbf{a}-\mathbf{b}}{2}\right|^p \leq \frac{1}{2}(|\mathbf{a}|^p + |\mathbf{b}|^p).$$

Arguing as in the proof of Theorem 4.3, the following Lemma can be proved:

Lemma 4.4. *[5, 18] Let $\Omega \subset \mathbb{R}^n$ be a nonempty measurable subset. Consider a measurable function $p : \Omega \longrightarrow [1,\infty]$ such that $p(x) < \infty$ a.e., and assume $p^- = \inf_{x\in\Omega} p(x) > 1$. The pseudo-modular function $\rho : L^{p(\cdot)}(\Omega) \longrightarrow [0,\infty]$ defined as*

$$\rho(u) = \rho_p(|\nabla u|) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx,$$

satisfies the (UC^*) property.

Theorem 4.3 in conjunction with Lemma 4.4 yields the following result:

Theorem 4.4. *[5, 18] Let $\Omega \subset \mathbb{R}^n$ be a nonempty measurable subset. Consider a measurable function $p : \Omega \longrightarrow [1,\infty]$ such that $p(x) < \infty$ a.e., and assume $p^- = \inf_{x\in\Omega} p(x) > 1$. The modular function $\rho_{1,p} : W^{1,p(\cdot)}(\Omega) \longrightarrow [0,\infty]$, defined by*

$$\rho_{1,p}(u) = \rho_p(u) + \rho_p(|\nabla u|) = \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)} dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx,$$

satisfies the (UC^*) property.

Proof. Let $\varepsilon > 0$ and $u, v \in W^{1,p(\cdot)}(\Omega)$ such that

$$\rho_{1,p}\left(\frac{u-v}{2}\right) \geq \varepsilon\ \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2}.$$

It is clear by definition of $\rho_{1,p}$ that

$$\rho_p\left(\frac{u-v}{2}\right) \geq \frac{\varepsilon}{2}\ \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2} \quad \text{or} \quad \rho_p\left(\frac{\nabla(u)-\nabla(v)}{2}\right) \geq \frac{\varepsilon}{2}\ \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2}.$$

Without loss of generality, assume that

$$\rho_p\left(\frac{u-v}{2}\right) \geq \frac{\varepsilon}{2}\ \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2}.$$

Based on Theorem 4.3, one can assert the existence of $\delta > 0$, which depends solely on ε, such that

$$\rho_p\left(\frac{u+v}{2}\right) \leq (1-\delta)\ \frac{\rho_p(u)+\rho_p(v)}{2}.$$

Thus

$$\begin{aligned}
\rho_{1,p}\left(\frac{u+v}{2}\right) &= \rho_p\left(\frac{u+v}{2}\right) + \rho_p\left(\frac{\nabla(u)+\nabla(v)}{2}\right) \\
&\leq (1-\delta)\ \frac{\rho_p(u)+\rho_p(v)}{2} + \frac{\rho_p(\nabla(u))+\rho_p(\nabla(v))}{2} \\
&= \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2} - \delta\ \frac{\rho_p(u)+\rho_p(v)}{2} \\
&\leq \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2} - \delta\ \frac{\rho_p(u-v)}{2} \\
&\leq \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2} - \frac{\delta\varepsilon}{2}\ \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2} \\
&= \left(1-\frac{\delta\varepsilon}{2}\right)\ \frac{\rho_{1,p}(u)+\rho_{1,p}(v)}{2},
\end{aligned}$$

which completes the proof of Theorem 4.4. □

5. Applications to boundary value problems

In this section, the modular concepts previously developed will be applied to deal with the solvability of the problem (1.2).

5.1. **Definitions and set up.**

The first task is to recall the definition of a weak solution to the problem (1.2). This is standard:

Definition 5.11. *A function $w \in W^{1,p(\cdot)}(\Omega)$ is said to be a weak solution to problem (1.2) iff for each $h \in C_0^\infty(\Omega)$, w satisfies the equality*

$$\int_\Omega |\nabla w(x)|^{p(x)-2} \nabla w(x) \nabla h(x)\, dx = 0 \tag{5.3}$$

and the boundary condition (see Definition 3.8)

$$u - \varphi \in V_0^{1,p(\cdot)}(\Omega). \tag{5.4}$$

Recall from Example 3.3 that for a nonempty domain $\Omega \subseteq \mathbb{R}^n$

$$|\nabla u| = \left(\sum_{k=1}^{n} \left(\frac{\partial u}{\partial x_k} \right)^2 \right)^{\frac{1}{2}}$$

and that one has the (UC^*) convex modular (Theorem 4.4) $\rho_{1,p} : W^{1,p(\cdot)}(\Omega) \to [0,\infty]$ given by

$$\rho_{1,p}(u) = \rho_p(u) + \rho_p(|\nabla u|) = \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx + \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)} dx.$$

The objective of this section is to prove the following theorem:

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $p : \Omega \to \mathbb{R}$ be a continuous function such that $p(x) < \infty$ a.e. and that $n < p_- = \inf_{x \in \Omega} p(x)$. Let $\varphi \in W^{1,p(\cdot)}(\Omega)$ satisfy $\int_\Omega p(x)^{-1} |\nabla \varphi(x)|^{p(x)} dx < \infty$. Then there exists a unique weak solution $w \in W^{1,p(\cdot)}(\Omega)$ to the Dirichlet problem*

$$\begin{cases} \Delta_{p(\cdot)}(w) = div\left(|\nabla w|^{p(\cdot)-2} \nabla w \right) = 0 & in\ \Omega, \\ w|_{\partial\Omega} = \varphi, \end{cases}$$

that satisfies the condition $\int_\Omega (p(x))^{-1}\, |\nabla w(x)|^{p(x)}\, dx < \infty$ and such that the inequality

$$\int_\Omega |\nabla w(x)|^{p(x)-2}\, \nabla w(x)\, \nabla (w + v - \varphi)(x)\, dx \le 0$$

holds for every $v \in V_0^{1,p(\cdot)}(\Omega)$ such that $\int_\Omega (p(x))^{-1}\, |\nabla (v - \varphi)(x)|^{p(x)}\, dx < \infty$.

In what follows, we set out to sketch the proof of this result.

5.2. **The Dirichlet energy integral.**

Let φ be as specified in the previous statement. The proof of Theorem 5.5 relies on the minimization of the Dirichlet integral $\mathscr{D} : V_0^{1,p(\cdot)}(\Omega) \to [0,\infty]$ defined by

$$\mathscr{D}(u) = \int_\Omega \frac{|\nabla (u - \varphi)(x)|^{p(x)}}{p(x)}\, dx. \tag{5.5}$$

It is clear that that $\mathscr{D}$ is bounded below and since $\mathscr{D}(0) = \int_\Omega \frac{|\nabla \varphi(x)|^{p(x)}}{p(x)}\, dx < \infty$, it must hold

$$\inf\left\{ \mathscr{D}(u);\ u \in V_0^{1,p(\cdot)}(\Omega) \right\} < \infty.$$

Moreover, $\mathscr{D}$ is differentiable in the following sense:

Lemma 5.5. *For $v \in W^{1,p(\cdot)}(\Omega)$ and $h \in C_0^\infty(\Omega)$ it holds that*

$$\lim_{t\to 0^+} \frac{\mathscr{D}(v+th)-\mathscr{D}(v)}{t} = \int_\Omega |\nabla(v-\varphi)(x)|^{p(x)-2}\nabla(v-\varphi)(x)\nabla h(x)\,dx.$$

Proof. The right-hand side is finite since h has compact support and p is continuous in Ω. The Lemma follows from an elementary estimate of $\mathscr{D}(v+th)-\mathscr{D}(v)$ and Lebesgue's dominated convergence theorem. Details are left to the reader (see [26]). □

Corollary 5.2. *If $u_0 \in V_0^{1,p(\cdot)}(\Omega)$ is a minimizer of $\mathscr{D}$ then u_0 it must satisfy (5.3) for every $h \in C_0^\infty(\Omega)$.*

Proof. The proof follows immediately by contradiction. □

It is worth observing at this point that the functional $\mathscr{D}$ is convex on $V_0^{1,p(\cdot)}(\Omega)$. Since $V_0^{1,p(\cdot)}(\Omega)$ is a linear subspace of $W_0^{1,p(\cdot)}(\Omega)$, for any $v \in V_0^{1,p(\cdot)}(\Omega)$, $t \in \mathbb{R}$ and $w \in C_0^\infty(\Omega)$ it holds that $tu_0 + (1-t)w \in V_0^{1,p(\cdot)}(\Omega)$. Assuming $u_0 \in V_0^{1,p(\cdot)}(\Omega)$ is a minimizer of $\mathscr{D}$, fix $u_0 \neq \omega \in C_0^\infty(\Omega)$ (hence, $\mathscr{D}(w) < \infty$). Then, for $(t_j) \subset (0,1)$,

$$\mathscr{D}(u_0) \le \mathscr{D}((1-t_j)u_0 + t_j w) \le (1-t_j)\mathscr{D}(u_0) + t_j\mathscr{D}(w).$$

Let $t_j \to 0$, to obtain $u_j = (1-t_j)u_0 + t_j w \overset{\rho_p}{\to} u_0$ and $\mathscr{D}(u_j) \to \mathscr{D}(u_0)$. In conclusion, there are no isolated minimizers of $\mathscr{D}$ on $V_0^{1,p(\cdot)}(\Omega)$.

A final observation is indispensable in the discussion of the minimization of $\mathscr{D}$:

Lemma 5.6. *[17] The following inclusion is modularly continuous:*

$$\mathscr{I} : \left(V_0^{1,p(\cdot)}(\Omega), \tau_{1,p}\right) \hookrightarrow \left(W_0^{1,p_-}(\Omega), \tau_{1,p_-}\right);$$

recall that $p_- = \inf_{x\in\Omega} p(x)$.

Proof. We sketch the proof, we refer the reader to [17] for the details.
It is shown in [17] that the inclusion $i_{p,p_-} : W^{1,p}(\Omega) \hookrightarrow W^{1,p_-}(\Omega)$ is modularly continuous. Consequently, if $K \subseteq W^{1,p_-}(\Omega)$ is norm- closed and contains $C_0^\infty(\Omega)$, the set $i_{p,p_-}^{-1}(K)$ must be $\rho_{1,p}$-closed and $C_0^\infty(\Omega) \subseteq i_{pp_-}^{-1}(K)$. Since p_- is constant, the family of norm-closed subsets of $W^{1,p_-}(\Omega)$ coincides with the class of modularly-closed subsets. By definition one concludes that $V_0^{1,p}(\Omega) \subseteq K \cap W^{1,p}(\Omega) \subseteq K$. Thus $V_0^{1,p}(\Omega)$ is contained in any norm-closed subset of $W^{1,p_-}(\Omega)$ that contains $C_0^\infty(\Omega)$. Hence,

$$V_0^{1,p}(\Omega) \subseteq W_0^{1,p_-}(\Omega). \tag{5.6}$$

□

Lemma 5.6 has the following profound implication:

Lemma 5.7. *[17] Any sequence $(v_j) \subset V_0^{1,p(\cdot)}(\Omega)$, with (∇v_j) ρ_p-Cauchy in $\left(L^{p(\cdot)}(\Omega)\right)^n$, must ρ_p-converge in $L^{p(\cdot)}(\Omega)$ to a function $v \in V_0^{1,p(\cdot)}(\Omega)$. Therefore, (v_j) must converge to $v \in V_0^{1,p(\cdot)}(\Omega)$ in the modular topology of $W^{1,p(\cdot)}(\Omega)$.*

Theorem 5.6 is the key point in the existence of the solution to the problem (1.2).

Theorem 5.6. *Under the hypotheses of Theorem 5.5, the functional $\mathscr{D}$ possesses a unique minimizer $u \in V_0^{1,p(\cdot)}(\Omega)$.*

Proof. We sketch the proof of this result, referring the reader to [17] for the details. Since $\mathscr{D}$ has a finite lower bound, one can extract a minimizing sequence $(u_j) \subset V_0^{1,p(\cdot)}(\Omega)$. An argument based on the uniform convexity of $\mathscr{D}$ (Lemma 4.4) yields that $\left(\frac{\nabla u_j}{2}\right)$ is ρ_p-Cauchy in $\left(L^{p(\cdot)}(\Omega)\right)^n$. On account of Lemma 5.7, $\left(\frac{u_j}{2}\right)$ must $\rho_{1,p}$-converge to a function $u \in W^{1,p(\cdot)}(\Omega)$. Since $V_0^{1,p(\cdot)}(\Omega)$ is closed in the modular topology of $W^{1,p(\cdot)}(\Omega)$, it follows that $u \in V_0^{1,p(\cdot)}(\Omega)$, which is a vector subspace of $W^{1,p(\cdot)}(\Omega)$ and hence $2u \in V_0^{1,p(\cdot)}(\Omega)$. Because of Lemma 3.1, no generality is lost by assuming that $\left(\frac{\nabla u_j}{2}\right)$ converges pointwise *a.e.* to ∇u. Fatou's Lemma then yields

$$\int_\Omega |\nabla(2u-\varphi)(x)|^{p(x)}\,dx \le \liminf_{j\to\infty} \int_\Omega \left|\nabla(u_j-\varphi)(x)\right|^{p(x)}\,dx.$$

It will be shown that $2u$ minimizes $\mathscr{F}$ in K. Indeed,

$$\begin{aligned} d &\le \int_\Omega \frac{|\nabla(\varphi-2u)(x)|^{p(x)}}{p(x)}dx \le \liminf_{k\to\infty} \int_\Omega \frac{\left|\nabla\left(\varphi-\left(\frac{u_k}{2}+u\right)\right)(x)\right|^{p(x)}}{p(x)}dx \\ &\le \liminf_{k\to\infty}\liminf_{l\to\infty} \int_\Omega \frac{\left|\nabla\left(\varphi-\left(\frac{u_k}{2}+\frac{u_l}{2}\right)\right)(x)\right|^{p(x)}}{p(x)}\,dx \\ &\le \liminf_{k\to\infty}\liminf_{l\to\infty} \frac{1}{2}\left(\int_\Omega \frac{|\nabla(\varphi-u_k)(x)|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{|\nabla(\varphi-u_l)(x)|^{p(x)}}{p(x)}\,dx\right) \\ &\le \liminf_{k\to\infty}\liminf_{l\to\infty} \frac{1}{2}\left(\mathscr{D}(u_k)+\mathscr{D}(u_l)\right) = d. \end{aligned}$$

The last inequality proves the claim. The uniqueness statement follows immediately from the arbitrariness of the minimizing sequence (u_j). □

In addition, the minimizer $2u$ satisfies the following inequality (see [17]):

Theorem 5.7. *For any $v \in V_0^{1,p(\cdot)}(\Omega)$ such that $\int_\Omega (p(x))^{-1}|\nabla(v-\varphi)(x)|^{p(x)}\,dx < \infty$, it holds that*

$$\int_\Omega |\nabla(2u-\varphi)(x)|^{p(x)-2}\,\nabla(2u-\varphi)(x)\,\nabla(v-2u)(x)\,dx \ge 0.$$

The following Corollary is immediate:

Corollary 5.3. *Let $2u$ be the unique minimizer of $\mathscr{D}$ on $V_0^{1,p(\cdot)}(\Omega)$ obtained in Theorem 5.6 and set $w=\varphi-2u$. Then the inequality*

$$\int_\Omega |\nabla w(x)|^{p(x)-2}\,\nabla w(x)\,\nabla(\xi+w-\varphi)(x)\,dx \le 0 \tag{5.7}$$

holds for every $\xi \in V_0^{1,p(\cdot)}(\Omega)$ such that $\int_\Omega (p(x))^{-1}\,|\nabla(\xi-\varphi)(x)|^{p(x)}\,dx < \infty$.

5.3. **Proof of Theorem 5.5.**

Existence in Theorem 5.5 follows immediately from Corollary 5.2 and Theorem 5.6. To see this, observe that if $2u$ is the minimizer obtained in Theorem 5.6, then the function w defined by $w = \varphi-2u$ satisfies (5.3) for every $h \in C_0^\infty(\Omega)$ and $w-\varphi \in V_0^{1,p(\cdot)}(\Omega)$, as required.

For uniqueness observe that by virtue of condition (5.7) any two solutions w_1 and w_2 must satisfy

$$\int_\Omega |\nabla w_1(x)|^{p(x)-2}\nabla w_1(x)\nabla(w_1-w_2)(x)\,dx \le 0 \tag{5.8}$$

and

$$\int_\Omega |\nabla w_2(x)|^{p(x)-2}\nabla w_2(x)\nabla(w_2-w_1)(x)\,dx \le 0.$$

Uniqueness follows from (5.8) coupled with the vector inequality

$$\langle |\mathbf{a}|^{\alpha-2}\mathbf{a} - |\mathbf{b}|^{\alpha-2}\mathbf{b}, \mathbf{a}-\mathbf{b}\rangle \ge \gamma(\alpha)|\mathbf{a}-\mathbf{b}|^{\alpha},$$

valid for all $\mathbf{a}\in\mathbb{R}^n$, $\mathbf{b}\in\mathbb{R}^n$, where $|\cdot|$ stands for the Euclidean norm and $\gamma(\alpha)$ is a positive constant dependig on α [22]. For more details, see [17].

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[1] Department of Mathematics, Khalifa University, Abu Dhabi, UAE
Email address: mohamed.khamsi@ku.ac.ae

[2] Center of Excellence in Theoretical and Computational Science (TaCS-CoE) and KMUTTFixed Point Research Laboratory, Room SCL 802 Fixed Point Laboratory Science Laboratory Building, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.
Email address: poom.kum@kmutt.ac.th

[3] Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, U.S.A
Email address: osmendez@utep.edu