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Cut-through connections of graphs

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ABSTRACT. A 4-regular plane graph G is cut-through connected if any two vertices of G are connected by a cut-through path (that is, the path with the property that every two consecutive edges are not consecutive in local rotation of their common vertex). In this paper, we present the complete characterization of cut-through connected 4-regular plane graphs in terms of Gauss Codes or Extended Gauss Codes.

1. INTRODUCTION

All graphs considered in this paper are plane (that is, drawn in Euclidean plane in the way their edges do not cross), possibly containing multiple edges or loops. We use a standard graph theory terminology according to Bondy and Murty [3]. However, we recall some specialized notation.

An edge-coloring of a graph G is *proper* if any two adjacent edges receive different colors. If G admits a proper coloring, then we say that G is properly colored.

An edge-colored graph G is called *rainbow connected* if any two vertices are connected by a path whose edges have different colors. The concept of rainbow connection in graphs was introduced by Chartrand et al. [7]. There is an extensive research concerning in this area, see e.g. [17, 18, 19, 20, 22, 23, 26].

As a modification of proper colorings and rainbow colorings of graphs, Andrews et al. [2] and independently Borozan et al. [4] introduced the concept of proper connection of graphs, where the edge coloring need not be proper, but any two vertices are connected by a properly colored path; for related results, see e.g. [1, 13, 16, 21, 25].

An edge-colored graph G is called *conflict-free connected* if any two vertices are connected by a path which contains at least one color used on exactly one of its edges. The concept of conflict-free connection was introduced in 2018 by Czap et al. [8] and then studied in several papers, see [5, 6, 24].

Motivated by the above mentioned three concepts and by the fact that all these properties concern colorings, we extend this area by a connection involving a structural property, introducing the cut-through connection:

Let G be a simple 4-regular plane graph and let e_1, e_2, e_3, e_4 be edges incident with a vertex v of G; assume that the indices match the clockwise local rotation of edges in the plane drawing of G. We say that pairs of edges e_1, e_3 and e_2, e_4 *cut-through* the vertex v. A path P in G is *cut-through path* (CT-path for short) if any two consecutive edges of P cut-through their common vertex on P. A graph G is *cut-through connected* if any two vertices of G are connected by a CT-path.

In this paper, we explore cut-through connectivity concept, focusing on the existence of cut-through connected graphs and their characterizations. In Section 2, we recall additional particular notation and prove general results on the existence of cut-through connected graphs as well as present the smallest cut-through Eulerian simple 4-regular

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plane graph (aka knot) which is not cut-through connected. In Section 3, we state the necessary and sufficient condition for a knot to be cut-through connected in terms of its Gauss code and expand this characterization to be applied on all 4-regular simple plane graphs, using the extension of Gauss code. The last Section 4 discusses possibilities to extend the obtained results for generalized cut-through property (called straight-ahead in [28]) in Eulerian plane graphs.

2. Preliminaries

Let $A = \{a, b, c, \ldots\}$ be an alphabet (whose elements are called letters). A *word* (*cyclic word*) is a finite sequence (cyclic sequence, respectively) of letters from A; a *subword* is a subsequence of a word or a cyclic word. For an ordered n-tuple $T = (a_1, a_2, \ldots, a_n)$ of letters from A, let $(a_1, a_2, \ldots, a_n)_A$ be a set of all words starting with a_1, ending with a_n, containing all letters $a_2, \ldots, a_{n-1}$ in the order specified by T such that, for every $i = 1, \ldots, n-1$, there is arbitrary word (including the empty one) over A between a_i and a_{i+1}. For example, the set $(aabc)_A$ contains, among others, the words $aabc$, $a\mathbf{a}ab\mathbf{c}c$ or $a\mathbf{b}a\mathbf{ca}bc$.

First, we state a useful sufficient condition for cut-through connectedness:

Lemma 2.1. *Let G be a 4-regular plane graph. If G contains three cut-through paths such that every vertex v of G is contained in at least two of them, then G is cut-through connected.*

Proof. Let P_1, P_2, and P_3 be above mentioned cut-through paths. Every vertex of G is contained in at least two of these paths; so, by pigeon-hole principle, every pair u, v of vertices is contained in at least one common path P_i. The (u, v)-subpath of P_i is thus the desired cut-through path between u and v. □

Theorem 2.1. *For every integer $n \geq 6, n \neq 7$, there exists cut-through connected 4-regular plane graph on n vertices*

Proof. Consider graphs $D_6, D_8, D_9, D_{10}, D_{11}, D_{12}$, and D_{13} on Figure 1.

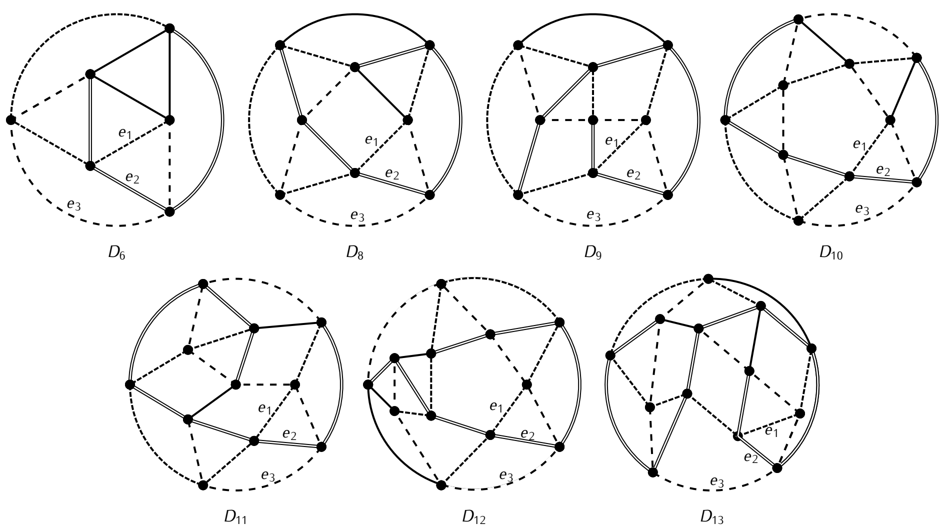

FIGURE 1. The graphs D_6 and D_8 - D_{13}

Each of them contains three cut-through paths satisfying the condition of Lemma 2.1 (dashed, long-dashed, and black-and-white, respectively). Thus, these graphs are cut-through connected, proving the theorem for $n \in S = \{6, 8, 9, 10, 11, 12, 13\}$. Now, let $n = 6p + q, p \geq 1, q \in S$. Modify the graph D_q in the following way: split the edges e_1, e_2, e_3 into half-edges e_1', e_1'', e_2', e_2'' and e_3', e_3'', and insert the configuration H (consisting of six half-edges and the "snake" of $6p - 2$ triangles) of Figure 2 such that the half-edges of H match the ones in D_q.

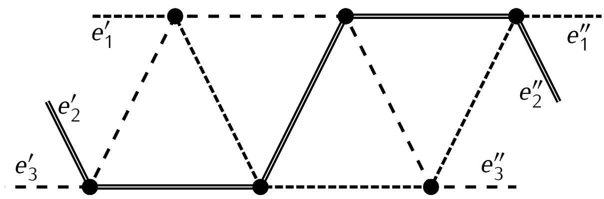

FIGURE 2. The configuration H

The resulting graph again satisfies the conditions of Lemma 2.1, hence, it is cut-through connected.

Note that there is no 4-regular plane simple graph on seven or less than six vertices, hence the constraints on n follow. □

Observe that, for $n \geq 8$, one may actually construct also cut-through connected n-vertex knots. Moreover, it follows that, up to 13 vertices, all knots are cut-through connected:

Lemma 2.2. *The smallest knot which is not cut-through connected has* 14 *vertices.*

Proof. Consider two 4-regular 14-vertex plane graphs on Figure 3; it is easy to verify that they are knots, but none of cut-through paths starting from the vertex x contains y.

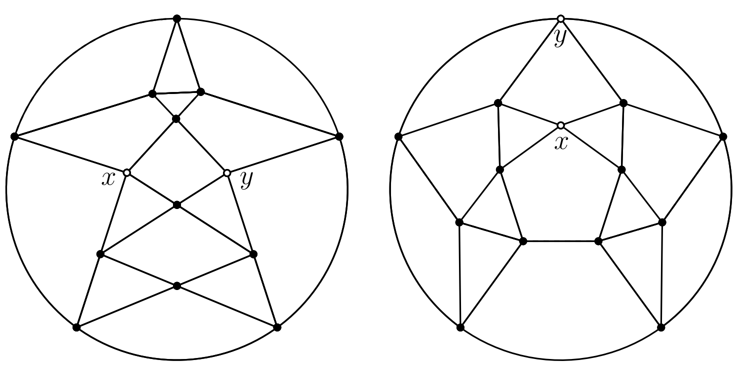

FIGURE 3. The smallest knots which are not cut-through connected

As for the 4-regular plane graphs on at most 13 vertices, we first used `plantri` graph generator to construct those ones which are 3-connected (they are obtained as duals of 3-connected plane quadrangulations; there are 38 of them), and, using the Maple computer algebra system and our custom procedures for handling cut-through property, we have selected from them the ones which are knots; there are altogether 12 of them and all were verified to be cut-through connected. It remains to check the graphs of connectivity at most 2: there are two such graphs on 12 vertices and three ones on 13 vertices, see Figure 4.

Anyway, regardless of their plane drawings, they are easily found to contain a cut-through cycle, hence, they are not knots. □

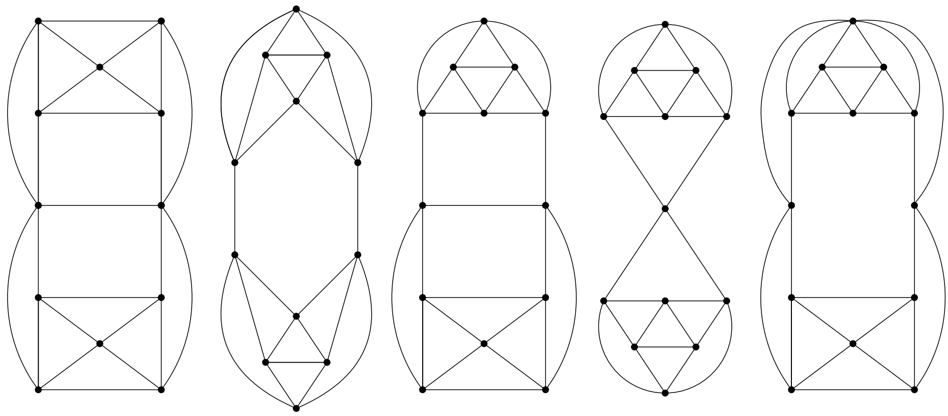

FIGURE 4. 1-connected and 2-connected quartic plane graphs on at most 13 vertices

3. KNOTS AND GAUSS CODES

In this section, we establish the relation between cut-through connected knots and their Gauss codes. Recall that a knot G can be represented by its Gauss diagram (also known as chord diagram) defined by C.F. Gauss [11], which is a graph consisting of cycle (whose vertices represent vertices of G in the order as visited following the cut-through Eulerian trail) and its chords, such that the endvertices of a chord represent the same vertex of G. These diagrams have some interesting properties: for example, every vertex in G is represented by exactly two vertices in Gauss diagram, or any arc-path of the cycle with terminal vertices representing the same vertex in G even number of internal vertices. The complete characterization of Gauss diagrams was made in several ways; for further reading see e.g. [9, 11, 12, 27].

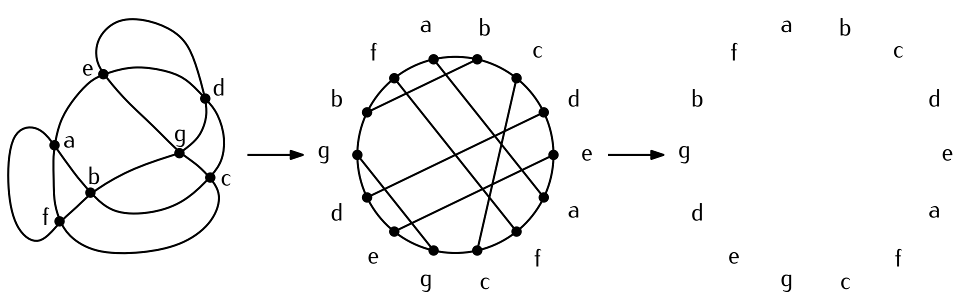

FIGURE 5. From knot to its Gauss diagram and Gauss code

Here, we use the following knot representation: label all vertices of a knot G by letters (from an alphabet). Then we obtain a cyclic word consisting of labels of vertices in the order as we visit them following the cut-through Eulerian trail of G. We call this cyclic word the Gauss code of G (see Figure 5).

All labels of Gauss Code occur twice. For the position of a pair of distinct labels, there are only two possibilities:

Case 1. If there is a subword $w \in (aabb)_A$ of Gauss code of G, we call the corresponding pair of vertices of G a loop-pair. In this case, these two vertices divide the cut-through Eulerian trail of G to $a-a$-, $a-b$-, $b-b$-, and $b-a$-subtrails (in this order).

Case 2. If there is a subword $w \in (abab)_A$ of Gauss Code of G, we call the corresponding pair of vertices of G a trail-pair. In this case, these two vertices divide the cut-through Eulerian trail of G to $a-b$-, $b-a$-, $a-b$-, and $b-a$-subtrails (in this order).

The next theorem gives the necessary and sufficient condition for a knot G to be cut-through connected regarding its Gauss Code:

Theorem 3.2. *Let G be a knot. Then G is cut-through connected if and only if there is no subword $w \in (aabbccdd)_A$ of its Gauss code.*

Proof. Let $w \in (aabbccdd)_A$ be a subword of Gauss code of G. As cut-through paths in G correspond to subsequences of the Gauss code of G, we get that there is no cut-through path between vertices a and c, because the vertex b (or d, respectively) appears on any $a-c$-subtrail twice. Thus there is no (a,c)-cut-through path in G, and G is not cut-through connected.

To prove the converse, assume that knot G is not cut-through connected. It means that there exists some pair of vertices u and v, such that there is no cut-through (u,v)-path in G. We consider two cases.

Case 1. If u, v is loop-pair in G, then there must be some vertex twice on $u-v$- and $v-u$-subtrails of the cut-through Eulerian trail. Let its labels be a and b, respectively. Then there is $w \in (uuaavvbb)_A$ subword of Gauss code of G.

Case 2. If u, v is trail-pair in G, then on every of $u-v$-, $v-u$-, $u-v$-, and $v-u$-subtrails of the cut-through Eulerian trail of G, there is a vertex which appears twice on that subtrail. Let the labels of these vertices (in the order given by traversing the Eulerian trail of G with respect to the u, v-trail-pair) be a, b, c, and d. Then there is $w \in (aabbccdd)_A$ subword of Gauss Code of G. □

Now, let G be a 4-regular plane graph (not necessarily a knot). Let $T_1, T_2, \ldots, T_k$ be a decomposition of $E(G)$ such that, for every $i = 1, 2, \ldots, k$, T_i is a closed trail such that every pair of its consecutive edges cut through their common vertex. From [28], it follows that such a decomposition is uniquely determined. The graph $G[T_i]$ induced by edges of T_i is either a cycle, or subdivision of a 4-regular plane pseudograph; all the subgraphs $G[T_i]$, $i = 1, 2, \ldots, k$ form the *cut-through components* of G, and $ct(G) = \{G[T_i], i = 1, 2, \ldots, k\}$ is the *cut-through decomposition* of G.

Cut-through components of cut-through decomposition can share common vertices. Because $ct(G)$ is uniquely determined, we can represent (in a unique way) the structure of cut-through components in $ct(G)$ by the following graph (denoted by $rep(G)$): its vertices correspond to cut-through components of $ct(G)$, and two vertices are connected by an edge if their corresponding cut-through components have at least one vertex of G in common. We will refer to vertices in $rep(G)$ corresponding to cycles as white vertices; all other vertices of $rep(G)$ are called black (see Figure 6).

It is easy to see that, for connected simple 4-regular plane graph G, it holds

Observation 1: $rep(G)$ is connected.

Now, let G be a cut-through connected simple 4-regular plane graph.

Observation 2: $rep(G)$ contains at most one black vertex.

Proof: Consider two black vertices in $rep(G)$ corresponding to cut-through components B_1 and B_2 from $ct(G)$ which are not cut-through cycles of G. Let v_i, $i = 1, 2$ be a vertex of B_i incident with four edges of B_i. Every cut-through path of G starting in v_1 contains only edges of B_1; thus, there is no cut-through (v_1, v_2)-path, because all four edges incident with v_2 are in B_2 only.

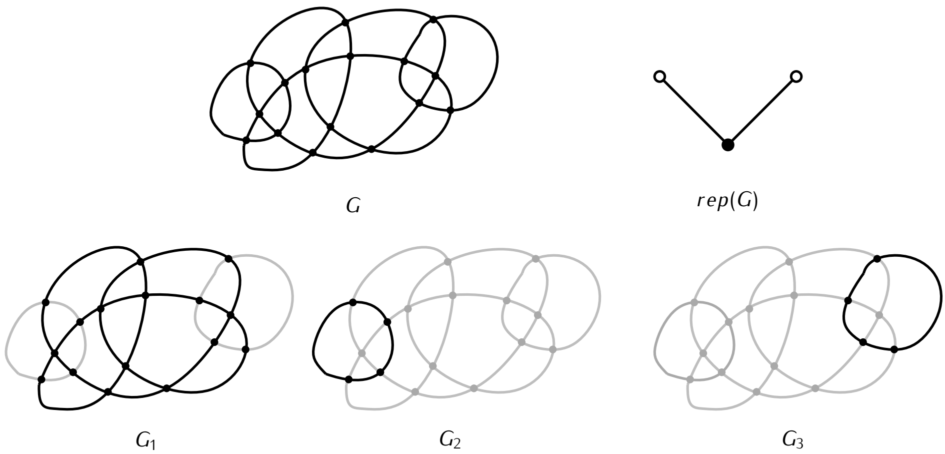

FIGURE 6. Finding cut-through decomposition and $rep(G)$ of G

Observation 3: If $rep(G)$ contains a black vertex, then no two white vertices are adjacent.
Proof: Consider the contrary, and take a black vertex in $rep(G)$ corresponding to a cut-through component B_1, and two adjacent white vertices corresponding to cut-through cycles W_1 and W_2 (hence, W_1 and W_2 have a common vertex v_2). Let v_1 be a vertex of B_1 that is incident with four edges of B_1. Every cut-through path that starts in v_1 contains only edges of B_1, thus, there is no cut-through (v_1, v_2)-path, because all four edges incident with v_2 are only in W_1 or W_2.
Observation 4: $rep(G)$ does not contain two non-adjacent edges with white endvertices.
Proof: Consider four white vertices in $rep(G)$ corresponding to cut-through cycles W_1, W_2, W_3, and W_4 such that W_1 and W_2 (W_3 and W_4, respectively) have a vertex v_1 (v_2, respectively) in common. Every cut-through path that starts in v_1 contains only edges of W_1 or W_2; thus, there is no cut-through (v_1, v_2)-path, because all four edges incident with v_2 belong only to W_3 or W_4.
From all the observations above, one can easily state the following

Corollary 3.1. *Let G be a simple 4-regular plane cut-through connected graph. Then $rep(G)$ is one of the following graphs:*
(i) *3-cycle consisting only of white vertices,*
(ii) *a star consisting only of white vertices,*
(iii) *a star with a central black vertex and white leaves.*

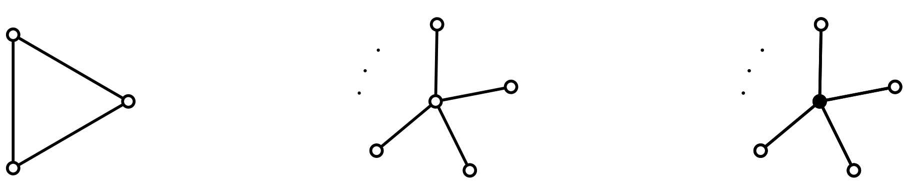

FIGURE 7. Three types of $rep(G)$ for a simple 4-regular cut-through connected G

Now, let G be a simple 4-regular plane graph (not necessarily cut-through connected) such that $rep(G)$ is a 3-cycle on white vertices. Then $ct(G)$ consists of three cut-through cycles and, in G, there exist three cut-through paths obtained from these cycles removing one edge from each. These paths satisfy the condition of Lemma 2.1, thus, G is cut-through connected.

Similarly, if $rep(G)$ is a star on white vertices, the decomposition $ct(G)$ contains a particular cut-through cycle C having common vertices with all other cut-through cycles of G. Moreover, all vertices in G are on C. Thus again, G is cut-through connected.

If $rep(G)$ is a star with a black central vertex, then G (referred lated as type *(iii)* graph) may or may be not cut-through connected. A construction of such G that is not cut-through connected is easy: take a plane drawing of a knot that is not cut-through connected (for example, one of those in Figure 3) and, around its arbitrary vertex, draw concentric cut-through cycles of length four. If the original knot contained two vertices u, v which were not connected by a cut-through path, then, in the obtained 4-regular plane graph, there is no cut-through path between u and v as well.

Similarly, a cut-through connected type *(iii)* graph can be constructed, for example from the multigraph in Figure 8 (on the left) by drawing concentric cut-through cycles of length four around one of its vertices (see Figure 8 on the right).

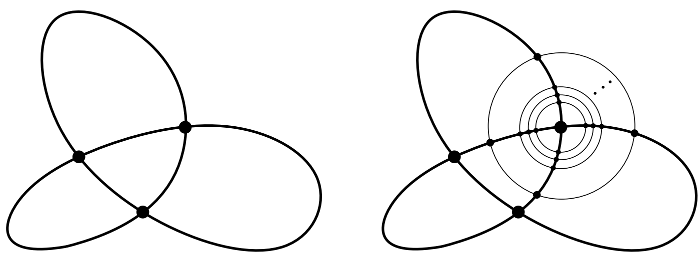

FIGURE 8. A construction of cut-through connected type *(iii)* graph

Consider a simple 4-regular plane graph G of type *(iii)*. Label all vertices of the single non-cycle component of $ct(G)$ by small letters and, for every cut-through cycle of $ct(G)$, label all its vertices by the same capital letter. Then we obtain a cyclic word (called the *extended Gauss Code* of G) consisting of labels of vertices in the order as we visit them traversing (with respect to cut-through property) the non-cycle component of $ct(G)$ (see Figure 9 for illustration).

Now, we are ready to state the sufficient and necessary condition for a general simple 4-regular plane graph to be cut-through connected:

Theorem 3.3. *Let G be a simple 4-regular plane graph of type (iii). Then G is cut-through connected if and only if its extended Gauss Code does not contain any of the following subwords:*

(a) $w_1 \in (aabbccdd)_A$,
(b) $w_2 \in (aabbccA)_A$,
(c) $w_3 \in (aaAbbB)_A$.

Proof. Assume that G is not cut-through connected, that is, there is a pair of vertices u, v of G such that there is no cut-through (u, v)-path.

If both u and v are incident only with edges belonging to non-cycle component of $ct(G)$, then this component itself (taking into account the cut-through property being true also for two edges incident with a vertex of degree 2) is not cut-through connected and, from Theorem 3.1, it follows that there is a subword $w_1 \in (uuaavvbb)_A$ in extended Gauss code of G, because there is no such a cut-through (u, v)-path.

If u is incident only with edges of non-cycle component of $ct(G)$ and v is incident with edges of some cut-through cycle of $ct(G)$ (thus, v is labeled with, say, label A), then there is a subword $w_2 \in (aauubbA)_A$ in extended Gauss code of G.

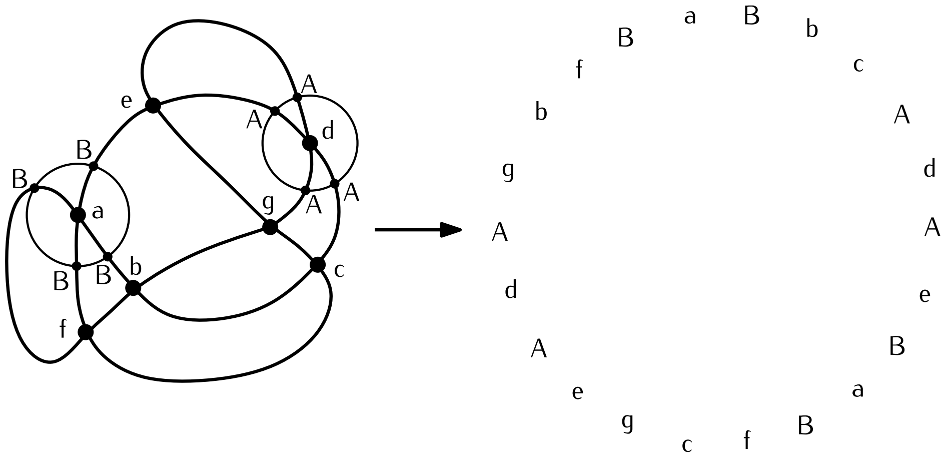

FIGURE 9. A 4-regular plane graph and its extended Gauss code

Finally, suppose that u and v are vertices incident only with edges belonging to cut-through cycles of $ct(G)$ (let u be labeled with A and v with B). Then u and v are not in the same cut-through cycle (because any two vertices contained in a common cut-through cycle are connected by a cut-through path of G). Thus, there is a subword $w_3 \in (aaAbbB)_A$ in extended Gauss code of G.

To prove the converse, suppose there is a subword w in extended Gauss code of G belonging to $(aabbccdd)_A$, $(aabbccA)_A$, or $(aaAbbB)_A$. It is easy to see, that, in the case *(a)*, there is no cut-through path between vertices labeled with a and b, and similarly, no such a path between any two vertices labeled by b and A (or A and B) in case *(b)* or case *(c)*, respectively. Hence, G is not cut-through connected. □

4. CONCLUDING REMARKS

The concept of cut-through neighborhood (also cut-through connectivity) can be considered not only for 4-regular plane graphs but in general for Eulerian plane graphs (by taking the "opposite" edge to an edge in the rotation around their common vertex). Consequently, for an Eulerian plane graph with a single cut-through component (a kind of "hyperknot"), an analogy of the Gauss code can be considered. Unfortunately, it seems that neither necessary nor sufficient conditions are known for a given sequence of vertex labels to be a Gauss code (that the necessary condition for the original Gauss code does not work for general Eulerian plane graphs can be seen, for example, on the graph of Figure 10 – there are occurrences of the same symbol yielding the subsequences of both even and odd length). Moreover, by slight modification of the multigraph from [28], we obtain a planar Eulerian graph which has two different plane drawings with different numbers of cut-through components (see Figure 11); as noted before, this is in sharp contrast with 4-regular plane graphs.

Another generalization of cut-through property (which, in principle, can be considered for graphs of minimum degree at least 4 with a prescribed rotation system) may be based on the assumption that the edges that are cut-through-related – in a generalized sense – are the adjacent edges which are not consecutive in the rotation around their common vertex. This allows more freedom to connect a pair of vertices with a generalized cut-through path (which need not be unique), but also brings complications with the definition of generalized cut-through-decomposition (it need not exist when requiring

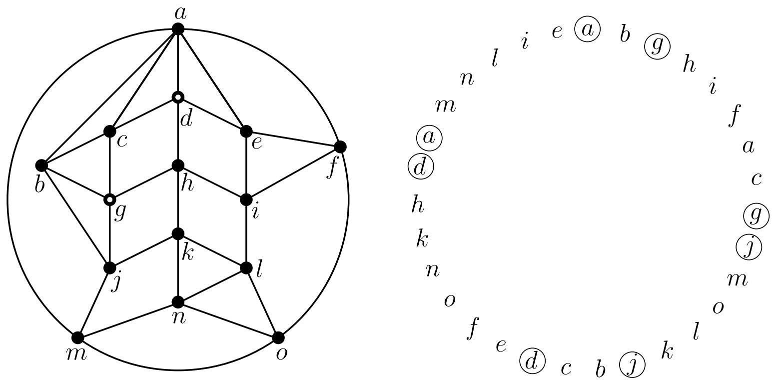

FIGURE 10. Eulerian "hyperknot", which is not cut-through connected, and its Gauss code

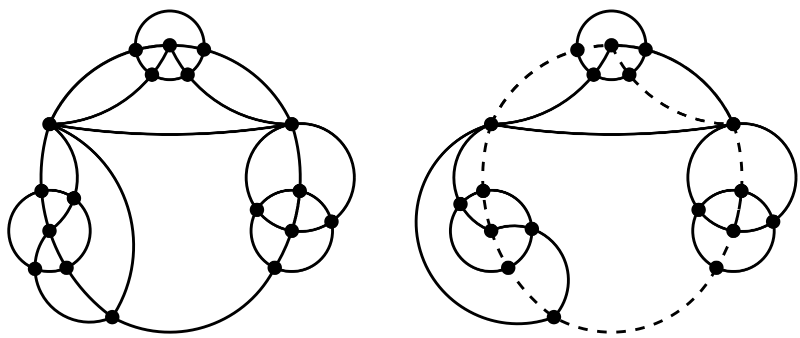

FIGURE 11. Two different plane embeddings of Eulerian planar graph with different numbers of cut-through components

edge-disjoint closed trails); thus, it requires different approaches and further investigation.

ACKNOWLEDGEMENTS

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REFERENCES

[1] Aardt, S.; Brause, C.; Burger, A.; Frick, M.; Kemnitz, A.; Schiermeyer, I. Proper connection and size of graphs. *Discrete Math.* **340** (2017), no 11., 2673–2677. https://doi.org/10.1016/j.disc.2016.09.021

[2] Andrews, E.; Lumduanhom, C.; Laforge, E.; Zhang, P. On proper-path colorings in graphs. *J. Combin. Math. Combin. Comput.* **97** (2016), 189–207.

[3] Bondy, J.; Murty, U. Graph theory. Graduate Texts in Mathematics, 244. *Springer, New York*, 2008. xii+651 pp. ISBN: 978-1-84628-969-9.

[4] Borozan, V.; Fujita, S.; Gerek, A.; Magnant, C.; Manoussakis, Y.; Montero, L.; Tuza, Z. Proper connection of graphs. *Discrete Math.* **312** (2012), no. 17, 2550–2560.

[5] Chang, H.; Doan, T.; Huang, Z.; Jendrol', S.; Li, X.; Schiermeyer, I. Graphs with conflict-free connection number two. *Graphs and Combin.* **34** (2018), no.6, 1553–1563. https://doi.org/10.1007/s00373-018-1954-0

[6] Chang, H.; Huang, Z.; Li, X.; Mao, Y.; Zhao, H. On conflict-free connection of graphs. *Discrete Appl. Math.* **255** (2019), 167–182. https://doi.org/10.1016/j.dam.2018.08.006

[7] Chartrand, G.; Johns, G.; McKeon, K.; Zhang, P. Rainbow connection in graphs. *Math. Bohem.* **133** (2008), no. 1, 85–98.

[8] Czap, J.; Jendrol', S.; Valiska, J. Conflict-free connections of graphs. *Discuss. Math. Graph Theory.* **38** (2018), no. 4, 911-920.

[9] Dehn, M. Über kombinatorische topologie. (German) *Acta Math.* **67** (1936), no. 1, 123–168. https://doi.org/10.1007/BF02401740

[10] Dudek, A.; Frieze, A. M.; Tsourakakis, C. E. Rainbow connection of random regular graphs. *SIAM J. Discrete Math.* **29** (2015), no. 4, 2255–2266. https://doi.org/10.48550/arXiv.1311.2299

[11] Gauss, C. Werke. *Band XIII. Hildesheim: Georg Olms Verlag* (1973). (Reprint of the 1900 original)

[12] Grinblat, A.; Lopatkin, V. On realizabilty of Gauss diagrams and constructions of meanders. *J. Knot Theory Ramifications* **29** (2020), no. 5, 2050031, 27 pp. https://doi.org/10.1142/S0218216520500315

[13] Gu, R.; Li, X.; Qin, Z. Proper connection number of random graphs. *Theoret. Comput. Sci.* **609** (2016),part 2, 336–343. https://doi.org/10.1016/j.tcs.2015.10.017

[14] Harborth, H. Eulerian straight ahead cycles in drawings of complete bipartite graphs. *Discrete Math.* **230** (2001), no 1-3, 45–48.

[15] Harborth, H.; Harborth, M. Straight ahead cycles in drawings of complete graphs. Proceedings of the 7th Fischland Colloquium, I (Wustrow, 1988). *Rostock. Math. Kolloq.* No. 38 (1989), 71–75.

[16] Huang, F.; Li, X.; Wang, S. Proper connection number and 2-proper connection number of a graph. (2015). https://doi.org/10.48550/arXiv.1507.01426

[17] Kamčev, N.; Krivelevich, M.; Sudakov, B. Some remarks on rainbow connectivity. *J. Graph Theory* **83** (2016), no. 4, 372–383.

[18] Frieze, A.; Tsourakakis, C. Rainbow connection of sparse random graphs. *Electron. J. Combin.* **19** (2012), no. 4, Paper 5, 19 pp.

[19] Kemnitz, A.; Schiermeyer, I. Graphs with rainbow connection number two. *Discuss. Math. Graph Theory* **31** (2011), no. 2, 313–320.

[20] Li, X.; Liu, M.; Schiermeyer, I. Rainbow connection number of dense graphs. *Discuss. Math. Graph Theory* **33** (2013), no. 3, 603–611.

[21] Li, X.; Magnant, C. Properly Colored Notions of Connectivity - A Dynamic Survey. *Theory Appl. Graphs* (2015). http://dx.doi.org/10.20429/tag.2015.000102

[22] Li, X.; Shi, Y.; Sun, Y. Rainbow connections of graphs: a survey. *Graphs Combin.* **29** (2013), no. 1, 1–38. http://dx.doi.org/10.1007/s00373-012-1243-2

[23] Li, X.; Sun, Y. Rainbow Connections of Graphs. *Springer New York* (2012). https://link.springer.com/book/10.1007/978-1-4614-3119-0

[24] Li, Z.; Wu, B. On the maximum value of conflict-free verex-connection number of graphs. *Discrete Math. Alg. Appl.* **10** (2017).https://doi.org/10.48550/arXiv.1709.01225

[25] Melville, R.; Goddard, W. Coloring graphs to produce properly colored walks. *Graphs Combin.* **33** (2017), no. 5, 1271–1281. https://doi.org/10.1007/s00373-017-1843-y

[26] Schiermeyer, I. Rainbow connection in graphs with minimum degree three. *Combinatorial algorithms*, 432–437, Lecture Notes in Comput. Sci., 5874, *Springer, Berlin*, 2009. https://doi.org/10.1007/978-3-642-10217-2_42

[27] Sossinsky, A. A classification of generic curves on oriented surfaces. *Russian J.Math. Phys.* **2** (1994), no. 2, 251–260.

[28] Pisanski, T.; Tucker, T.; Žitnik, A. Straight-ahead walks in Eulerian graphs. *Discrete Math.* **281** (2004), no. 1-3, 237–246.

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