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Properties of isocompact spaces in topological groups

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ABSTRACT. This paper's primary purpose is to seek properties of isocompact spaces by topological groups. In this work, we propose and say that a topological space X has the $isoc$ property if each family of isocompact subsets in X is weakly hereditarily closure-preserving. Our first result shows that each T_2 topological group with a locally compact subgroup, where the quotient group is isocompact, is isocompact if it has the $isoc$ property. Our second result provides a necessary and sufficient condition for an ω narrow to be Lindelöf.

1. INTRODUCTION

In studying the compactness of countably compact spaces, Bacon [3] introduced the notion of isocompact space. Due to the importance of isocompact spaces, many topologists used mappings to study isocompact spaces, and several new facts concerning isocompact spaces were achieved. It is well known that we connect various class spaces using mappings as a linkage.

In [3], Bacon showed that if f is a closed compact mapping from a space X into an isocompact space Y, then X is an isocompact space. Since each finite-to-one closed mapping is perfect [15], it follows that if f is a finite-to-one closed mapping from a space X into an isocompact space Y, X is an isocompact space. In [5], Buhagiar and Lin showed that if $f : X \to Y$ is a closed mapping with lindelöf fibres, and X is a strong Σ space, then Y is an isocompact Σ space. In [8], Dube, etc., showed that if $f : X \to Y$ is a proper mapping of locales, and Y is isocompact, then X is isocompact. In [18], Miller showed that if f is a closed continuous from a space X into a T_1 isocompact space Y, and $f^{-1}(y)$ is isocompact in X for each y in Y, then X is isocompact. In addition, if $f : X \to Y$ is a closed continuous mapping, X is a T_2 isocompact wM space, and Y is a regular q space, then Y is a T_2 isocompact wM space [15].

The relationships among these mappings are illustrated in Figure 1.

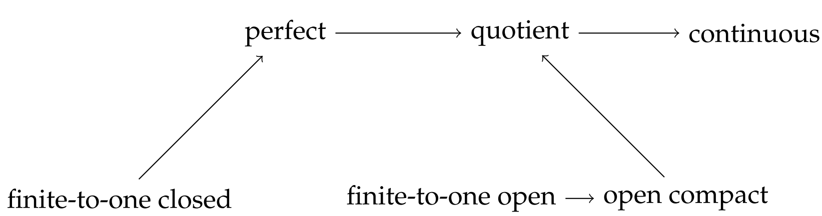

Figure 1 : *Relationships of mappings*

Naturally, we have the following question.

Question 1.1. *Suppose X is a topological space, H is a subspace such that X/H is isocompact, and f is a quotient mapping from X in to X/H. Is X isocompact?*

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However, in [10], Gittings showed that although f is a finite-to-one open mapping from a space X into an isocompact space Y, X is not necessarily an isocompact space. Since each finite-to-one open mapping is a quotient mapping [15], it follows that the answer to Question 1.1 may be negative.

Naturally, we have the following question.

Question 1.2. *If H is a subspace of a space X, under what conditions on H and X/H is X isocompact?*

It is well known that topological properties are transferred from topological group G to quotient group G/H and some others from G/H to G. To address Question 1.2, we will connect isocompact spaces and topological groups. Here are the conclusions regarding the relationships among these topological spaces and quotient groups.

In [2], Arhangel'skii showed that if X is a homogeneous compact space, F is a compact subspace of X, and a is a point of F such that $X/\{a\}$ is isocompact and ω-Lindelöf, then X is first-countable if the tightness of F is countable and the space F is first-countable at a. In [14], Higgins showed that if G is a topological group, and H is a compact subgroup such that the quotient space G/H is compact, then G is compact; hence, it is isocompact. If G is a topological group, and H is a locally compact subgroup such that the quotient space G/H is paracompact, then G is paracompact [1]; hence, it is isocompact.

On the other hand, many topologists have used quotient spaces to study the properties of topological spaces related to isocompact spaces, such as θ-refinable spaces and weakly $\delta\theta$-refinable spaces, and have achieved significant results ([4], [6], [16], [20], [22], [23]). The relationships among these topological spaces are illustrated in Figure 2.

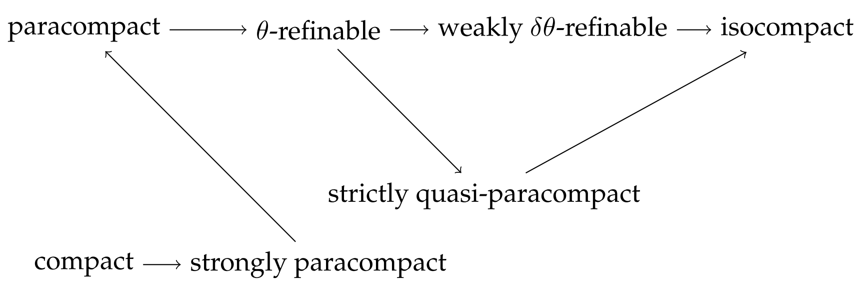

Figure 2 : Relationships of spaces

According to the research of previous topologists and the relationships of these topological spaces, we know that a large number of topological spaces imply isocompactness. Moreover, they used the properties of quotient groups to obtain some interesting conclusions about these topological spaces. The following question, however, remains open:

Question 1.3. *If H is a subgroup of a topological group G, under what conditions on H and G/H is G isocompact?*

In this work, we aim to address Question 1.3, and we will obtain the answer in Theorem 3.10. Furthermore, we propose the following question, and will obtain the answer in Theorem 3.11.

Question 1.4. *If H is a subgroup of a topological group G, under what conditions is G an isocompact space equivalent to G/H being an isocompact space?*

2. PRELIMINARIES

In this section, we introduce the necessary notations and terminology. Throughout this paper, X and Y are always topological spaces. The ω_1 denotes the first uncountable ordinal. Let $\mathscr{A}$ be a family in X. Then $st(x, \mathscr{A}) = \cup\{A \in \mathscr{A}, x \in A\}$ and $st(B, \mathscr{A}) = \cup\{C \in \mathscr{A}, C \cap B \neq \emptyset\}$. Unless otherwise stated, no separation axioms are assumed. The set of positive integers is denoted as $\mathbb{N}$, and the real line is denoted as $\mathbb{R}$. G denotes an abelian group endowed with a topology. If G is a group, then e denotes its identity element and $\sim$ denotes an equivalence relation. For definitions not defined here, we refer the reader to [15].

Definition 2.1. *[3] A space X is called an $isocompact$ space if every closed countably compact set in X is a compact set.*

Obviously, any topological property that makes a countably compact space compact also implies isocompactness. It is clear that each paracompact space is isocompact, and there is a locally compact space that is not isocompact, for example, ω_1 [15].

Proposition 2.1. *Suppose X is a T_2 isocompact space and $f : X \to Y$ is a continuous injective mapping. If each countably compact set is closed in Y, then each countably compact set is compact in X.*

Proof. Suppose A is countably compact in X. It is easy to verify that $f(A)$ is countably compact in Y. Hence, $f(A)$ is closed in Y. Since f is continuous and injective, it follows that $A = f^{-1}f(A)$ is closed in X. Since X is isocompact, it follows that A is compact. □

To address Question 1.3, we need some new definitions. Suppose $\mathscr{A}$ is a family of sets in a space X. $\mathscr{A}$ is said to be closure-preserving if, for any $\mathscr{A}_1 \subset \mathscr{A}$, $\overline{\cup\mathscr{A}_1} = \cup\overline{\mathscr{A}_1}$. $\mathscr{A}$ is said to be weakly hereditarily closure-preserving if, for every $x(A) \in A \in \mathscr{A}$, the family $\{\{x(A)\} : A \in \mathscr{A}\}$ is closure-preserving [15]. Clearly, the weakly hereditarily closure-preserving families are preserved by closed continuous mappings [15]. The following example shows that closure-preserving closed families fails to be weakly hereditarily closure-preserving.

Example 2.1. *Let $X = \mathbb{R}$ with the standard topology. Then $\{[0, 1/n]\}_{n\in\mathbb{N}}$ is closure-preserving. We take $1 \in [0, 1], 1/2 \in [0, 1/2], \cdots, 1/n \in [0, 1/n], \cdots$. Thus, $\{\{1/n\}\}_{n\in\mathbb{N}}$ is not closure-preserving. Hence, $\{[0, 1/n]\}_{n\in\mathbb{N}}$ is not weakly hereditarily closure-preserving.*

Definition 2.2. *A topological space X is said to have the $isoc$ property if each family of isocompact subsets in X is weakly hereditarily closure-preserving.*

The following example shows that topological space with the $isoc$ property need not be isocompact.

Example 2.2. *Let $X = [0, \omega_1)$ with the order topology. We assume that X has the $isoc$ property. Since X is countably compact and not compact [11], it follows that X is not isocompact.*

The spaces with the $isoc$ property are very rich. Here is an example of the $isoc$ property.

Example 2.3. *Let $X = \mathbb{R}$ with the standard topology. Suppose $\mathscr{A}$ is an open cover of X. It is well known that X is a regular paracompact [9], and each regular paracompact space has a closed locally finite refinement cover $\mathscr{A}_1$ [17]. Since each paracompact is isocompact, and the isocompact spaces are closed hereditarily [3], it follows that $\mathscr{A}_1$ is isocompact. Since each locally finite family is weakly hereditarily closure-preserving, it follows that $\mathscr{A}_1$ is weakly hereditarily closure-preserving. Thus, X has an isocompact family $\mathscr{A}_1$, which is weakly hereditarily closure-preserving.*

This is another example of the property of $isoc$.

Example 2.4. *Suppose $X = \{a, b, c, d, e\}$. Let*

$$\mathscr{A} = \{X, \emptyset, \{c\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}\}.$$

Then $(X, \mathscr{A})$ is a topological space. It is easy to prove that X has the $isoc$ property.

Theorem 2.5. *[9] Every perfect preimage of a compact space is a compact space.*

Proposition 2.2. *If X has the $isoc$ property and $f : X \to Y$ is a perfect injection, then Y has the $isoc$ property.*

Proof. Let $\mathscr{A} = \{A_\alpha : \alpha \in I\}$ and each A_α is isocompact. Take any $y_\alpha \in A_\alpha$ and $x_\alpha \in f^{-1}(y_\alpha)$ for each $\alpha \in I$. Suppose B is closed countably compact in $f^{-1}(A_\alpha)$. It is easy to verify that $f(B)$ is closed countably compact in $ff^{-1}(A_\alpha) = A_\alpha$. Hence, $f(B)$ is compact. According to Theorem 2.5, $f^{-1}f(B) = B$ is compact. Thus, $f^{-1}(A_\alpha)$ is isocompact for each $\alpha \in I$. Since X has the $isoc$ property, it follows that $\{\overline{\{x_\alpha\}} : \alpha \in I\}$ is closure-preserving. Hence, $\cup_{\alpha\in I}\overline{\{x_\alpha\}} = \overline{\cup_{\alpha\in I}\{x_\alpha\}}$ and $f(\cup_{\alpha\in I}\overline{\{x_\alpha\}}) = f(\overline{\cup_{\alpha\in I}\{x_\alpha\}})$.

Since f is closed, it follows that $f(\cup_{\alpha\in I}\overline{\{x_\alpha\}})$ is closed in Y, and $f(\cup_{\alpha\in I}\{x_\alpha\})$ is a subset of $f(\cup_{\alpha\in I}\overline{\{x_\alpha\}})$. Since $\overline{f(\cup_{\alpha\in I}\{x_\alpha\})}$ is the intersection of all closed subsets of Y that contain $f(\cup_{\alpha\in I}\{x_\alpha\})$, it follows that

$$\overline{f(\cup_{\alpha\in I}\{x_\alpha\})} \subset f(\cup_{\alpha\in I}\overline{\{x_\alpha\}}) = \cup_{\alpha\in I}f(\overline{\{x_\alpha\}}).$$

Since f is continuous, it follows that $f^{-1}(\overline{\cup_{\alpha\in I}f(\{x_\alpha\})})$ is closed in X, and

$$f^{-1}(\cup_{\alpha\in I}f(\{x_\alpha\})) \subset f^{-1}(\overline{\cup_{\alpha\in I}f(\{x_\alpha\})}).$$

Since $\overline{f^{-1}(\cup_{\alpha\in I}f(\{x_\alpha\}))}$ is the intersection of all closed sets that contain $f^{-1}(\cup_{\alpha\in I}f(\{x_\alpha\}))$, it follows that

$$f^{-1}(\overline{\cup_{\alpha\in I}f(\{x_\alpha\})}) \supset \overline{f^{-1}(\cup_{\alpha\in I}f(\{x_\alpha\}))} = \overline{\cup_{\alpha\in I}f^{-1}f(\{x_\alpha\})} = \overline{\cup_{\alpha\in I}\{x_\alpha\}} = \cup_{\alpha\in I}\overline{\{x_\alpha\}}.$$

Thus,

$$ff^{-1}\overline{\cup_{\alpha\in I}f(\{x_\alpha\})} = \overline{\cup_{\alpha\in I}f(\{x_\alpha\})} \supset f(\cup_{\alpha\in I}\overline{\{x_\alpha\}}).$$

Hence,

$$\overline{\cup_{\alpha\in I}f(\{x_\alpha\})} \supset f(\cup_{\alpha\in I}\overline{\{x_\alpha\}}) = \cup_{\alpha\in I}f(\overline{\{x_\alpha\}}).$$

Thus, $\cup_{\alpha\in I}\overline{f(\{x_\alpha\})} = \overline{\cup_{\alpha\in I}f(\{x_\alpha\})}$ and $\cup_{\alpha\in I}\overline{\{y_\alpha\}} = \overline{\cup_{\alpha\in I}\{y_\alpha\}}$. Then $\mathscr{A}$ is weakly hereditarily closure-preserving. Therefore, Y has the $isoc$ property. □

The proof of the following proposition are straightforward, and thus omitted.

Proposition 2.3. *Suppose X has the $isoc$ property and Y is a subspace in X. Then Y has the $isoc$ property.*

Theorem 2.6. *[3] If X is an isocompact space and Y is a closed subset in X, then Y is isocompact*

According to Theorem 2.6, the following remark is straightforward.

Remark 2.1. *Suppose X has the $isoc$ property, $\mathscr{A}$ is an isocompact family, and $\mathscr{A}_1$ is closed in $\mathscr{A}$. Then $\mathscr{A}_1$ is weakly hereditarily closure-preserving.*

Definition 2.3. *[21] A space X is a $w\Delta$ space if and only if there is a sequence $\{\mathscr{A}_n : n \in \mathbb{N}\}$ of open covers of X such that if $x \in X$ and $\{x_n : n \in \mathbb{N}\}$ is a sequence in X with $x_n \in st(x, A_n)$, then $\{x_n : n \in \mathbb{N}\}$ has an accumulation point in X. The sequence $\{\mathscr{A}_n : n \in \mathbb{N}\}$ is called $w\Delta$ sequence.*

Definition 2.4. *[7] A topological group G is said to be ω narrow if for every open neighbourhood V of the neutral element in G, there exists a countable set A in G such that $VA = AV = G$.*

Definition 2.5. *[15] A topological space is said to be σ compact if it is the union of countably many compact subspaces.*

3. PROPERTIES OF ISOCOMPACT SPACES

It is well known that many topologists use various compact spaces to study topological groups. In this section, we obtain some properties of isocompact spaces through topological groups and explore some applications of isocompact spaces to topological groups. At the same time, we will show how the topological properties of isocompact spaces are transferred from G/H to G. Furthermore, we establish the relationships between Lindelöf and ω narrow.

Theorem 3.7. *[3] If a space X is the union of a countable family of closed isocompact subsets then X is isocompact.*

Theorem 3.8. *[15] A space is countably compact if and only if every countably family of closed subsets having the finite intersection property has non-empty intersection.*

Theorem 3.9. *[13] Let G be a T_2 topological group. Then any locally compact subgroup is closed.*

Lemma 3.1. *Suppose G is a topological group, H is a subgroup, and G/H is a quotient space. If S is a subset of G, then $SH = \cup\{aH : aH \cap S \neq \emptyset\}$.*

Proof. Suppose $bH \subset SH = \cup_{b\in S} bH$ and $b \in S$. Since H is a subgroup, it follows that $b \in bH$. Thus, $bH \cap S \neq \emptyset$ and $bH \subset \cup\{aH : aH \cap S \neq \emptyset\}$. Suppose $aH \cap S \neq \emptyset$. Let $c \in aH \cap S$. Then $c \sim a$ and $c^{-1}a \in H$. Thus, $c^{-1}aH \subset H^2 = H$ and $aH \subset cH$. Since $(c^{-1}a)^{-1} \in H$, it follows that $a^{-1}c \in H$ and $a^{-1}cH \subset H^2 = H$. Hence, $cH \subset aH$ and $aH = cH$. Therefore, $aH \subset \cup_{b\in S} bH = SH$ and $SH = \cup\{aH : aH \cap S \neq \emptyset\}$. □

Theorem 3.10. *Suppose G is a T_2 topological group with a locally compact subgroup H such that G/H is isocompact. If G has the isoc property, then G is isocompact.*

Proof. Suppose P is a closed countably compact subset of G and H is a locally compact subgroup. Let $f : G \times G \to G$ be a multiplication mapping, and $h : G \to G/H$ be a natural quotient mapping of G onto the quotient space G/H. Thus, there exists an open neighbourhood A of e such that $\overline{A}$ is compact. Then $f^{-1}(A)$ is an open neighbourhood of (e, e). Thus, there exist open neighbourhoods B_1 and B_2 such that $B_1 \times B_2 \subset f^{-1}(A)$. Then there exists an open neighbourhood B_3 such that $B_3 \subset B_1 \cap B_2$. Thus, $(e, e) \in B_3 \times B_3 \subset f^{-1}(A)$ and $B_3^2 \subset A$. Let $B = B_3 \cap B_3^{-1}$. Then B is an open neighbourhood of e such that $B = B^{-1}$ and $B^2 \subset A$. Suppose $a \in \overline{B}$. Then Ba is an open neighbourhood of a such that $Ba \cap B \neq \emptyset$. Hence, there exist b_1 and b_2 in B such that $b_1a = b_2$. Thus, $a = b_1^{-1}b_2 \in B^{-1}B = B^2 \subset A$ and $e \in \overline{B} \subset A$. Since H is a locally compact subgroup, and according to Theorem 3.9, H is closed in G. Hence, $H \cap \overline{B}$ is closed. Thus, $H \cap \overline{B} \subset H \cap A \subset \overline{H \cap A} \subset \overline{A}$. Since $\overline{A}$ is compact, it follows that $H \cap \overline{B}$ is compact.

Let $g : G \times G \to G$, $(x, y) \to x^2y$ be a mapping. Then g is continuous, and $g^{-1}(B)$ is an open neighbourhood of (e, e). Hence, there exist open neighbourhoods C_1 and C_2 such that $C_1 \times C_2 \subset g^{-1}(B)$. Then there exists an open neighbourhood C_3 such that $C_3 \subset C_1 \cap C_2$ and $C_3^3 \subset B$. Let $C = C_3 \cap C_3^{-1}$. Thus, $e \in C^3 \subset B$ and $e \in C = C^{-1}$. Let $f_1 : G \to G, x \to x^{-1}$ be an inverse mapping. Then f_1 is continuous. Thus, $f_1(\overline{C}) \subset \overline{f_1(C)}$, that is, $\overline{C}^{-1} \subset \overline{C^{-1}} = \overline{C}$. Hence, $f_1(\overline{C}^{-1}) \subset f_1(\overline{C})$. Thus, $(\overline{C}^{-1})^{-1} \subset (\overline{C})^{-1}$, that is, $\overline{C} \subset (\overline{C})^{-1}$ and $\overline{C} = (\overline{C})^{-1}$. Since f is continuous, it follows that $f(\overline{C} \times \overline{C} \times \overline{C}) = f(\overline{C \times C \times C}) \subset \overline{f(C \times C \times C)}$, that is, $\overline{C}^3 \subset \overline{C^3}$. Then $\overline{C}^3 \cap H \subset \overline{C^3} \cap H \subset \overline{B} \cap H$ and $\overline{\overline{C}^3} \cap H$ is closed. Thus, $\overline{\overline{C}^3} \cap H$ is compact.

Let $f_2 = h \mid_{\overline{C}}: \overline{C} \to h(\overline{C})$. Now, we will show that $f_2(\overline{C})$ is isocompact. Suppose $c \in \overline{C_0} \subset \overline{C}$ and $f_2(c) \in \overline{f_2(\overline{C_0})}$. We have to show that $f_2(c) \in f_2(\overline{C_0})$. According to Lemma 3.1, we must show that $cH \cap \overline{C_0} \neq \emptyset$. Assume that $cH \cap \overline{C_0} = \emptyset$. Then $cH \cap \overline{C_0} \cap$

$\overline{C_0}^2 = \emptyset$. Since $c^{-1}\overline{C_0}^2 \subset C_0\overline{C_0}^2 \subset \overline{C_0}^3$, it follows that $c^{-1}\overline{C_0}^2 \cap H$ is a closed subset in $\overline{C_0}^3 \cap H$. Since $\overline{C}^3 \cap H$ is compact, it follows that $c^{-1}\overline{C_0}^2 \cap H$ is compact and $\overline{C_0}^2 \cap cH$ is compact.

Suppose $d \in \overline{C_0}^2 \cap cH$. Since $cH \cap \overline{C_0} = \emptyset$, it follows that $d \notin \overline{C_0}$ and there exists an open neighbourhood E_d of e such that $E_d d \cap \overline{C_0} = \emptyset$. Thus, $f^{-1}(E_d)$ is an open neighbourhood of (e, e), and there exists an open neighbourhood F_d of e such that $F_d \times F_d \subset f^{-1}(E_d)$. Hence, $F_d^2 \subset E_d$ and $\{F_d d : d \in \overline{C_0}^2 \cap cH\}$ is an open cover of $\overline{C_0}^2 \cap cH$. Then there exists a finite subcover $\mathscr{F} = \{F_{d_1}d_1, F_{d_2}d_2, \cdots, F_{d_n}d_n\}$ of $\overline{C_0}^2 \cap cH$. Let $F = (\cap_{i=1}^n F_{d_i} \cap C) \cap (\cap_{i=1}^n F_{d_i} \cap C)^{-1}$. Then F is an open neighbourhood of e and $F \subset \overline{C_0}$. Suppose $l \in \overline{C_0}^2 \cap cH$. Then there exists $F_{d_i}d_i \in \mathscr{F}$ such that $l \in F_{d_i}d_i$. Thus,

$$Fl \subset FF_{d_i}d_i \subset F_{d_i}F_{d_i}d_i \subset E_d d_i.$$

Since $E_{d_i}d_i \cap \overline{C_0} = \emptyset$, it follows that $Fl \cap \overline{C_0} = \emptyset$ and $(F(\overline{C_0}^2 \cap cH)) \cap \overline{C_0} = \emptyset$. Since $h(Fc)$ is an open neighbourhood of $h(c)$ and $h(c) \in \overline{h(\overline{C_0})}$, it follows that $h(\overline{C_0}) \cap h(Fc) \neq \emptyset$. Thus, there exist points $p_1 \in \overline{C_0}$ and $p_2 \in F$ such that $h(p_1) = h(p_2c)$. Hence, there exist points q_1 and q_2 in H such that $p_1q_1 = p_2cq_2$ and $p_1 = p_2cq_2q_1^{-1}$. Since $p_2^{-1} \in F^{-1} = F \subset \overline{C_0}$ and $cq_2q_1^{-1} \in cH$, it follows that $cq_2q_1^{-1} = p_2^{-1}p_1 \in \overline{C_0}^2$ and $cq_2q_1^{-1} \in cH \cap \overline{C_0}^2$. Thus, $p_1 = p_2bq_2q_1^{-1} \in F(cH \cap \overline{C_0}^2)$. Hence,

$$p_1 \in \overline{C_0} \cap (F(cH \cap \overline{C_0}^2)).$$

This contradicts the assumption, and hence $f_2(c) \in f_2(\overline{C_0})$ and $f_2(\overline{C_0})$ is closed. Thus, f_2 is closed, and $f_2(\overline{C}) = h(\overline{C})$ is closed. Since G/H is isocompact, and according to Theorem 2.6, $f_2(\overline{C})$ is isocompact.

Now, we will show that $\overline{C}$ is isocompact. Let L be a closed, countably compact subset of $\overline{C}$. We have to show that L is compact. Since f_2 is a closed mapping, it follows that $f_2(L)$ is closed. Since f_2 is onto and continuous, it follows that $f_2(L)$ is a closed, countably compact subset of $f_2(\overline{C})$. Hence, $f_2(L)$ is compact. Now, we will show that $f_2^{-1}(f_2(L))$ is compact. Suppose $\mathscr{M} = \{M_\alpha : \alpha \in I\}$ is an open cover of $f_2^{-1}(f_2(L))$ and $m \in L$. Thus, $f_2^{-1}(f_2(m)) = mH \cap \overline{C}$. Since $m^{-1} \in (\overline{C})^{-1} = \overline{C}$, it follows that $H \cap m^{-1}\overline{C} \subset H \cap (\overline{C})^2 \subset H \cap \overline{C}^3$. Hence, $H \cap m^{-1}\overline{C}$ is compact. Thus, $f_2^{-1}(f_2(m)) = mH \cap \overline{C}$ is compact. Then there exists a finite subcover $\mathscr{M}_m = \{M_1, M_2, \cdots, M_n\}$ of $f_2^{-1}(f_2(m))$. Let $\cup_{i=1}^n M_i = M_m$. Then $f_2^{-1}(f_2(m)) \subset M_m$. Let

$$O_m = f_2(\overline{C}) - f_2(\overline{C} - M_m).$$

Hence, O_m is an open set in $f_2(\overline{C})$ such that $f_2^{-1}(f_2(m)) \subset O_m \subset M_m$, $O_m = f_2^{-1}(f_2(O_m))$ and $f_2(O_m)$ is open. Thus, $\{f_2(O_m) : m \in L\}$ is an open cover of $f_2(L)$. Then there exists a finite subcover $\{f_2(O_{m_1}), f_2(O_{m_2}), \cdots, f_2(O_{m_r})\}$ of $f_2(L)$. Thus,

$$f_2^{-1}(f_2(L)) \subset \cup_{i=1}^r O_{m_i} \subset \cup_{i=1}^r M_{m_i}.$$

Hence, $f_2^{-1}(f_2(L))$ is compact. Since L is closed in $f_2^{-1}(f_2(L))$, it follows that L is compact. Thus, $\overline{C}$ is isocompact.

Thus, there exists a closed cover $\mathscr{C} = \{\overline{C}_x : x \in G\}$ of G and each $\overline{C}_x$ is isocompact. Suppose $P \cap \overline{C}_{x_1} \neq \emptyset$ and $y_1 \in P \cap \overline{C}_{x_1}$. Assume that $P - \overline{C}_{x_1}$ is an infinite set, then there exists a set $\overline{C}_{x_2}$ such that $(P - \overline{C}_{x_1}) \cap \overline{C}_{x_2} \neq \emptyset$. Suppose $y_2 \in (P - \overline{C}_{x_1}) \cap \overline{C}_{x_2}$. Arguing by induction, there exists a point $y_n \in (P - \cup_{i<n}\overline{C}_{x_i}) \cap \overline{C}_{x_n}$ and a sequence $\{y_n : n \in \mathbb{N}\} = Y$

in P. Let $Q_n = \{y_{n+i} : i = 0, 1, 2, \cdots\}$ and $n \in \mathbb{N}$. Thus, $\{\overline{Q_n} : n \in \mathbb{N}\}$ is a closed subset family having the finite intersection property.

According to Theorem 3.8, $\{\overline{Q_n} : n \in \mathbb{N}\}$ has a non-empty intersection. Then there exists a point $y \in \overline{Q_n}$ and $n \in \mathbb{N}$. Thus, y is an accumulation point of Y. Suppose y_n in Y. Take an arbitrary point y_k in Y and $y_n \neq y_k$. Thus, $y_n \in \overline{C}_{x_n}$ and $y_k \in P - \overline{C}_{x_n} \subset G - \overline{C}_{x_n}$. Hence, $f^{-1}(G - \overline{C}_{x_n})$ is an open neighbourhood of (y_k, y_k). Thus, there exists an open neighbourhood Y_1 of y_k such that $Y_1 \times Y_1 \subset f^{-1}(G - \overline{C}_{x_n})$.

Let $Y_2 = Y_1 \cap Y_1^{-1}$. Then $Y_2^2 \subset G - \overline{C}_{x_n}$ and $Y_2 = Y_2^{-1}$. Suppose $s \in \overline{Y}_2$. Thus, $Y_2 s \cap Y_2 \neq \emptyset$. Hence, there exist points s_1 and s_2 in Y_2 such that $s_1 s = s_2$. Thus,

$$s = s_1^{-1} s_2 \in Y_2^{-1} Y_2 = Y_2^2 \subset G - \overline{C}_{x_n}$$

and $\overline{Y_2} \subset G - \overline{C}_{x_n}$. Then $y_k \in Y_2 \subset G - \overline{C}_{x_n}$ and $Y_2 \cap \overline{C}_{x_n} = \emptyset$. Thus, $\{y_n\}$ is closed. Since $\overline{C}_x$ is isocompact and G is a topological group with the property $isoc$, it follows that $\mathscr{C}$ is weakly hereditarily closure-preserving. Then Y is closure-preserving, and

$$y \in Y = \cup\{y_n\} = \cup\overline{\{y_n\}} = \overline{\cup\{y_n\}} = \overline{Y}.$$

Hence, each subset in Y is closed, and Y is discrete. It is a contradiction. Thus, $P - \overline{C}_{x_1}$ is a finite set. Then there exists a finite subcover $\mathscr{C}_1 = \{\overline{C}_{x_2}, \overline{C}_{x_3}, \cdots, \overline{C}_{x_t}\} \subset \mathscr{C}$ of $P - \overline{C}_{x_1}$. Thus, $\{\overline{C}_{x_1}, \overline{C}_{x_2}, \cdots, \overline{C}_{x_t}\}$ is a finite subcover of P. According to Theorem 3.7, $\cup_{i=1}^{t} \overline{C}_{x_i}$ is isocompact. Therefore, P is compact, and G is isocompact. □

The following example [10] shows that even though G is a T_2 topological group with a locally compact subgroup H such that G/H is isocompact, G need not be isocompact. Hence, the hypothesis that G with the $isoc$ property is essential in Theorem 3.10.

Example 3.5. *Suppose G is a T_2 topological group, and $H = [0, \omega_1)$ is a subgroup, where G is the topological sum of $[0, \omega_1)$ and $[0, \omega_1]$, and G/H is homeomorphic to $[0, \omega_1]$. Define $f : G \to [0, \omega_1]$ by sending each point of G to the naturally corresponding point in $[0, \omega_1]$. Then f is a finite-to-one open mapping, and H is locally compact. Hence, f is a quotient mapping. However, G/H and $[0, \omega_1]$ are isocompact, and G is not isocompact.*

According to Proposition 2.2 and Theorem 3.10, the following corollary is direct.

Corollary 3.1. *Suppose G_1 is a topological group with the $isoc$ property, and G_2 is a T_2 topological group with a locally compact subgroup H_2 such that G_2/H_2 is isocompact. If $f : G_1 \to G_2$ is a perfect injection, then G_2 is isocompact.*

It is well known that the mapping $f : G_1 \to G_2$ is called a quasi-k mapping if the preimage of every countably compact subset is countably compact [19]. It is easy to prove that if f is continuous and quasi-k, and G_1 is isocompact, then G_2 is isocompact. According to Theorem 2.6, Theorem 3.9, and Theorem 3.10, the following corollaries are direct.

Corollary 3.2. *Suppose G is a T_2 topological group with a locally compact subgroup H such that G/H is isocompact. If G has the $isoc$ property, then H is isocompact.*

Corollary 3.3. *Suppose G_1 is a T_2 topological group with a locally compact subgroup H_1 such that G_1/H_1 is isocompact, and G_1 has the $isoc$ property. If $f : G_1 \to G_2$ is continuous onto quasi-k mapping, then G_2 is isocompact.*

Since quotient mapping is continuous, and according to Theorem 3.10, it follows that the following remark is direct.

Remark 3.2. *Suppose G is a T_2 topological group with a locally compact subgroup H, and G has the $isoc$ property. If the quotient mapping is quasi-k, then G is isocompact if and only if G/H is isocompact.*

Lemma 3.2. *Suppose G is a topological group with a subgroup H. If G is isocompact and the quotient mapping is injection, then G/H is isocompact.*

Proof. Suppose A is closed countably compact in G. Let $f : G \to G/H$ be a quotient mapping. Then f is continuous and open. Thus, $f^{-1}(A)$ is closed in G/H. Suppose $\{B_n : n \in \mathbb{N}\}$ is an open cover of $f^{-1}(A)$. Then $\{f(B_n) : n \in \mathbb{N}\}$ is an open cover of A. Hence, there exists a finite subcover $\{f(B_1), f(B_2), \cdots, f(B_r)\}$ of A. Since the quotient mapping f is an injection, it follows that $\{B_1, B_2, \cdots, B_r\}$ is a subcover of $f^{-1}(A)$. Thus, $f^{-1}(A)$ is countably compact in G. Since G isocompact, it follows that $f^{-1}(A)$ is compact in G. It is easy to verify that A is compact in G/H. Therefore, G/H is isocompact. □

According to Theorem 3.10 and Lemma 3.2, we obtain the following theorem directly.

Theorem 3.11. *Suppose G is a T_2 topological group with the $isoc$ property, and H is a locally compact subgroup. If the quotient mapping is injection, then G/H is isocompact if and only if G is isocompact.*

4. APPLICATIONS OF ISOCOMPACT SPACES

Below are some applications of isocompact spaces in topological groups. In [24], Wicke and Worrell established the connections between weakly $\delta\theta$-refinable spaces and isocompact spaces.

Theorem 4.12. *[24] Every closed and countably compact subset of a weakly $\delta\theta$-refinable space is compact.*

Obviously, each weakly $\delta\theta$-refinable space is isocompact. According to Theorem 3.10 and Theorem 4.12, we obtain the following theorem directly.

Theorem 4.13. *Suppose G is a T_2 topological group with a locally compact subgroup H such that G/H is weakly $\delta\theta$-refinable. If G has the $isoc$ property, then G is isocompact.*

Since each paracompact (respectively, metacompact, θ-refinable, weakly θ-refinable, paralindelöf, metalindelöf, and $\delta\theta$-refinable) space is weakly $\delta\theta$-refinable, and the relationships among these topological spaces are illustrated in Figure 3 of [24], we obtain the following corollary directly.

Corollary 4.4. *Suppose G is a T_2 topological group with a locally compact subgroup H such that G/H is paracompact (respectively, metacompact, θ-refinable, weakly θ-refinable, paralindelöf, metalindelöf, and $\delta\theta$-refinable). If G has the $isoc$ property, then G is isocompact.*

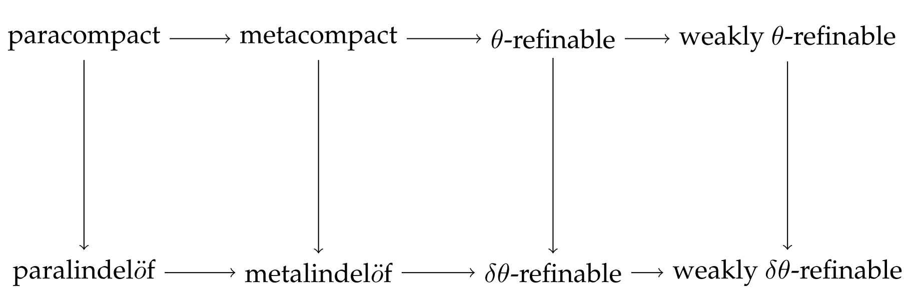

Figure 3 : Relationships of spaces

Since each weakly $\delta\theta$-refinable space is isocompact, we have the following question.

Question 4.14. *Suppose G is a T_2 topological group with a locally compact subgroup H such that G/H is isocompact, and G has the $isoc$ property. Is G weakly $\delta\theta$-refinable?*

The following example shows that the answer to Question 4.14 may be negative and also illustrates that an isocompact space is not necessarily a weakly $\delta\theta$-refinable space.

Example 4.6. *Suppose G is a T_2 topological group, where G is the ordered topological space $[0, \omega_1)$, and G/G is a singleton. Thus, G has the isoc property, and G/G is isocompact. It is well known that G is locally compact, and not weakly $\delta\theta$-refinable [15].*

Since the ω narrow topological groups can be characterized as topological subgroups of arbitrary topological products of second-countable topological groups, and each second-countable space is Lindelöf, and it is natural for us to look for the relationships between Lindelöf and ω narrow, and pose the following question.

Question 4.15. *Suppose G is a T_2 topological group with a locally compact subgroup. Under what conditions is G Lindelöf if and only if G is ω narrow?*

The following theorem shows the connection between Lindelöf and ω narrow.

Theorem 4.16. *Suppose G is a T_2 topological group with the isoc property. If G has a locally compact subgroup H such that G/H is isocompact, and G is a $w\Delta$ space, then G is Lindelöf if and only if G is ω narrow.*

Proof. Suppose $\mathscr{A}$ is an open cover of G. Let $f : G \times G \to G$ be a multiplication mapping. Now, we will show that G is Lindelöf. Suppose $\{\mathscr{B}_n : n \in \mathbb{N}\}$ is a $w\Delta$ sequence of G. Thus, there exists an open set $B_1 \in \mathscr{B}_1$ such that $e \in B_1$. Then $(e, e) \in f^{-1}(B_1)$ and there exists an open neighborhood C_1 of e such that $C_1 \times C_1 \subset f^{-1}(B_1)$ and $C_1^2 \subset B_1$. Let $D_1 = C_1 \cap C_1^{-1}$. Hence, $D_1^2 \subset B_1$ and $D_1 = D_1^{-1}$. Suppose $a \in \overline{D_1}$. Since D_1 is an open neighborhood of e, it follows that $D_1 a \cap D_1 \neq \emptyset$, and there exists points a_1 and a_2 in D_1 such that $a_1 a = a_2$. Hence, $a = a_1^{-1} a_2 \in D_1^2 \subset B_1$ and $\overline{D_1} \subset B_1$.

Similarly, there exists an open neighborhood E_1 such that

$$e \in E_1 \subset \overline{E}_1 \subset D_1 \subset \overline{D}_1 \subset B_1.$$

Hence, there exists an open set $B_2 \in \mathscr{B}_2$ such that $e \in B_2$. Similarly, there exists an open neighborhood F_1 such that $e \in F_1 \subset \overline{F}_1 \subset B_2$. Let $D_2 = F_1 \cap E_1$. Then $e \in D_2 \subset \overline{D}_2 \subset B_2$. Similarly, there exists an open neighborhood E_2 such that

$$e \in E_2 \subset \overline{E}_2 \subset D_2 \subset \overline{D}_2 \subset B_2.$$

Arguing by induction, there exists an open neighborhood E_n such that $e \in E_n \subset \overline{E}_n \subset D_n \subset \overline{D}_n \subset B_n$ for n in $\mathbb{N}$. Hence, for each n in $\mathbb{N}$, we have

$$E_{n+1} \subset \overline{E}_{n+1} \subset D_{n+1} \subset \overline{D}_{n+1} \subset \overline{E}_n \subset D_n.$$

Thus, $e \in \cap_{n\in\mathbb{N}} D_n = \cap_{n\in\mathbb{N}} \overline{D}_n$ and $\{D_n : n \in \mathbb{N}\}$ is a decreasing sequence. Then there exists an open cover $\mathscr{D}_n = \{D_n x : x \in G\}$ of G such that $\mathscr{D}_n$ refines $\mathscr{B}_n$ for n in $\mathbb{N}$. According to Theorem 3.8, $\cap_{n\in\mathbb{N}} D_n$ is countably compact. According to Theorem 3.10, G is isocompact. Then $\cap_{n\in\mathbb{N}} D_n$ is compact and $e \in \cap_{n\in\mathbb{N}} D_n$.

Then there exists a finite subcover $\mathscr{A}_1 \subset \mathscr{A}$ of $\cap_{n\in\mathbb{N}} D_n$. Let $\mathscr{A}_1 = \{A_1, A_2, \cdots, A_r\}$. Thus, $\cap_{n\in\mathbb{N}} D_n = \cup_{i=1}^{r} A_i$. Hence, there exists an open set $A_i \in \mathscr{A}_1$ such that $e \in A_i$. Since G is ω narrow, it follows that there exists a countable set $H \subset G$ such that $HA_i = G$. Then $H \cap_{n\in\mathbb{N}} D_n = G$. Let $H = \{h_1, h_2, \cdots, h_j, \cdots\}$. Thus, $h_j \cap_{n\in\mathbb{N}} D_n$ is compact for each $h_j \in H$. Hence, there exists a finite subcover $\mathscr{A}_j \subset \mathscr{A}$ of $h_j \cap_{n\in\mathbb{N}} D_n$ for each $j \in \mathbb{N}$. Therefore, there exists a countable subcover $\mathscr{A}_0 \subset \mathscr{A}$ of $H \cap_{n\in\mathbb{N}} D_n$, and G is a Lindelöf space.

Now, we will show that G is ω narrow. Suppose L is an open open neighborhood of e. Then $\{Lx : x \in G\}$ is an open cover of G, and there exists an open countable subcover $\mathscr{L}_1 = \{Lx_n : n \in \mathbb{N}\}$. Let $M = \{x_n : n \in \mathbb{N}\}$. It is clear that $LM \subset G$. For each $y \in G$, there exists a set $Lx_n \in \mathscr{L}_1$ such that $y \in Lx_n$. Since $Lx_n \subset LM$, it follows that $y \in LM$ and $G \subset LM$. Therefore, $LM = G$, and G is ω narrow. □

According to Theorem 4.16, the proof of the following corollaries are straightforward, and thus omitted.

Corollary 4.5. *Suppose G is a T_2 topological group with the isoc property. If G has a locally compact subgroup H such that G/H is isocompact, and G is $w\Delta$ ω narrow, then G is σ compact.*

Corollary 4.6. *Suppose G is a T_2 topological group with the isoc property. If G has a locally compact subgroup H such that G/H is isocompact, and G is $w\Delta$ ω narrow, then any quotient group of G is Lindelöf.*

Theorem 4.17. *[12] A topological group G is ω narrow if and only if G is topologically isomorphic to a subgroup of a product of second countable topological groups.*

According to Theorem 4.16 and Theorem 4.17, we obtain the following theorem directly.

Theorem 4.18. *Suppose G is a T_2 topological group with the isoc property. If G has a locally compact subgroup H such that G/H is isocompact, and G is a $w\Delta$ space, then G is Lindelöf if and only if G is topologically isomorphic to a subgroup of a product of second countable topological groups.*

Proposition 4.4. *Suppose G is a T_2 topological group with the isoc property. If G has a locally compact subgroup H such that G/H is isocompact, and G is $w\Delta$ ω narrow, then H is σ compact.*

Proof. According to Theorem 4.16, G is Lindelöf. Since H is a locally compact subgroup, it follows that there exists an open compact neighbourhood A of e. Thus, $\{Ax : x \in H\}$ is an open cover of H. According to Theorem 3.9, H is closed. It is easy to verify that Lindelöf spaces are closed hereditary, then H is Lindelöf. Hence, there exists an countable subcover $\{Ax_n : n \in \mathbb{N}\}$ of H. Therefore, $H = \cup_{n\in\mathbb{N}} Ax_n$ and H is σ compact. □

Proposition 4.5. *Suppose G is a T_2 $w\Delta$ ω narrow topological group with the isoc property. If H is a discrete subset such that G/H is isocompact, then H is a countable set.*

Proof. Since H is discrete, it follows that H is locally compact. According to Theorem 3.9, H is closed. According to Theorem 4.16, G is Lindelöf. It is easy to verify that Lindelöf spaces are closed hereditary, then H is Lindelöf. Thus, H is a countable set that became quite clear. □

5. Conclusion and future direction

In this work, we obtain some properties of isocompact spaces. At the same time, we establish the relationships between isocompact spaces and locally compact subgroups.

Our first result shows the relationships between locally compact topological groups and isocompact spaces. Furthermore, we show that if a T_2 topological group with the $isoc$ property has a locally compact subgroup where the quotient mapping is injective, then the topological group is isocompact if and only if the quotient group is isocompact. At last, we have determined the conditions under which ω narrow and Lindelöf are equivalent.

Several directions for future research are discussed below. For example, to obtain different results in further research, we propose the following questions.

Question 5.19. *Suppose G is a T_2 topological group with a paracompact subgroup H such that G/H is isocompact. If G has the property $isoc$, is G isocompact?*

Question 5.20. *Suppose G is a T_2 paratopological group with a locally compact subgroup H such that G/H is isocompact. If G has the property $isoc$, is G isocompact?*

The work initiated here is the starting point for continuing work towards that direction and motivate others to do so.

6. Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

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