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The approximate solution of the general split variational inequality problem by intermixed iteration

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ABSTRACT. This paper first introduces a new problem that is the general split variational inequality problem and invents a mathematical tool for solving our new problem which is Lemma 2.5. Then, we establish and prove a strong convergence theorem aimed at finding an element of the set of the solution of the general split variational inequality problem. Furthermore, we apply our main theorem to demonstrate a strong convergence theorem for finding solutions to the split variational inequality problem, the split feasibility problem, and the minimization problem. Finally, we provide numerical examples to advocate our main result.

1. INTRODUCTION

Throughout this article, let H, H_1, and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be a nonempty closed convex subset of H_1 and H_2, respectively, and let $A, B : H_1 \to H_2$ be bounded linear operators. We denote weak convergence and strong convergence by notations "$\rightharpoonup$" and "$\to$", respectively.

The split feasibility problem (SFP) originated from modeling and inverse problems, phase retrievals, and medical image reconstruction [3]. The SFP can also be used in various disciplines such as image restoration, computer tomography, signal processing, and radiation therapy treatment planning; see more detail [3, 4, 5]. Recall that the split feasibility problem is to find a point $x \in H_1$ such that

$$x \in C \qquad \text{and} \qquad Ax \in Q, \tag{1.1}$$

The set of all solutions of (1.1) is denoted by Γ. Later, Censor et al. [4] modified the SFP that is *the multiple-set split feasibility problem (MSSFP)*, which is to find a point $x^* \in H_1$ with the property:

$$x^* \in \bigcap_{i}^{p} C_i \qquad \text{and} \qquad Ax^* \in \bigcap_{j}^{r} Q_j. \tag{1.2}$$

If we put $p = r = 1$ in (1.2), then the MSSFP reduces to the SFP. Over the past decade, many mathematicians have introduced new problems derived from the split feasibility problem, including the split variational inequality problem, the split common null point problem, the split common fixed point problem, the split equilibrium problem, the split equality problem, and the split general system of variational inequalities problem. Approximations of solutions to these problems have been developed in Banach and Hilbert spaces. For further details, see [15, 13, 14, 12, 11, 6, 10, 16, 9, 8, 7].

Based on the split feasibility problem and the split variational inequality problem, we

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introduce *the general split variational inequality problem*, which is to find a point $x^* \in C$ such that

$$(1.3) \qquad \langle y - x^*, A_C x^* \rangle \geq 0,$$

for all $y \in C$, and find $y_A^* = Ax^*, y_B^* = Bx^* \in Q$ such that

$$(1.4) \qquad \langle z - y_A^*, A_Q y_A^* \rangle \geq 0,$$

and

$$(1.5) \qquad \langle z - y_B^*, A_Q y_B^* \rangle \geq 0,$$

for all $z \in Q$, where $A_C : C \to H_1$ and $A_Q : Q \to H_2$ are mappings. The set of all the above solutions is denoted by $\Phi = \{x^* \in VI(C, A_C) : Ax^*, Bx^* \in VI(Q, A_Q)\}$. Indeed, the general split variational inequality problem is a generalization of the SFP and the split variational inequality problem (SVIP). That is, if we put $A \equiv B$, $y_A^* = y_B^*$ in (1.4) and (1.5), then the general split variational inequality problem reduces to the SVIP introduced by [19], which the SVIP has been studied and modified in many pieces of literature; see more detail in [27, 28, 29, 30, 31]. Furthermore, if we put $A \equiv B$, $y_A^* = y_B^*$, and $A_C \equiv A_Q \equiv 0$ in (1.3), (1.4), and (1.5), then the general split variational inequality problem reduces to the SFP.

The following definition is important to our main theorem and its applications.

Definition 1.1. *Let $T : H \to H$ be a mapping. Then*

(i) a mapping T is called a strongly positive linear bounded operator if there exists a constant $\alpha > 0$ with the property

$$\langle Tx, x \rangle \geq \alpha \|x\|^2, \qquad \text{for all } x \in H,$$

(ii) a mapping T is called lipschitz continuous on C if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \text{for all } x, y \in C.$$

If $L \in [0, 1)$, then T is called a contraction. Obviously, if $L = 1$, T is a nonexpansive. Moreover, it is also known that if T is a nonexpansive mapping of H into itself, we have

$$\langle Ty - Tx, (I - T)x - (I - T)y \rangle \leq \frac{1}{2}\|(I - T)x - (I - T)y\|^2,$$

for all $x, y \in H$,

(iii) a mapping T is called α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \qquad \text{for all } x, y \in C,$$

(iv) a mapping T is called firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \qquad \text{for all } x, y \in C.$$

The fixed point theorem based on the contraction principle has studied the existence and uniqueness of the solutions. This theory was developed extensively and applied in various fields. For example, the performances of the fixed point method are applied to ridge regression in statistics and are used in communication engineering as a tool to solve problems. *The fixed point problem* for the mapping $\varphi : H \to C$ is to find $x \in H$ such that

$$\varphi(x) = x,$$

we denote the set of fixed point of a mapping φ by $F(\varphi)$.

For many years, numerous mathematicians have developed methods for solving fixed point problems by constructing sequences $\{x_n\}$ in various forms, such as the Mann iteration [32], the Halpern iteration [33], the Ishikawa iteration [34], and others. Additionally, such sequences have been extended to solve other problems beyond fixed point problems as well.

Using the technique for creating sequences $\{x_n\}$ as Mann and Viscosity iteration, Yao et al. [17] introduced the following sequences $\{x_n\}$ and $\{y_n\}$ which is called *the intermixed algorithm* by the definition of the sequences $\{x_n\}$ and $\{y_n\}$ depend on each other as the following algorithm:

Algorithm 1. *For arbitrarily given $x_1, y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

(1.6)
$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n], & for\ all\ n \in \mathbb{N}, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n], & for\ all\ n \in \mathbb{N}, \end{cases}$$

where $S, T : C \to C$ is a λ-strictly pseudo-contraction, $f : C \to H$ is a ρ_1-contraction and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1-\lambda)$ is a constant and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $(0,1)$. Furthermore, under some control conditions, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.19) converge to $x^* = P_{F(T)}f(y^*)$ and $y^* = P_{F(S)}g(x^*)$, respectively.

Inspired by Algorithm 1, Khuangsatung and Kangtunyakarn [20] proposed a novel intermixed algorithm that leverages viscosity techniques to address the problem of finding a common solution for the combination of mixed variational inequality problems and fixed-point problems involving nonexpansive mappings in a real Hilbert space. Their algorithm is outlined as follows:

Theorem 1.1. *Let C be a nonempty, closed, and convex subset of H. For every $i = 1, 2$, let $f_i : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function, let $A_i, B_i : C \to H$ be δ_i^A- and δ_i^B-inverse strongly monotone operators, respectively, with $\delta_i = \min\{\delta_i^A, \delta_i^B\}$ and let $T_i : C \to C$ be nonexpansive mappings. Assume that $\Omega_i = F(T_i) \cap VI(C, A_i, f_i) \cap VI(C, B_i, f_i) \neq \emptyset$, for all $i = 1, 2$. Let $g_1, g_2 : H \to H$ be σ_1- and σ_2- contraction mappings with $\sigma_1, \sigma_2 \in (0,1)$ and $\sigma = \max\{\sigma_1, \sigma_2\}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1, y_1 \in C$ and*

$$\begin{cases} w_n &= b_2 y_n + (1-b_2) T_2 y_n, \\ y_{n+1} &= (1-\beta_n) w_n + \beta_n P_c(\alpha_n g_2(x_n) + (1-\alpha_n) J_{\gamma f}^2 (y_n - \gamma_2(a_2 A_2 + (1-a_2) B_2) y_n)), \\ z_n &= b_1 x_n + (1-b_1) T_1 x_n, \\ x_{n+1} &= (1-\beta_n) z_n + \beta_n P_c(\alpha_n g_1(y_n) + (1-\alpha_n) J_{\gamma f}^1 (x_n - \gamma_1 (a_1 A_1 + (1-a_1) B_1) x_n)), \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$, $\gamma_i \in (0, 2\delta_i)$, $a_i, b_i \in (0,1)$, and $J_{\gamma f}^i : H \to H$ defined as $J_{\gamma f}^i = (I + \gamma_i \nabla f_i)^{-1}$ is the resolvent operator for all $i = 1, 2$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \bar{l} \leq \beta_n \leq l$ *for all $n \in \mathbb{N}$ and for some $\bar{l}, l > 0$,*
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{\Omega_i} g_1(y^*)$ and $y^* = P_{\Omega_2} g_2(x^*)$, respectively.

In contemporary research, the intermixed algorithm has undergone numerous modifications and enhancements to address a diverse range of problems, as evidenced in the works of [21, 22, 23, 24, 25, 26] and many others.

We establish a convergence theorem for identifying an element of the solution set of the general split variational inequality problem. This theorem utilizes the solution technique of the intermixed algorithm, as demonstrated in Theorem 3.2 of Section 3. Moreover, we apply our main theorem to prove a strong convergence theorem for finding solutions to the SVIP, the SFP, and the minimization problem. Finally, we provide numerical examples to illustrate and support our main result.

2. Preliminaries

This section states some basic properties and lemmas used in our convergence theorems and applications. For every $x \in H$, there exist a unique nearest point $P_C x$ in C such that

$$P_C x := \arg\min_{y\in C} \|y - x\|.$$

Such an operator P_C is called *the metric projection* of H onto C. Moreover, P_C is a firmly nonexpansive mapping, that is

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle,$$

for all $x, y \in H$.

Lemma 2.1. *(See* [1]*) For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \qquad \forall v \in C.$$

Lemma 2.2. *(See* [1]*) Let T be a mapping of C into H. Let $u \in C$, then for $\lambda > 0$,*

$$u = P_C(I - \lambda T)u \Leftrightarrow u \in VI(C, T),$$

where P_C is the metric projection of H onto C.

Lemma 2.3. *(See* [2]*) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \qquad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

1) $\sum_{n=1}^{\infty} \alpha_n = \infty$; *2)* $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$. *Then* $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *(See* [18]*) Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.5. *Let $A_C : C \to H_1$, $A_Q : Q \to H_2$ be α_C, α_Q-inverse strongly monotone operater. Assume that $\Phi \neq \emptyset$. Then the following conditions are equivalent:*

(i) $x^* \in \Phi$,

(ii) $x^* = P_C(I - \lambda_C A_C)(I - a(\frac{A^*(I-P_Q(I-\lambda_Q A_Q))A}{2} + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))B}{2}))x^*$,

where $\lambda_C \in (0, 2\alpha_C)$, $\lambda_Q \in (0, 2\alpha_Q)$, $a \in (0, \frac{1}{L})$ and $L = \max\{L_A, L_B\}$ with L_A, L_B are the spectral radius of the operator A^*A and B^*B, respectively.

Proof. i) $\Rightarrow$ ii). Let $x^* \in \Phi$, we have $x^* \in VI(C, A_C)$ and $Ax^*, Bx^* \in VI(Q, A_Q)$.
From Lemma 2.2 and $\lambda_C, \lambda_Q > 0$, we have $P_C(I - \lambda_C A_C)x^* = x^*$ and $Ax^* = P_Q(I - \lambda_Q A_Q)Ax^*$, $Bx^* = P_Q(I - \lambda_Q A_Q)Bx^*$.
It follows that

$$x^* = P_C(I - \lambda_C A_C)(I - a(\frac{A^*(I - P_Q(I - \lambda_Q A_Q))A}{2} + \frac{B^*(I - P_Q(I - \lambda_Q A_Q))B}{2}))x^*.$$

ii) $\Rightarrow$ i). Let $x^* = P_C(I - \lambda_C A_C)(I - a(\frac{A^*(I-P_Q(I-\lambda_Q A_Q))A}{2} + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))B}{2}))x^*$ and $w \in \Phi$. From i) $\Rightarrow$ ii), we have

$$w = P_C(I - \lambda_C A_C)(I - a(\frac{A^*(I - P_Q(I - \lambda_Q A_Q))A}{2} + \frac{B^*(I - P_Q(I - \lambda_Q A_Q))B}{2}))w.$$

Put $M = \frac{A^*(I-P_Q(I-\lambda_Q A_Q))A}{2} + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))B}{2}$.
It implies that

$$\begin{aligned}\|x^* - w\|^2 \leq & \|x^* - w - a(Mx^* - Mw)\|^2 \\ = & \|x^* - w\|^2 - 2a\langle x^* - w, Mx^* - Mw\rangle + a^2\|Mx^* - Mw\|^2. \end{aligned} \tag{2.7}$$

From definition of M and $L = \max\{L_A, L_B\}$, we have

$$\begin{aligned} -2a&\langle x^* - w, Mx^* - Mw\rangle \\ &= a\langle Aw - Ax^*, (I - P_Q(I - \lambda_Q A_Q))Ax^*\rangle \\ &\quad + a\langle Bw - Bx^*, (I - P_Q(I - \lambda_Q A_Q))Bx^*\rangle \\ &= a\langle Aw - P_Q(I - \lambda_Q A_Q)Ax^*, (I - P_Q(I - \lambda_Q A_Q))Ax^*\rangle \\ &\quad + a\langle P_Q(I - \lambda_Q A_Q)Ax^* - Ax^*, (I - P_Q(I - \lambda_Q A_Q))Ax^*\rangle \\ &\quad + a\langle Bw - P_Q(I - \lambda_Q A_Q)Bx^*, (I - P_Q(I - \lambda_Q A_Q))Bx^*\rangle \\ &\quad + a\langle P_Q(I - \lambda_Q A_Q)Bx^* - Bx^*, (I - P_Q(I - \lambda_Q A_Q))Bx^*\rangle \\ &\leq \frac{a}{2}\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2 - a\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2 \\ &\quad + \frac{a}{2}\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2 - a\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2 \\ &= -\frac{a}{2}\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2 - \frac{a}{2}\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2, \end{aligned} \tag{2.8}$$

and
(2.9)

$$a^2\|Mx^* - Mw\|^2 \leq \frac{a^2}{2}(L)\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2 + \frac{a^2}{2}(L)\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2.$$

From (2.7), (2.8) and (2.9), we have

$$\begin{aligned}\|x^* - w\|^2 \leq & \|x^* - w\|^2 - \frac{a}{2}(1 - aL)\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2 \\ & - \frac{a}{2}(1 - aL)\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2.\end{aligned}$$

It implies that

$$Ax^* = P_Q(I - \lambda_Q A_Q)Ax^*, Bx^* = P_Q(I - \lambda_Q A_Q)Bx^*. \tag{2.10}$$

From (2.10) and Lemma 2.2, we have

$$Ax^*, Bx^* \in VI(Q, A_Q). \tag{2.11}$$

From (2.10), we have

$$\begin{aligned} x^* = & P_C(I - \lambda_C A_C)(I - a(\frac{A^*(I - P_Q(I - \lambda_Q A_Q))A}{2} + \frac{B^*(I - P_Q(I - \lambda_Q A_Q))B}{2}))x^* \\ = & P_C(I - \lambda_C A_C)x^*. \end{aligned}$$

From Lemma 2.2, we have

$$x^* \in VI(C, A_C). \tag{2.12}$$

From (2.11) and (2.12), we can conclude that $x^* \in \Phi$. □

Remark 2.1. *From a part of the proof of Lemma 2.5, it is clear that*

$$\begin{aligned}&\|P_C(I-\lambda_C A_C)(I-aM)x - P_C(I-\lambda_C A_C)(I-aM)y\|^2\\ &\le \|x-y\|^2 - \frac{a}{2}(1-aL)\|(I-P_Q(I-\lambda_Q A_Q))Ax - (I-P_Q(I-\lambda_Q A_Q))Ay\|^2\\ &\quad - \frac{a}{2}(1-aL)\|(I-P_Q(I-\lambda_Q A_Q))Bx - (I-P_Q(I-\lambda_Q A_Q))By\|^2,\end{aligned}$$

where $M = \frac{A^*(I-P_Q(I-\lambda_Q A_Q))A}{2} + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))B}{2}$ *and for all* $x, y \in C$.

3. Main results

In this section, we prove a strong convergence theorem for finding an element of the set of the solution of the general split variational inequality problem.

Theorem 3.2. *Let* A_C *and* A_Q *define as the same in Lemma 2.5, and* A^*, B^* *are adjoint of* A, B, *respectively, with* $L = \max\{L_A, L_B\}$, *where* L_A, L_B *are spectral radius of* A^*A, B^*B, *respectively. Assume that* $\Phi \neq \emptyset$. *Let* $f, g : H_1 \to H_1$ *be* a_f *and* a_g*-contraction mappings with* $a_f, a_g \in (0, \frac{1}{2})$ *and* $\bar{a} = \max\{a_f, a_g\}$. *Let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by* $x_1, y_1 \in C$ *and*

$$\begin{cases} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C(\beta_n f(y_n) + (1-\beta_n)P_C(I-\lambda_C A_C)(I-aM)x_n),\\ y_{n+1} &= (1-\alpha_n)y_n + \alpha_n P_C(\beta_n g(x_n) + (1-\beta_n)P_C(I-\lambda_C A_C)(I-aM)y_n),\end{cases} \tag{3.13}$$

where $M = \frac{A^*(I-P_Q(I-\lambda_Q A_Q))A}{2} + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))B}{2}$, $\lambda_C \in (0, 2\alpha_C)$, $\lambda_Q \in (0, 2\alpha_Q)$, $a \in (0, \frac{1}{L})$ *and* $\{\alpha_n\}, \{\beta_n\}$ *are a sequence in* $(0,1)$, *for all* $n \in \mathbb{N}$. *Suppose the following conditions hold:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,
(ii) $0 < \bar{\theta} \le \alpha_n \le \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\bar{\theta}, \theta > 0$,
(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_\Phi f(y^*)$ *and* $y^* = P_\Phi g(x^*)$, *respectively.*

Proof. Let $x, y \in C$ and $M = \frac{A^*(I-P_Q(I-\lambda_Q A_Q))A}{2} + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))B}{2}$.
Since A_Q is α_Q-inverse strongly monotone mapping, we have

$$\begin{aligned}\|(I-\lambda_Q A_Q)x - (I-\lambda_Q A_Q)y\|^2 &= \|x-y-\lambda_Q(A_Q x - A_Q y)\|^2\\ &\le \|x-y\|^2 - \lambda_Q(2\alpha_Q - \lambda_Q)\|A_Q x - A_Q y\|^2\\ &\le \|x-y\|^2.\end{aligned}$$

Hence $(I-\lambda_Q A_Q)$ is a nonexpansive mapping. Then, we obtain that $P_Q(I-\lambda_Q A_Q)$ is a nonexpansive mapping.
Since $M = \frac{A^*(I-P_Q(I-\lambda_Q A_Q))A}{2} + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))B}{2}$, we have

$$\begin{aligned}&\|M(x)-M(y)\|^2\\ &= \Big\|\frac{A^*(I-P_Q(I-\lambda_Q A_Q))Ax}{2} - \frac{A^*(I-P_Q(I-\lambda_Q A_Q))Ay}{2}\\ &\quad + \frac{B^*(I-P_Q(I-\lambda_Q A_Q))Bx}{2} - \frac{B^*(I-P_Q(I-\lambda_Q A_Q))By}{2}\Big\|^2\\ &\le \frac{1}{2}\|A^*(I-P_Q(I-\lambda_Q A_Q))Ax - A^*(I-P_Q(I-\lambda_Q A_Q))Ay\|^2\\ &\quad + \frac{1}{2}\|B^*(I-P_Q(I-\lambda_Q A_Q))Bx - B^*(I-P_Q(I-\lambda_Q A_Q))By\|^2\\ &\le \frac{L}{2}\|(I-P_Q(I-\lambda_Q A_Q))Ax - (I-P_Q(I-\lambda_Q A_Q))Ay\|^2\end{aligned}$$

$$+\frac{L}{2}\|(I-P_Q(I-\lambda_Q A_Q))Bx-(I-P_Q(I-\lambda_Q A_Q))By\|^2. \tag{3.14}$$

Since

$$\begin{aligned}
&\|(I-P_Q(I-\lambda_Q A_Q))Ax-(I-P_Q(I-\lambda_Q A_Q))Ay\|^2\\
&=\langle (I-P_Q(I-\lambda_Q A_Q))Ax-(I-P_Q(I-\lambda_Q A_Q))Ay, Ax-Ay\rangle\\
&\quad-\langle (I-P_Q(I-\lambda_Q A_Q))Ax-(I-P_Q(I-\lambda_Q A_Q))Ay, P_Q(I-\lambda_Q A_Q)Ax\\
&\quad-P_Q(I-\lambda_Q A_Q)Ay\rangle\\
&=\langle A^*(I-P_Q(I-\lambda_Q A_Q))Ax-A^*(I-P_Q(I-\lambda_Q A_Q))Ay, x-y\rangle\\
&\quad+\langle (I-P_Q(I-\lambda_Q A_Q))Ax-(I-P_Q(I-\lambda_Q A_Q))Ay, P_Q(I-\lambda_Q A_Q)Ay\\
&\quad-P_Q(I-\lambda_Q A_Q)Ax\rangle\\
&\le\langle A^*(I-P_Q(I-\lambda_Q A_Q))Ax-A^*(I-P_Q(I-\lambda_Q A_Q))Ay, x-y\rangle\\
&\quad+\frac{1}{2}\|(I-P_Q(I-\lambda_Q A_Q))Ax-(I-P_Q(I-\lambda_Q A_Q))Ay\|^2,
\end{aligned}$$

then

$$\begin{aligned}
&\|(I-P_Q(I-\lambda_Q A_Q))Ax-(I-P_Q(I-\lambda_Q A_Q))Ay\|^2\\
&\le 2\langle A^*(I-P_Q(I-\lambda_Q A_Q))Ax-A^*(I-P_Q(I-\lambda_Q A_Q))Ay, x-y\rangle.
\end{aligned} \tag{3.15}$$

By using the same process as (3.15), we have

$$\begin{aligned}
&\|(I-P_Q(I-\lambda_Q A_Q))Bx-(I-P_Q(I-\lambda_Q A_Q))By\|^2\\
&\le 2\langle B^*(I-P_Q(I-\lambda_Q A_Q))Bx-B^*(I-P_Q(I-\lambda_Q A_Q))By, x-y\rangle.
\end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), then

$$\begin{aligned}
&\|M(x)-M(y)\|^2\\
&\le L\langle A^*(I-P_Q(I-\lambda_Q A_Q))Ax-A^*(I-P_Q(I-\lambda_Q A_Q))Ay, x-y\rangle\\
&\quad+L\langle B^*(I-P_Q(I-\lambda_Q A_Q))Bx-B^*(I-P_Q(I-\lambda_Q A_Q))By, x-y\rangle\\
&=2L\langle \frac{A^*(I-P_Q(I-\lambda_Q A_Q))Ax+B^*(I-P_Q(I-\lambda_Q A_Q))Bx}{2}\\
&\quad-(\frac{A^*(I-P_Q(I-\lambda_Q A_Q))Ay+B^*(I-P_Q(I-\lambda_Q A_Q))By}{2}), x-y\rangle\\
&=2L\langle M(x)-M(y), x-y\rangle.
\end{aligned}$$

So, we have M is $\frac{1}{2L}$-inverse strongly monotone.
From the definition of M, we obtain

$$\begin{aligned}
&\|(I-aM)x-(I-aM)y\|^2\\
&=\|x-y\|^2-2a\langle M(x)-M(y), x-y\rangle+a^2\|M(x)-M(y)\|^2\\
&\le\|x-y\|^2-\frac{a}{L}\|M(x)-M(y)\|^2+a^2\|M(x)-M(y)\|^2\\
&=\|x-y\|^2-a(\frac{1}{L}-a)\|M(x)-M(y)\|^2\\
&\le\|x-y\|^2,
\end{aligned}$$

then

$$\|(I-aM)x-(I-aM)y\|\le\|x-y\|, \qquad \text{for all } x,y\in C. \tag{3.17}$$

We will separate the proofs in the following order for the orderliness of proof.
Step 1. Show that $\{x_n\}$ and $\{y_n\}$ are bounded.

Let $z \in \Phi$. By Lemma 2.5, we have $z = P_C(I - \lambda_C A_C)(I - aM)z$.
From (3.13), we have

$$\begin{aligned}
&\|x_{n+1} - z\| \\
=&\|(1-\alpha_n)(x_n - z) + \alpha_n(P_C(\beta_n f(y_n) + (1-\beta_n)P_C(I - \lambda_C A_C)(I - aM)x_n) - z)\| \\
\leq&(1-\alpha_n)\|x_n - z\| + \alpha_n\|\beta_n f(y_n) + (1-\beta_n)P_C(I - \lambda_C A_C)(I - aM)x_n - z\| \\
\leq&(1-\alpha_n)\|x_n - z\| \\
&+ \alpha_n(\beta_n\|f(y_n) - z\| + (1-\beta_n)\|P_C(I - \lambda_C A_C)(I - aM)x_n - z\|) \\
\leq&(1-\alpha_n)\|x_n - z\| + \alpha_n(\beta_n\|f(y_n) - z\| + (1-\beta_n)\|x_n - z\|) \\
\leq&(1-\alpha_n\beta_n)\|x_n - z\| + \alpha_n\beta_n(\|f(y_n) - f(z)\| + \|f(z) - z\|) \\
\leq&(1-\alpha_n\beta_n)\|x_n - z\| + \alpha_n\beta_n(a_f\|y_n - z\| + \|f(z) - z\|) \\
\leq&(1-\alpha_n\beta_n)\|x_n - z\| + \alpha_n\beta_n\bar{a}\|y_n - z\| + \alpha_n\beta_n\|f(z) - z\|.
\end{aligned} \tag{3.18}$$

From (3.13) and by using the same method as (3.18), we have

$$\|y_{n+1} - z\| \leq (1-\alpha_n\beta_n)\|y_n - z\| + \alpha_n\beta_n\bar{a}\|x_n - z\| + \alpha_n\beta_n\|g(z) - z\|. \tag{3.19}$$

Combining (3.18) and (**??**), then

$$\begin{aligned}
&\|x_{n+1} - z\| + \|y_{n+1} - z\| \\
&\leq(1-\alpha_n\beta_n)\|x_n - z\| + \alpha_n\beta_n\bar{a}\|y_n - z\| + \alpha_n\beta_n\|f(z) - z\| \\
&\quad + (1-\alpha_n\beta_n)\|y_n - z\| + \alpha_n\beta_n\bar{a}\|x_n - z\| + \alpha_n\beta_n\|g(z) - z\| \\
&=(1-\alpha_n\beta_n(1-\bar{a}))(\|x_n - z\| + \|y_n - z\|) \\
&\quad + \frac{\alpha_n\beta_n(1-\bar{a})}{1-\bar{a}}(\|f(z) - z\| + \|g(z) - z\|) \\
&\leq\bar{M},
\end{aligned}$$

where $\bar{M} = \max\{\|x_1 - z\| + \|y_1 - z\|, \frac{\|f(z)-z\|+\|g(z)-z\|}{1-\bar{a}}\}$.
Such that $\|x_n - z\| + \|y_n - z\| \leq \bar{M}$.
Hence, $\{x_n\}$ and $\{y_n\}$ are bounded.

Step 2. Show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.
From (3.13), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
\leq&(1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&+ \alpha_n\|\beta_n f(y_n) + (1-\beta_n)P_C(I - \lambda_C A_C)(I - aM)x_n \\
&- (\beta_{n-1}f(y_{n-1}) + (1-\beta_{n-1})P_C(I - \lambda_C A_C)(I - aM)x_{n-1})\| \\
&+ |\alpha_n - \alpha_{n-1}|\|P_C(\beta_{n-1}f(y_{n-1}) + (1-\beta_{n-1})P_C(I - \lambda_C A_C)(I - aM)x_{n-1})\| \\
\leq&(1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&+ \alpha_n(\beta_n\|f(y_n) - f(y_{n-1})\| + |\beta_n - \beta_{n-1}|\|f(y_{n-1})\| \\
&+ (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n|\|P_C(I - \lambda_C A_C)(I - aM)x_{n-1}\|) \\
&+ |\alpha_n - \alpha_{n-1}|\|P_C(\beta_{n-1}f(y_{n-1}) + (1-\beta_{n-1})P_C(I - \lambda_C A_C)(I - aM)x_{n-1})\| \\
\leq&(1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&+ \alpha_n(\beta_n\bar{a}\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(y_{n-1})\| \\
&+ |\beta_{n-1} - \beta_n|\|P_C(I - \lambda_C A_C)(I - aM)x_{n-1}\|)
\end{aligned} \tag{3.20}$$

$$+|\alpha_n-\alpha_{n-1}|\|P_C(\beta_{n-1}f(y_{n-1})+(1-\beta_{n-1})P_C(I-\lambda_C A_C)(I-aM)x_{n-1})\|. \tag{3.21}$$

By using the same process as (3.20), we have

$$\begin{aligned}
&\|y_{n+1}-y_n\|\\
\le&(1-\alpha_n\beta_n)\|y_n-y_{n-1}\|+|\alpha_{n-1}-\alpha_n|\|y_{n-1}\|\\
&+\alpha_n(\beta_n\bar{a}\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|g(x_{n-1})\|+|\beta_{n-1}-\beta_n|\|P_C(I-\lambda_C A_C)(I-aM)y_{n-1}\|)\\
&+|\alpha_n-\alpha_{n-1}|\|P_C(\beta_{n-1}g(x_{n-1})+(1-\beta_{n-1})P_C(I-\lambda_C A_C)(I-aM)y_{n-1})\|.
\end{aligned} \tag{3.22}$$

From (3.20) and (3.22), we have

$$\begin{aligned}
&\|x_{n+1}-x_n\|+\|y_{n+1}-y_n\|\\
\le&(1-\alpha_n\beta_n(1-\bar{a}))(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|)\\
&+|\alpha_{n-1}-\alpha_n|(\|x_{n-1}\|+\|y_{n-1}\|\\
&+\|P_C(\beta_{n-1}f(y_{n-1})+(1-\beta_{n-1})P_C(I-\lambda_C A_C)(I-aM)x_{n-1})\|\\
&+\|P_C(\beta_{n-1}g(x_{n-1})+(1-\beta_{n-1})P_C(I-\lambda_C A_C)(I-aM)y_{n-1})\|)\\
&+|\beta_{n-1}-\beta_n|(\|f(y_{n-1})\|+\|g(x_{n-1})\|+\|P_C(I-\lambda_C A_C)(I-aM)x_{n-1}\|\\
&+\|P_C(I-\lambda_C A_C)(I-aM)y_{n-1}\|).
\end{aligned}$$

Applying Lemma 2.3 and the conditions i), iii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.23}$$

Step 3. Show that $\lim_{n\to\infty}\|x_n-W_n\|=0$ where $W_n=\beta_n f(y_n)+(1-\beta_n)P_C(I-\lambda_C A_C)(I-aM)x_n$ and $\lim_{n\to\infty}\|y_n-V_n\|=0$ where $V_n=\beta_n g(x_n)+(1-\beta_n)P_C(I-\lambda_C A_C)(I-aM)y_n$.

Let $\tilde{x},\tilde{y}\in\Phi$. From (3.13), we obtain that

$$\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2\\
=&(1-\alpha_n)\|x_n-\tilde{x}\|^2+\alpha_n\|P_C W_n-\tilde{x}\|^2-\alpha_n(1-\alpha_n)\|x_n-P_C W_n\|^2\\
\le&(1-\alpha_n)\|x_n-\tilde{x}\|^2+\alpha_n\|W_n-\tilde{x}\|^2-\alpha_n(1-\alpha_n)\|x_n-P_C W_n\|^2\\
=&(1-\alpha_n)\|x_n-\tilde{x}\|^2\\
&+\alpha_n\|\beta_n(f(y_n)-P_C(I-\lambda_C A_C)(I-aM)x_n)+P_C(I-\lambda_C A_C)(I-aM)x_n-\tilde{x}\|^2\\
&-\alpha_n(1-\alpha_n)\|x_n-P_C W_n\|^2\\
\le&(1-\alpha_n)\|x_n-\tilde{x}\|^2+\alpha_n(\|P_C(I-\lambda_C A_C)(I-aM)x_n-\tilde{x}\|^2\\
&+2\beta_n\langle f(y_n)-P_C(I-\lambda_C A_C)(I-aM)x_n,W_n-\tilde{x}\rangle)-\alpha_n(1-\alpha_n)\|x_n-P_C W_n\|^2\\
\le&(1-\alpha_n)\|x_n-\tilde{x}\|^2+\alpha_n(\|x_n-\tilde{x}\|^2+2\beta_n\|f(y_n)-P_C(I-\lambda_C A_C)(I-aM)x_n\|\|W_n-\tilde{x}\|)\\
&-\alpha_n(1-\alpha_n)\|x_n-P_C W_n\|^2\\
=&\|x_n-\tilde{x}\|^2+2\alpha_n\beta_n\|f(y_n)-P_C(I-\lambda_C A_C)(I-aM)x_n\|\|W_n-\tilde{x}\|\\
&-\alpha_n(1-\alpha_n)\|x_n-P_C W_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\|x_n-P_C W_n\|^2\\
\le&\|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2+2\alpha_n\beta_n\|f(y_n)-P_C(I-\lambda_C A_C)(I-aM)x_n\|\|W_n-\tilde{x}\|\\
\le&\|x_n-x_{n+1}\|(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|)\\
&+2\alpha_n\beta_n\|f(y_n)-P_C(I-\lambda_C A_C)(I-aM)x_n\|\|W_n-\tilde{x}\|.
\end{aligned}$$

By (3.23), the conditions i) and ii), then we get

$$\lim_{n\to\infty} \|x_n - P_C W_n\| = 0. \tag{3.24}$$

By using the same process as (3.24), we have

$$\lim_{n\to\infty} \|y_n - P_C V_n\| = 0. \tag{3.25}$$

From firmly nonexpansiveness of P_C, we have

$$\begin{aligned}\|P_C W_n - \tilde{x}\|^2 =& \|P_C W_n - P_C \tilde{x}\|^2 \\ \leq& \langle W_n - \tilde{x}, P_C W_n - \tilde{x}\rangle \\ =& \frac{1}{2}(\|W_n - \tilde{x}\|^2 + \|P_C W_n - \tilde{x}\|^2 - \|W_n - P_C W_n\|^2).\end{aligned}$$

It implies that

$$\|P_C W_n - \tilde{x}\|^2 \leq \|W_n - \tilde{x}\|^2 - \|W_n - P_C W_n\|^2. \tag{3.26}$$

By using the same process as (3.26), we have

$$\|P_C V_n - \tilde{y}\|^2 \leq \|V_n - \tilde{y}\|^2 - \|V_n - P_C V_n\|^2. \tag{3.27}$$

From the definitions of W_n and V_n, we have

$$\begin{aligned}\|W_n - \tilde{x}\|^2 \leq& \beta_n \|f(y_n) - \tilde{x}\|^2 + (1-\beta_n)\|P_C(I - \lambda_C A_C)(I - aM)x_n - \tilde{x}\|^2 \\ \leq& \beta_n \|f(y_n) - \tilde{x}\|^2 + (1-\beta_n)\|x_n - \tilde{x}\|^2. \end{aligned} \tag{3.28}$$

And

$$\begin{aligned}\|V_n - \tilde{y}\|^2 \leq& \beta_n \|g(x_n) - \tilde{y}\|^2 + (1-\beta_n)\|P_C(I - \lambda_C A_C)(I - aM)y_n - \tilde{y}\|^2 \\ \leq& \beta_n \|g(x_n) - \tilde{y}\|^2 + (1-\beta_n)\|y_n - \tilde{y}\|^2. \end{aligned} \tag{3.29}$$

From (3.26) and (3.28), we have

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 \leq& (1-\alpha_n)\|x_n - \tilde{x}\|^2 + \alpha_n \|P_C W_n - \tilde{x}\|^2 \\ \leq& (1-\alpha_n)\|x_n - \tilde{x}\|^2 + \alpha_n(\|W_n - \tilde{x}\|^2 - \|W_n - P_C W_n\|^2) \\ \leq& (1-\alpha_n)\|x_n - \tilde{x}\|^2 \\ &+ \alpha_n(\beta_n \|f(y_n) - \tilde{x}\|^2 + (1-\beta_n)\|x_n - \tilde{x}\|^2 - \|W_n - P_C W_n\|^2) \\ =& (1-\alpha_n\beta_n)\|x_n - \tilde{x}\|^2 + \alpha_n\beta_n\|f(y_n) - \tilde{x}\|^2 - \alpha_n\|W_n - P_C W_n\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\alpha_n\|W_n - P_C W_n\|^2 \leq& \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 + \alpha_n\beta_n\|f(y_n) - \tilde{x}\|^2 \\ \leq& \|x_n - x_{n+1}\|(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n\beta_n\|f(y_n) - \tilde{x}\|^2.\end{aligned}$$

From (3.23) and condition i), we have

$$\lim_{n\to\infty} \|W_n - P_C W_n\| = 0. \tag{3.30}$$

From the definition of V_n and utilizing the similar technique as (3.30), it can conclude that

$$\lim_{n\to\infty} \|V_n - P_C V_n\| = 0. \tag{3.31}$$

Since

$$\|x_n - W_n\| \leq \|x_n - P_C W_n\| + \|P_C W_n - W_n\|.$$

From (3.24) and (3.30), we have

$$\lim_{n\to\infty} \|x_n - W_n\| = 0. \tag{3.32}$$

Similarly

$$\|y_n - V_n\| \leq \|y_n - P_C V_n\| + \|P_C V_n - V_n\|.$$

From (3.25) and (3.31), we have

$$\lim_{n\to\infty} \|y_n - V_n\| = 0. \tag{3.33}$$

Step 4. Show that $\lim_{n\to\infty} \|x_n - P_C(I - \lambda_C A_C)(I - aM)x_n\| = 0$ and $\lim_{n\to\infty} \|y_n - P_C(I - \lambda_C A_C)(I - aM)y_n\| = 0$.
Since

$$\begin{aligned} W_n - x_n =& \beta_n f(y_n) + (1 - \beta_n)P_C(I - \lambda_C A_C)(I - aM)x_n - x_n \\ =& \beta_n(f(y_n) - x_n) + (1 - \beta_n)(P_C(I - \lambda_C A_C)(I - aM)x_n - x_n), \end{aligned}$$

then

$$(1 - \beta_n)\|P_C(I - \lambda_C A_C)(I - aM)x_n - x_n\| \leq \beta_n\|f(y_n) - x_n\| + \|W_n - x_n\|.$$

From condition i) and (3.32), we have

$$\lim_{n\to\infty} \|x_n - P_C(I - \lambda_C A_C)(I - aM)x_n\| = 0. \tag{3.34}$$

Similarly

$$\begin{aligned} V_n - y_n =& \beta_n g(x_n) + (1 - \beta_n)P_C(I - \lambda_C A_C)(I - aM)y_n - y_n \\ =& \beta_n(g(x_n) - y_n) + (1 - \beta_n)(P_C(I - \lambda_C A_C)(I - aM)y_n - y_n), \end{aligned}$$

then

$$(1 - \beta_n)\|P_C(I - \lambda_C A_C)(I - aM)y_n - y_n\| \leq \beta_n\|g(x_n) - y_n\| + \|V_n - y_n\|.$$

From condition i) and (3.33), we have

$$\lim_{n\to\infty} \|y_n - P_C(I - \lambda_C A_C)(I - aM)y_n\| = 0. \tag{3.35}$$

Step 5. Show that $\lim_{n\to\infty} \sup\langle f(y^*) - x^*, W_n - x^*\rangle \leq 0$ and $\lim_{n\to\infty} \sup\langle g(x^*) - y^*, V_n - y^*\rangle \leq 0$, where $x^* = P_\Phi f(y^*)$ and $y^* = P_\Phi g(x^*)$.

Indeed, take a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that

$$\lim_{n\to\infty} \sup\langle f(y^*) - x^*, W_n - x^*\rangle = \lim_{k\to\infty} \sup\langle f(y^*) - x^*, W_{n_k} - x^*\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$ where $\hat{x} \in C$. We obtain $W_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$.
Assume that $\hat{x} \neq P_C(I - \lambda_C A_C)(I - aM)\hat{x}$.
From Remark 2.1, (3.17), (3.34) and Opial's property, we have

$$\begin{aligned} \lim_{k\to\infty} \inf \|x_{n_k} - \hat{x}\| &< \lim_{k\to\infty} \inf \|x_{n_k} - P_C(I - \lambda_C A_C)(I - aM)\hat{x}\| \\ &\leq \lim_{k\to\infty} \inf(\|x_{n_k} - P_C(I - \lambda_C A_C)(I - aM)x_{n_k}\| \\ &\quad + \|P_C(I - \lambda_C A_C)(I - aM)x_{n_k} - P_C(I - \lambda_C A_C)(I - aM)\hat{x}\|) \\ &\leq \lim_{k\to\infty} \inf \|x_{n_k} - \hat{x}\|. \end{aligned}$$

This is a contradiction, and then, we have

$$\hat{x} = P_C(I - \lambda_C A_C)(I - aM)\hat{x},$$

then

$$\hat{x} \in F(P_C(I - \lambda_C A_C)(I - aM)). \tag{3.36}$$

Since $W_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$, (3.36) and Lemma 2.1, we can derive that

$$\begin{aligned}\lim_{n\to\infty} \sup\langle f(y^*) - x^*, W_n - x^*\rangle &= \lim_{k\to\infty} \sup\langle f(y^*) - x^*, W_{n_k} - x^*\rangle \\ &= \langle f(y^*) - x^*, \hat{x} - x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.37}$$

Similarly, indeed, take a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\lim_{n\to\infty} \sup\langle g(x^*) - y^*, V_n - y^*\rangle = \lim_{k\to\infty} \sup\langle g(x^*) - y^*, V_{n_k} - y^*\rangle.$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$ where $\hat{y} \in C$. We obtain $V_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$.
Following the same method as (3.37), we easily obtain that

$$\lim_{n\to\infty} \sup\langle g(x^*) - y^*, V_n - y^*\rangle \leq 0. \tag{3.38}$$

Step 6. Show that $\{x_n\}$ converses strongly to x^*, where $x^* = P_\Phi f(y^*)$ and $\{y_n\}$ converses strongly to y^*, where $y^* = P_\Phi g(x^*)$.
From (3.13), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ \leq&(1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|P_C W_n - x^*\|^2 \\ \leq&(1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|W_n - x^*\|^2 \\ =&(1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|\beta_n(f(y_n) - x^*) + (1-\beta_n)(P_C(I-\lambda_C A_C)(I-aM)x_n - x^*)\|^2 \\ \leq&(1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n((1-\beta_n)\|P_C(I-\lambda_C A_C)(I-aM)x_n - x^*\|^2 \\ &+ 2\beta_n\langle f(y_n) - x^*, W_n - x^*\rangle) \\ \leq&(1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n((1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\langle f(y_n) - x^*, W_n - x^*\rangle) \\ =&(1-\alpha_n\beta_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n(\langle f(y_n) - f(y^*), W_n - x^*\rangle + \langle f(y^*) - x^*, W_n - x^*\rangle) \\ \leq&(1-\alpha_n\beta_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n\|f(y_n) - f(y^*)\|(\|W_n - x_{n+1}\| + \|x_{n+1} - x^*\|) \\ &+ 2\alpha_n\beta_n\langle f(y^*) - x^*, W_n - x^*\rangle \\ \leq&(1-\alpha_n\beta_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n\bar{a}\|y_n - y^*\|(\|W_n - x_{n+1}\| + \|x_{n+1} - x^*\|) \\ &+ 2\alpha_n\beta_n\langle f(y^*) - x^*, W_n - x^*\rangle \\ \leq&(1-\alpha_n\beta_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n\bar{a}\|y_n - y^*\|\|W_n - x_{n+1}\| \\ &+ \alpha_n\beta_n\bar{a}(\|y_n - y^*\|^2 + \|x_{n+1} - x^*\|^2) + 2\alpha_n\beta_n\langle f(y^*) - x^*, W_n - x^*\rangle,\end{aligned}$$

it implies that

$$\begin{aligned}(1-\alpha_n\beta_n\bar{a})\|x_{n+1} - x^*\|^2 \leq&(1-\alpha_n\beta_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n\bar{a}\|y_n - y^*\|\|W_n - x_{n+1}\| \\ &+ \alpha_n\beta_n\bar{a}\|y_n - y^*\|^2 + 2\alpha_n\beta_n\langle f(y^*) - x^*, W_n - x^*\rangle,\end{aligned}$$

then

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\leq\frac{1-\alpha_n\beta_n}{1-\alpha_n\beta_n\bar{a}}\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\bar{a}}{1-\alpha_n\beta_n\bar{a}}\|y_n - y^*\|\|W_n - x_{n+1}\| \\ &\quad+ \frac{\alpha_n\beta_n\bar{a}}{1-\alpha_n\beta_n\bar{a}}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\bar{a}}\langle f(y^*) - x^*, W_n - x^*\rangle \\ &=\left(1 - \frac{\alpha_n\beta_n(1-\bar{a})}{1-\alpha_n\beta_n\bar{a}}\right)\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\bar{a}}{1-\alpha_n\beta_n\bar{a}}\|y_n - y^*\|\|W_n - x_{n+1}\|\end{aligned}$$

$$+\frac{\alpha_n\beta_n\bar{a}}{1-\alpha_n\beta_n\bar{a}}\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\bar{a}}\langle f(y^*)-x^*,W_n-x^*\rangle. \tag{3.39}$$

Similarly, as derived above, we also have

$$\begin{aligned}&\|y_{n+1}-y^*\|^2\\ &\leq\left(1-\frac{\alpha_n\beta_n(1-\bar{a})}{1-\alpha_n\beta_n\bar{a}}\right)\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n\bar{a}}{1-\alpha_n\beta_n\bar{a}}\|x_n-x^*\|\|V_n-y_{n+1}\|\\ &\quad+\frac{\alpha_n\beta_n\bar{a}}{1-\alpha_n\beta_n\bar{a}}\|x_n-x^*\|^2+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\bar{a}}\langle g(x^*)-y^*,V_n-y^*\rangle.\end{aligned} \tag{3.40}$$

From (3.39) and (3.40), we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\ &\leq(1-\frac{\alpha_n\beta_n(1-2\bar{a})}{1-\alpha_n\beta_n\bar{a}})(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &\quad+\frac{2\alpha_n\beta_n\bar{a}}{1-\alpha_n\beta_n\bar{a}}(\|x_n-x^*\|\|V_n-y_{n+1}\|+\|y_n-y^*\|\|W_n-x_{n+1}\|)\\ &\quad+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\bar{a}}(\langle f(y^*)-x^*,W_n-x^*\rangle+\langle g(x^*)-y^*,V_n-y^*\rangle).\end{aligned} \tag{3.41}$$

By (3.23), (3.32), (3.33), (3.37), (3.38), the condition i), and Lemma 2.3, this implies by (3.41) that the sequences $\{x_n\}$ and $\{y_n\}$ converge to $x^*=P_\Phi f(y^*)$ and $y^*=P_\Phi g(x^*)$, respectively. This completes the proof. □

4. Applications

4.1. **The split variational inequality problem and the split feasibility problem.** In 2012, Censor et al. [19] introduced *the split variational inequality* (SVIP), which is to find $\widehat{x}\in C$ such that

$$\langle f_1\widehat{x},x-\widehat{x}\rangle\geq 0,\qquad \forall x\in C, \tag{4.42}$$

and find $\widehat{y}=D\widehat{x}\in Q$ such that

$$\langle f_2\widehat{y},y-\widehat{y}\rangle\geq 0,\qquad \forall y\in Q, \tag{4.43}$$

where $f_1:C\to H_1$ and $f_2:Q\to H_2$ are nonlinear mappings and $D:H_1\to H_2$ is a bounded linear operator. The set of all solution of the SVIP is denoted by $\phi=\{\widehat{x}\in VI(C,f_1):\widehat{y}\in VI(C,f_2)\}$. The SVIP reduces to the split feasibility problem (SFP) if $f_1\equiv f_2\equiv 0$.

Corollary 4.1. *Let C, Q, A, A_C, A_Q, f, and g define as the same in Theorem 3.2. Assume that $\phi\neq\emptyset$. For given $x_1,y_1\in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.13), where $M=A^*(I-P_Q(I-\lambda_Q A_Q))A$, $a=(0,\frac{1}{L_A})$ and parameters a_f, a_g, $\bar{a}$, λ_C, λ_Q, $\{\alpha_n\}$, $\{\beta_n\}$, and the conditions (i)-(iii) define as the same in Theorem 3.2. Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^*=P_\phi f(y^*)$ and $y^*=P_\phi g(x^*)$, respectively.*

Proof. If we put $A\equiv B$, in Theorem 3.2. The conclusion of Corollary 4.1 can be obtained from Theorem 3.2. □

Corollary 4.2. *Let C, Q, A, f, and g define as the same in Theorem 3.2. Assume that $\Gamma\neq\emptyset$. For given $x_1,y_1\in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ generated by*

$$\begin{cases}x_{n+1}= & (1-\alpha_n)x_n+\alpha_nP_C(\beta_nf(y_n)+(1-\beta_n)P_C(I-a(A^*(I-P_Q)A))x_n),\\ y_{n+1}= & (1-\alpha_n)y_n+\alpha_nP_C(\beta_ng(x_n)+(1-\beta_n)P_C(I-a(A^*(I-P_Q)A))y_n),\end{cases} \tag{4.44}$$

where $a \in (0, \frac{1}{L_A})$ *and parameters* a_f, a_g, $\bar{a}$, $\{\alpha_n\}$, $\{\beta_n\}$, *and the conditions (i)-(iii) define as the same in Theorem 3.2. Then,* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_\Gamma f(y^*)$ *and* $y^* = P_\Gamma g(x^*)$, *respectively.*

Proof. If we put $A_C \equiv A_Q \equiv 0$, in Corollary 4.1, we obtain the desired conclusion. □

4.2. **The minimization problem.** The *constrained minimization problem* is to find $x^* \in C$ such that

$$h(x^*) = \min_{x \in C} h(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2, \tag{4.45}$$

where $h : H_1 \to \mathbb{R}$ is a continuous differentiable function.

After that, Kantunyakarn [31] introduced the general constrained minimization problem as follows:

$$\min_{x \in C} h(x) := \frac{\|\left(I - P_Q\right)Ax\|^2}{4} + \frac{\|(I - P_Q)\,Bx\|^2}{4}. \tag{4.46}$$

The set of all solution of (4.46) is denoted by $\bar{\Gamma}_h = \{x^* \in C : h\,(x^*) \leq h\,(x)\,,\ \forall x \in C\}$.

By using the concepts of (4.45) and (4.46), we introduce the *modified general constrained minimization problem* as follows;

$$\min_{x \in C} h(x) := \frac{\|(I - P_Q(I - \lambda_Q A_Q))Ax\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))Bx\|^2}{4}. \tag{4.47}$$

The set of all solution of (4.47) is denoted by $\Gamma_h = \{x^* \in C : h(x^*) \leq h(x), \forall x \in C\}$.

From (1.3), if we put $A_C \equiv 0$, then we have a new problem to find $x^* \in C$ and $y_A^* = Ax^*, y_B^* = Bx^* \in Q$ such that

$$\langle z - y_A^*, A_Q y_A^* \rangle \geq 0, \tag{4.48}$$

and

$$\langle z - y_B^*, A_Q y_B^* \rangle \geq 0, \tag{4.49}$$

for all $z \in Q$. The set of all solutions of (4.48) and (4.49) is denoted by $\bar{\theta} = \{x^* \in C : Ax^*, Bx^* \in VI(Q, A_Q)\}$.

By applying Lemma 2.5, we get a lemma that expresses the relationship between the constrained minimization problem and problems (4.48) and (4.49) as follow:

Lemma 4.6. *Let* A^*, B^* *are adjoint of A and B, respectively, and let* $h : H_1 \to \mathbb{R}$ *be a continuous differentiable function defined by* $h(x) = \frac{\|(I - P_Q(I - \lambda_Q A_Q))Ax\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))Bx\|^2}{4}$, *for all* $x \in H_1$. *Assume that* $\bar{\theta} \neq \emptyset$. *Then the followings are equivalent.*

(i) $x^* \in \bar{\theta}$,
(ii) $x^* \in \Gamma_h$,
(iii) $x^* = P_C(I - a(\frac{A^*(I - P_Q(I - \lambda_Q A_Q))A}{2} + \frac{B^*(I - P_Q(I - \lambda_Q A_Q))B}{2}))x^*$.

Proof. i) ⇔ iii). If we put $A_C \equiv 0$ in Lemma 2.5, it is easy to see that i) equivalent iii).

ii) ⇒ i). Let $x^* \in \Gamma_h$ and let $\bar{x} \in \bar{\theta}$, we get $\bar{x} \in C$ and $A\bar{x}, B\bar{x} \in VI(Q, A_Q)$.
Since $x^* \in \Gamma_h$, we have

$$\begin{aligned}
&\frac{\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2}{4} \\
&\leq \frac{\|(I - P_Q(I - \lambda_Q A_Q))Ay\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))By\|^2}{4},
\end{aligned}$$

for all $y \in C$.
Since $\bar{x} \in C$, we have

$$\begin{aligned}
&\frac{\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2}{4} \\
&\leq \frac{\|(I - P_Q(I - \lambda_Q A_Q))A\bar{x}\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))B\bar{x}\|^2}{4}. \qquad (4.50)
\end{aligned}$$

Since $A\bar{x}, B\bar{x} \in VI(Q, A_Q)$, we have $A\bar{x} = P_Q(I - \lambda_Q A_Q)A\bar{x}$ and $B\bar{x} = P_Q(I - \lambda_Q A_Q)B\bar{x}$.
From (4.50), we have

$$\frac{\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2}{4} = 0.$$

It implies that $Ax^* = P_Q(I - \lambda_Q A_Q)Ax^* \in VI(Q, A_Q)$ and $Bx^* = P_Q(I - \lambda_Q A_Q)Bx^* \in VI(Q, A_Q)$.
Since $x^* \in \Gamma_h$ then $x^* \in C$.
Hence $x^* \in \bar{\theta}$.

i) $\Rightarrow$ ii). Let $x^* \in \bar{\theta}$, we have $x^* \in C$ and $Ax^*, Bx^* \in VI(Q, A_Q)$. Then, we have

$$\begin{aligned}
&\frac{\|(I - P_Q(I - \lambda_Q A_Q))Ax^*\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))Bx^*\|^2}{4} = 0 \\
&\leq \frac{\|(I - P_Q(I - \lambda_Q A_Q))Ay\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))By\|^2}{4},
\end{aligned}$$

for all $y \in C$. It implies that $x^* \in \Gamma_h$. □

Remark 4.2. *From (4.47), we observe that* $\nabla h = \frac{A^*(I - P_Q(I - \lambda_Q A_Q))A}{2} + \frac{B^*(I - P_Q(I - \lambda_Q A_Q))B}{2}$, *where* A^* *and* B^* *are adjoint of* A *and* B*, respectively, and* ∇h *is a gradient of* h.

We get the result of ∇h to prove Theorem 4.3 as follow:

Theorem 4.3. *Let* A_C, A_Q, f, *and* g *define as the same in Theorem 3.2. Let the function* $h : H_1 \to \mathbb{R}$ *be a continuous differentiable function defined by* $h(x) = \frac{\|(I - P_Q(I - \lambda_Q A_Q))Ax\|^2}{4} + \frac{\|(I - P_Q(I - \lambda_Q A_Q))Bx\|^2}{4}$, *for all* $x \in H_1$. *Assume that* $\Gamma_h \neq \emptyset$. *For given* $x_1, y_1 \in C$ *and let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by*

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n P_C(\beta_n f(y_n) + (1 - \beta_n)P_C(I - a\nabla h)x_n), \\ y_{n+1} &= (1 - \alpha_n)y_n + \alpha_n P_C(\beta_n g(x_n) + (1 - \beta_n)P_C(I - a\nabla h)y_n), \end{cases} \qquad (4.51)$$

where parameters a_f, a_g, $\bar{a}$, λ_C, λ_Q, $\{\alpha_n\}$, $\{\beta_n\}$, *and the conditions (i)-(iii) define as the same in Theorem 3.2. Then,* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\Gamma_h} f(y^*)$ *and* $y^* = P_{\Gamma_h} g(x^*)$, *respectively.*

Proof. From Theorem 3.2 and Lemma 4.6, we can conclude Theorem 4.3. □

5. Examples

In this section, we present two distinct numerical examples: one in $\mathbb{R}^2$ and another in l_2.

Example 5.1. *Let* $\mathbb{R}$ *be the set of real numbers, and let* $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *be an inner product defined by* $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2$ *for all* $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ *and a usual norm* $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$, *where* $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. *Let* $H_1 = H_2 = \mathbb{R}^2$, $C = [-100, 100] \times [-100, 100]$, $Q = [-200, 200] \times [-200, 200]$. *Let* $A_C : C \to H_1$ *and* $A_Q : Q \to H_2$ *be defined by*

$$A_C\mathbf{x} = (\frac{x_1}{2}, \frac{x_2}{3}) \qquad \text{and} \qquad A_Q\mathbf{x} = (\frac{x_1}{4}, \frac{x_2}{5}),$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$*. Let the mappings* $f, g : H_1 \to H_1$ *be defined by*

$$f(\mathbf{x}) = (\frac{x_1}{4}, \frac{x_2}{5}),$$

and

$$g(\mathbf{x}) = (\frac{x_1 - 2}{5}, \frac{x_2 + 3}{6}),$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$*. Let the mappings* $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$A(\mathbf{x}) = (2x_1 - x_2, x_1 + 2x_2) \text{ and } B(\mathbf{x}) = (x_1 + x_2, x_1 - x_2),$$

and let $A^*, B^* : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$A^*(\mathbf{x}) = (2x_1 + x_2, -x_1 + 2x_2) \text{ and } B^*(\mathbf{x}) = (x_1 + x_2, x_1 - x_2),$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$*.*

From the definitions of A_C*,* A_Q*,* f*, and* g*, then* A_C*,* A_Q *are* $\frac{1}{2}$ *and* $\frac{1}{3}$*-inverse strongly monotone, respectively, and* f*,* g *are* $\frac{1}{3}$ *and* $\frac{1}{4}$*-contraction mappings, respectively, with* $\lambda_C \in (0, 1)$*,* $\lambda_Q \in (0, \frac{2}{3})$*, and* $\bar{a} = \max\{\frac{1}{3}, \frac{1}{4}\} = \frac{1}{3}$*. Moreover, the spectral radius of the operators* A^*A*,* B^*B *are* 5 *and* 2*, respectively, thus* $L = \max\{5, 2\} = 5$*. Define* $M : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$M(\mathbf{x}) = \frac{A^*(I - P_Q(I - \lambda_Q A_Q))A}{2} + \frac{B^*(I - P_Q(I - \lambda_Q A_Q))B}{2}, \qquad \forall \mathbf{x} \in \mathbb{R}^2.$$

Let $x_1 = (x_1^1, x_1^2), y_1 = (y_1^1, y_1^2) \in C$ *and the sequences* $\{\mathbf{x_n}\}$*,* $\{\mathbf{y_n}\}$ *generated by (3.13), where* $\mathbf{x_n} = (x_n^1, x_n^2)$*,* $\mathbf{y_n} = (y_n^1, y_n^2)$*,* $\alpha_n = \frac{1}{n^2}$ *and* $\beta_n = \frac{1}{n}$*, for every* $n \in \mathbb{N}$*. By the definitions of* A, B, A_C, A_Q, f *and* g*, we have* $\{(0, 0)\} \in \Phi$*. For every* $n \in \mathbb{N}$*, we can rewrite (3.13) as follows:*

$$\mathbf{x_{n+1}} = (1 - \frac{1}{n^2})\mathbf{x_n} + (\frac{1}{n^2})P_C(\frac{1}{n} \cdot f(\mathbf{y_n}) + (1 - \frac{1}{n})P_C(I - \lambda_C A_C)(I - (\frac{1}{8})M)\mathbf{x_n}),$$
$$\mathbf{y_{n+1}} = (1 - \frac{1}{n^2})\mathbf{y_n} + (\frac{1}{n^2})P_C(\frac{1}{n} \cdot g(\mathbf{x_n}) + (1 - \frac{1}{n})P_C(I - \lambda_C A_C)(I - (\frac{1}{8})M)\mathbf{y_n}),$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$ *and* $P_Q(x_1, x_2) = (\max\{\min\{x_1, 200\}, -200\}, \max\{\min\{x_2, 200\}, -200\})$*.*

The following Table 1 and Figure 1 shows the values of $\{\mathbf{x_n}\}$ *and* $\{\mathbf{y_n}\}$ *with* $x_1 = (x_1^1, x_1^2) = (-1, 1)$*,* $y_1 = (y_1^1, y_1^2) = (-1, 1)$*,* $\lambda_C = \frac{1}{2}$*,* $\lambda_Q = \frac{1}{3}$ *and* $n = N = 500$*.*

TABLE 1. The values of $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ with $x_1 = (x_1^1, x_1^2) = (-1, 1)$, $y_1 = (y_1^1, y_1^2) = (-1, 1)$, and $n = N = 500$.

n	$\mathbf{x_n} = (x_n^1, x_n^2)$	$\mathbf{y_n} = (y_n^1, y_n^2)$
1	(-1.0000,1.0000)	(-1.0000,1.0000)
2	(-0.2500,0.2000)	(-0.6000,0.6667)
3	(-0.1748,0.1531)	(-0.4637,0.5556)
4	(-0.1359,0.1278)	(-0.3963,0.4997)
⋮	⋮	⋮
250	(-0.0024,0.0076)	(-0.0267,0.0724)
⋮	⋮	⋮
496	(-0.0011,0.0045)	(-0.0155,0.0484)
497	(-0.0011,0.0045)	(-0.0155,0.0483)
498	(-0.0011,0.0045)	(-0.0154,0.0483)
499	(-0.0011,0.0045)	(-0.0154,0.0482)
500	(-0.0011,0.0045)	(-0.0154,0.0481)

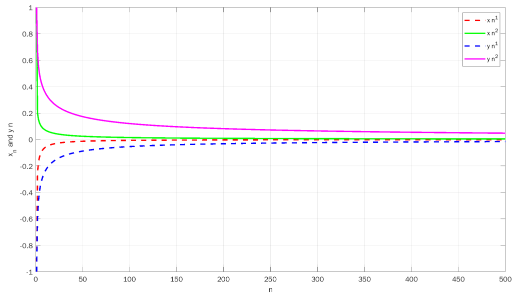

FIGURE 1. The convergence of $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ with $x_1 = (x_1^1, x_1^2) = (-1, 1)$, $y_1 = (y_1^1, y_1^2) = (-1, 1)$, and $n = N = 500$.

From Theorem 3.2 we can guarantee that the sequences $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ approach to $(0, 0)$.

Example 5.2. *Let $\mathbb{R}$ be the set of real numbers. Let $H_1 = l_2 := \{\mathbf{x} = (x_1, x_2, x_3, ...) : x_k \in \mathbb{R}, k = 1, 2, ...$ and $\sum_{k=1}^{\infty} x_k^2 < \infty\}$ with the inner product defined by*

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{k=1}^{\infty} x_k y_k, \qquad \text{for all } \mathbf{x} = (x_1, x_2, ...),\ \mathbf{y} = (y_1, y_2, ...) \in H_1,$$

and let $C = H(\alpha, 0) := \{\mathbf{z} = (z_1, z_2, ...) \in l_2 : \langle \alpha, z \rangle = 0\} = \{\mathbf{z} \in l_2 : \sum_{k=1}^{\infty} \alpha_k z_k = 0\}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, ...)$, we obtain

$$P_{H(\alpha,0)}\mathbf{x} = \mathbf{x} - \frac{\sum_{k=1}^{\infty} \alpha_k x_k}{\sum_{k=1}^{\infty} \alpha_k^2} \cdot \alpha.$$

TABLE 2. The values of $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ with $\mathbf{x_1} = (x_1^1, x_1^2, x_1^3, x_1^4, ...) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}, ...)$ and $\mathbf{y_1} = (y_1^1, y_1^2, y_1^3, y_1^4, ...) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{20}}, ...)$, and $n = N = 20$.

n	$\mathbf{x_n} = (x_n^1, x_n^2, x_n^3, x_n^4, ...)$	$\mathbf{y_n} = (y_n^1, y_n^2, y_n^3, y_n^4, ...)$
1	(0.7071,0.4082,0.2887,0.2236,...)	(-0.7071,-0.4082,-0.2887 ,-0.2236,...)
2	(-0.3536,-0.2041,-0.1443,-0.1118,...)	(0.1768,0.1021,0.0722,0.0559,...)
3	(-0.1316,-0.0760,-0.0556,-0.0431,...)	(0.0550, 0.0317 ,0.0206 ,0.0159,...)
4	(-0.0811,-0.0468,-0.0354,-0.0275,...)	(0.0320, 0.0185 ,0.0109 ,0.0084,...)
⋮	⋮	⋮
10	(-0.0271,-0.0156,-0.0134,-0.0105,...)	(0.0119,0.0069 ,0.0027 ,0.0020,...)
⋮	⋮	⋮
17	(-0.0166,-0.0096,-0.0089,-0.0070,...)	(0.0086, 0.0050 ,0.0017 ,0.0012,...)
18	(-0.0158,-0.0091,-0.0085,-0.0067,...)	(0.0084, 0.0049 ,0.0016 ,0.0011,...)
19	(-0.0151,-0.0087,-0.0082,-0.0065,...)	(0.0081, 0.0047 ,0.0015 ,0.0011,...)
20	(-0.0145,-0.0083,-0.0079,-0.0062,...)	(0.0079, 0.0046, 0.0015 ,0.0010,...)

Let $H_2 = \mathbb{R}^4$ with the inner product defined by $\langle \bar{x}, \bar{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, for all $\bar{x} = (x_1, x_2, x_3, x_4), \bar{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, and let $Q = [-100, 100] \times [-100, 100] \times [-100, 100] \times [-100, 100]$, we have $P_Q\bar{x} = (\max\{\min\{x_1, 100\}, -100\}, ..., \max\{\min\{x_4, 100\}, -100\})$, where $\bar{x} = (x_1, x_2, x_3, x_4)$. Let $A_C : C \to H_1$ be defined by

$$A_C(\mathbf{z}) = (\frac{z_1}{2}, \frac{z_2}{2}, ...), \qquad \text{for all } \mathbf{z} = (z_1, z_2, ...) \in C,$$

and let $A_Q : Q \to H_2$ defined by

$$A_Q(\bar{r}) = (r_1, \frac{r_2}{2}, \frac{r_3}{3}, \frac{r_4}{4}), \qquad \textit{for all } \bar{r} = (r_1, r_2, r_3, r_4) \in Q.$$

Let $A, B : H_1 \to H_2$ be defined by

$$A(\mathbf{x}) = (0, x_1, x_2, x_3), \qquad \textit{for all } \mathbf{x} = (x_1, x_2, ...) \in l_2,$$

and

$$B(\mathbf{x}) = (0, 0, x_1, x_2), \qquad \textit{for all } \mathbf{x} = (x_1, x_2, ...) \in l_2,$$

and let $A^*, B^* : H_2 \to H_1$ be defined by

$$A^*(\bar{x}) = (x_2, x_3, x_4, 0, 0, ...), \qquad \textit{for all } \bar{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,$$

and

$$B^*(\bar{x}) = (x_3, x_4, 0, 0, 0, ...), \qquad \textit{for all } \bar{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,$$

where $L_A = \sup\{|\lambda| : (A^*A - \lambda I)x = 0\} = 1$ and $L_B = \sup\{|\lambda| : (B^*B - \lambda I)x = 0\} = 1$ with $L = \max\{L_A, L_B\} = 1$ and $a \in (0, 1)$.

Let $f, g : H_1 \to H_1$ be defined by

$$f(\mathbf{h}) = (\frac{h_1}{2}, \frac{h_2}{2}, ...) \qquad \textit{and} \qquad g(\mathbf{h}) = (\frac{h_1}{4}, \frac{h_2}{4}, ...), \qquad \textit{for all } \mathbf{h} = (h_1, h_2, ...) \in l_2.$$

From the definitions of A_C, A_Q, f, and g, then A_C, A_Q are $\frac{1}{2}$ and $\frac{1}{3}$-inverse strongly monotone, respectively, and f, g are $\frac{1}{2}$ and $\frac{1}{4}$-contraction mappings, respectively, with $\lambda_C \in (0, 1)$, $\lambda_Q \in (0, \frac{2}{3})$, and $\bar{a} = \max\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{2}$. Let $M : H_1 \to H_1$ be defined by

$$M(\mathbf{x}) = \frac{A^*(I - P_Q(I - \lambda_Q A_Q))A(\mathbf{x})}{2} + \frac{B^*(I - P_Q(I - \lambda_Q A_Q))B(\mathbf{x})}{2},$$

for all $\mathbf{x} = (x_1, x_2, ...) \in l_2$. By the definitions of A, B, A_C, A_Q, f, and g, we have

$$P_C(I - \lambda_C A_C)(I - aM)(x_1, x_2, x_3, ...) = (x_1, x_2, x_3, ...),$$

then

$$(x_1, x_2, x_3, ...) = (0, 0, 0, 0, ...).$$

Hence, $(0, 0, 0, 0, ...)$ is fixed point of $P_C(I - \lambda_C A_C)(I - aM)$.

Let $\mathbf{x_1} = (x_1^1, x_1^2, x_1^3, ...), \mathbf{y_1} = (y_1^1, y_1^2, y_1^3, ...) \in C$, and the sequences $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ generated by (3.13), where $\mathbf{x_n} = (x_n^1, x_n^2, x_n^3, ...)$, $\mathbf{y_n} = (y_n^1, y_n^2, y_n^3, ...)$ for all $n \in \mathbb{N}$ and the parameters $\{\alpha_n\}$ and $\{\beta_n\}$ define as the same in Example 5.1 and we choose $a = \frac{1}{3}$, $\lambda_C = \frac{1}{4}$, and $\lambda_Q = \frac{1}{3}$. For every $n \in \mathbb{N}$, we can rewrite (3.13) as follows:

$$\begin{aligned} \mathbf{x_{n+1}} &= (1 - \frac{1}{n^2})\mathbf{x_n} + (\frac{1}{n^2})P_C(\frac{1}{n} \cdot f(\mathbf{y_n}) + (1 - \frac{1}{n})P_C(I - \frac{1}{4}A_C)(I - \frac{1}{3}M)\mathbf{x_n}), \\ \mathbf{y_{n+1}} &= (1 - \frac{1}{n^2})\mathbf{y_n} + (\frac{1}{n^2})P_C(\frac{1}{n} \cdot g(\mathbf{x_n}) + (1 - \frac{1}{n})P_C(I - \frac{1}{4}A_C)(I - \frac{1}{3}M)\mathbf{y_n}). \end{aligned}$$

From Theorem 3.2, we can conclude that the sequences $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ converge strongly to $(0, 0, 0, ...)$. To illustrate the numerical example, we choose

$$\alpha_k = \begin{cases} -\frac{1}{\sqrt{(k+1)(k+2)}} & ; k = 2n - 1, \\ \frac{1}{\sqrt{k(k-1)}} & ; k = 2n, \end{cases}$$

where $n \in \mathbb{N}$, we have $C = \{\mathbf{z} = (z_1, z_2, z_3, ...) \in H_1 \mid \sum_{k=1}^{\infty} \alpha_k z_k = 0\}$ and $P_C\mathbf{x} = \mathbf{x} - \frac{\sum_{k=1}^{\infty} \alpha_k x_k}{\sum_{k=1}^{\infty} \alpha_k^2} \cdot \alpha = \mathbf{x} - \alpha \cdot \sum_{k=1}^{\infty} \alpha_k x_k$, where $\mathbf{x} = (x_1, x_2, x_3, ...)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, ...)$.

To demonstrate the convergence of the sequences $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$, where $\mathbf{x_n} = (x_n^1, x_n^2, x_n^3, x_n^4, ...)$, $\mathbf{y_n} = (y_n^1, y_n^2, y_n^3, y_n^4, ...)$ for all $n \in \mathbb{N}$, we observe that the values $x_n^5, x_n^6, x_n^7, ...$ and $y_n^5, y_n^6, y_n^7, ...$

exhibit the same convergence behavior as $x_n^1, x_n^2, x_n^3, x_n^4$ *and* $y_n^1, y_n^2, y_n^3, y_n^4$*, respectively. Therefore, we only show the first four values of the sequences* $\{\mathbf{x_n^k}\}$ *and* $\{\mathbf{y_n^k}\}$*. The following Table 2 and Figure 2 show the values of* $\{\mathbf{x_n}\}$ *and* $\{\mathbf{y_n}\}$ *with* $\mathbf{x_1} = (x_1^1, x_1^2, x_1^3, x_1^4, \ldots)$ *and* $\mathbf{y_1} = (y_1^1, y_1^2, y_1^3, y_1^4, \ldots)$*, where* $\mathbf{x_1^k} = \frac{1}{\sqrt{k(k+1)}}$ *and* $\mathbf{y_1^k} = -\frac{1}{\sqrt{k(k+1)}}$*, for all* $k \in \mathbb{N}$*, and* $n = N = 20$*.*

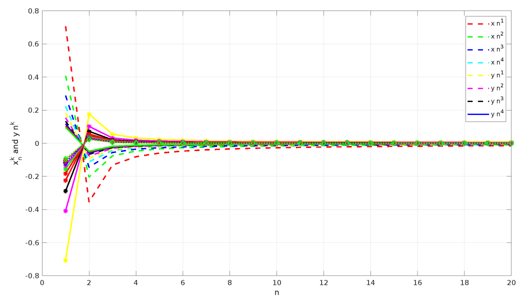

FIGURE 2. The convergence of the sequences $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ with $\mathbf{x_1} = (x_1^1, x_1^2, x_1^3, x_1^4, \ldots) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}, \ldots)$ and $\mathbf{y_1} = (y_1^1, y_1^2, y_1^3, y_1^4, \ldots) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{20}}, \ldots)$, and $n = N = 20$.

From Theorem 3.2 we can guarantee that the sequences $\{\mathbf{x_n}\}$ *and* $\{\mathbf{y_n}\}$ *approach to* $(0, 0, 0, \ldots)$*, where* $\mathbf{x_n} = (x_n^1, x_n^2, x_n^3, x_n^4, \ldots)$ *and* $\mathbf{y_n} = (y_n^1, y_n^2, y_n^3, y_n^4, \ldots)$*.*

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