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Algorithmic and Analytical Approach for a System of Generalized Multi-valued Resolvent Equations- Part II: Algorithms and Convergence

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ABSTRACT. The concept of resolvent operator associated with a P-η-accretive mapping is used in constructing of a new iterative algorithm for solving a new system of generalized multi-valued resolvent equations in the framework of Banach spaces. The convergence analysis of the sequences generated by our proposed iterative algorithm under some appropriate conditions is studied. The results presented in this paper are new, and improve and generalize many known corresponding results.

1. INTRODUCTION

Building on the foundational work established in Part I [1], this section delves into the convergence properties of the proposed iterative algorithm for solving the system of generalized multi-valued resolvent equations within Banach spaces. Part I introduced the concept of the resolvent operator associated with a $P-\eta$-accretive mapping and demonstrated its Lipschitz continuity, accompanied by an estimate of its Lipschitz constant under novel conditions. These results provided a solid theoretical framework and generalized several existing findings in the field. Additionally, Part I laid out definitions, examples, and the primary theoretical advancements, which collectively highlighted the versatility and applicability of the newly proposed algorithm. Also, we reviewed some definitions and theoretical developments of the η-accretive, strictly η-accretive, r-strongly η-accretive, ϱ-Lipschitz continuous, k-strongly η-accretive and generalized m-accretive (or η-m-accretive) for vector-valued mapping and multi-valued mapping. Also, we derived our main result showing that if P is a δ-strongly η-accretive mapping, η is a τ-Lipschitz continuous mapping and M is a P-η-accretive mapping this implies that P-η-proximal-point mapping J_ρ^M is $\frac{\tau}{\delta}$-Lipschitz continuous.

In this part, we extend the analysis by focusing on the convergence of the sequences generated by the algorithm. By imposing appropriate conditions, we aim to ensure rigorous convergence results, further enhancing the algorithm's utility and reliability in addressing complex variational inclusion problems.

In 2009, Ahmad and Yao [4] considered and studied a system of generalized resolvent equations (for short, SGRE) in a uniformly smooth Banach space setting. They established an equivalence relation between the SGRE and a system of variational inclusions and then by employing the obtained equivalence formulation, they suggested some iterative algorithms for finding the approximate solutions of the SGRE. Finally, they discussed the existence of the solution of the SGRE and studied the convergence analysis of the sequences generated by their proposed iterative algorithms.

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Inspired and motivated by recent research going on in this fascinating and interest fields, in this paper, using the notion of resolvent operator associated with a P-η-accretive mapping, we first construct a new iterative algorithm for finding the approximate solution of a system of generalized multi-valued resolvent equations (for short, SGMRE) in a real Banach space setting. Then, we discuss the existence of the solution of the SGMRE and study the convergence analysis of the sequences generated by our iterative algorithm under some appropriate conditions. The results derived in this paper are new, and improve and generalize the results presented in [4] and many known corresponding results.

2. Preliminary Notions and Results

In order to make the paper self-contained we begin by introducing some preliminary notions. Consider E a real Banach space and E^* its topological dual space. For the sake of simplicity, the norms in E and E^* will be designated by the same symbol $\|.\|$, and the metric induced by the norm $\|.\|$ will be denoted by d. As usual, the notation x^* stands for the weak* topology in E^*, while by $\langle x, x^*\rangle$ we denote the value of the inner continuous functional $x^* \in E^*$ at $x \in E$. We also use the symbol $CB(E)$ (resp. 2^E) to represent the set of all nonempty closed and bounded (resp., all nonempty) subsets of E. We define the graph and range of a given multi-valued mapping $M : E \to 2^E$ by

$$\mathrm{Graph}(M) := \{(x, u) \in E \times E : u \in M(x)\}$$

and

$$\mathrm{Range}(M) \quad := \quad \{y \in E : \exists x \in E : (x, y) \in \mathrm{Graph}(M)\} = \bigcup_{x\in E} M(x),$$

respectively. We shall denote by S_E and B_E respectively the unite sphere and the unit ball in E.

Let us recall that a normed space E is called strictly convex if S_E is strictly convex, that is, the inequality $\|x + y\| < 2$ holds for all distinct unit vectors x and y in E. It is said to be smooth if for every vector x in E there exists a unique $x^* \in E^*$ such that $\|x^*\| = \langle x, x^*\rangle = 1$.

It is known that E is smooth if E^* is strictly convex, and that E is strictly convex if E^* is smooth.

Definition 2.1. *A normed space E is said to be uniformly convex if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B_E$ with $\|x - y\| \geq \varepsilon$ the inequality $\|x + y\| \leq 2(1 - \delta)$ holds.*

The modulus of convexity of E is a function $\delta_E : [0, 2] \to [0, 1]$ defined in the following way:

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in B_E, \|x - y\| \geq \varepsilon\}.$$

It should be pointed out that in the definition of $\delta_E(\varepsilon)$ we can as well take the infimum over all vectors $x, y \in S_E$ with $\|x - y\| = \varepsilon$, see for example [5]. The functional δ_E is continuous and increasing on the interval $[0, 2]$ and $\delta_E(0) = 0$. Obviously, invoking the definition of the function δ_E, a normed space E is uniformly convex if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$. For any Banach space E, its modulus of convexity is bounded from above by the modulus of convexity of a Hilbert space $\mathcal{H}$, $\delta_E(\varepsilon) \leq \delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$. This means that Hilbert spaces are the most convex among all Banach spaces.

Definition 2.2. *A normed space E is called uniformly smooth if for any given $\varepsilon > 0$ there exists $\tau > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1$ and $\|y\| \leq \tau$, the inequality $\|x + y\| + \|x - y\| \leq 2 + \varepsilon\|y\|$ holds.*

The function $\rho_E : [0, +\infty) \to [0, +\infty)$ defined by the formula

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in S_E\}$$

is called the modulus of smoothness of E. Note, in particular, that the function ρ_E is convex, continuous and increasing on the interval $[0, +\infty)$ and $\rho_E(0) = 0$. In addition $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$. In the light of the definition of the function ρ_E, a normed space is uniformly smooth if $\lim\limits_{\tau \to 0} \tau^{-1}\rho_E(\tau) = 0$.

Any uniformly convex and any uniformly smooth Banach space is reflexive. A Banach space E is uniformly convex (resp., uniformly smooth) if and only if E^* is uniformly smooth (resp., uniformly convex). The spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex as well as uniformly smooth, see [6,8,13]. At the same time, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, can be found in [6,8,13].

Let us recall that the normalized duality mapping $\mathcal{F} : E \to 2^{E^*}$ is defined by

$$\mathcal{F}(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall x \in E.$$

We observe immediately that if $E = \mathcal{H}$, a Hilbert space, then $\mathcal{F}$ is the identity mapping on $\mathcal{H}$. Furthermore, it is an immediate consequence of the Hahn-Banach theorem that $\mathcal{F}(x)$ is nonempty for each $x \in E$. In the sequel, the notation j will be used to represent a selection of the normalized duality mapping $\mathcal{F}$.

Definition 2.3. *Let $P : E \to E$ and $\eta : E \times E \to E$ be two vector-valued mappings and $\mathcal{F} : E \to 2^{E^*}$ be the normalized duality mapping. Then P is said to be*

(i) *η-accretive if,*

$$\langle P(x) - P(y), j(\eta(x, y))\rangle \geq 0, \qquad \forall x, y \in E, j(\eta(x, y)) \in \mathcal{F}(\eta(x, y));$$

(ii) *strictly η-accretive if, P is η-accretive and equality holds if and only if $x = y$;*

(iii) *r-strongly η-accretive if there exists a constant $r > 0$ such that*

$$\langle P(x) - P(y), j(\eta(x, y))\rangle \geq r\|x - y\|^2, \quad \forall x, y \in E, j(\eta(x, y)) \in \mathcal{F}(\eta(x, y));$$

(iv) *ϱ-Lipschitz continuous if there exists a constant $\varrho > 0$ such that*

$$\|P(x) - P(y)\| \leq \varrho\|x - y\|, \quad \forall x, y \in E.$$

It should be pointed that if $\eta(x, y) = x - y$, for all $x, y \in E$, then parts (i) to (iii) of Definition 2.3 reduce to the definitions of accretivity, strict accretivity and strong accretivity of the mapping P, respectively.

Definition 2.4. *Let $\eta : E \times E \to E$ be a vector-valued mapping, $M : E \to 2^E$ be a multi-valued mapping, and let $\mathcal{F} : E \to 2^{E^*}$ be the normalized duality mapping. Then M is said to be*

(i) *η-accretive if*

$$\langle u - v, j(\eta(x, y))\rangle \geq 0, \quad \forall (x, u), (y, v) \in \text{Graph}(M), \quad j(\eta(x, y)) \in \mathcal{F}(\eta(x, y));$$

(ii) *strictly η-accretive if, M is η-accretive and equality holds if and only if $x = y$;*

(iii) *k-strongly η-accretive if there exists a constant $k > 0$ such that*

$$\langle u - v, j(\eta(x, y))\rangle \geq k\|x - y\|^2, \quad \forall (x, u), (y, v) \in \text{Graph}(M), \quad j(\eta(x, y)) \in \mathcal{F}(\eta(x, y));$$

(iv) *generalized m-accretive (or η-m-accretive) if M is η-accretive and $(I + \rho M)(E) = E$ holds for every real constant $\rho > 0$, where I stands for identity mapping.*

It should be remarked that if $\eta(x,y) = x - y$ for all $x, y \in E$, then parts (i) to (iv) of Definition 2.4 reduce to the definitions of accretivity, strict accretivity, strong accretivity and m-accretivity of the mapping M, respectively.

We note that M is a generalized m-accretive (or η-m-accretive) mapping if and only if M is η-accretive and there is no other η-accretive mapping whose graph contains strictly $\mathrm{Graph}(M)$. The generalized m-accretivity is to be understood in terms of inclusion of graphs. If $M : E \to 2^E$ is a generalized m-accretive mapping, then adding anything to its graph so as to obtain the graph of a new multi-valued mapping, destroys the η-accretivity. In fact, the extended mapping is no longer η-accretive. In other words, for every pair $(x,u) \in E \times E \setminus \mathrm{Graph}(M)$ there exist $(y,v) \in \mathrm{Graph}(M)$ and $j(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$ such that $\langle u - v, j(\eta(x,y))\rangle < 0$. In the light of the above-mentioned discussion, a necessary and sufficient condition for a multi-valued mapping $M : E \to 2^E$ to be generalized m-accretive is that for any $(x,u) \in E \times E$, the property

$$\langle u - v, j(\eta(x,y))\rangle \geq 0, \quad \forall (y,v) \in \mathrm{Graph}(M), \quad j(\eta(x,y)) \in \mathcal{F}(\eta(x,y))$$

is equivalent to $(x,u) \in \mathrm{Graph}(M)$. The above characterization of generalized m-accretive mappings provides us a useful and manageable way for recognizing that an element u belongs to $M(x)$.

Definition 2.5. *Let $P : E \to E$ and $\eta : E \times E \to E$ be vector-valued mappings, and $M : E \to 2^E$ be a multi-valued mapping. M is said to be P-η-accretive if M is η-accretive and $(P + \rho M)(E) = E$ holds for every real constant $\rho > 0$.*

3. Formulations, Iterative Algorithms and Convergence Results

Let for each $i \in \{1,2\}$, E_i be a real Banach space, $\eta_i : E_i \times E_i \to E_i$ be a vector-valued mapping and $P_i : E_i \to E_i$ be a strictly η_i-accretive mapping. Suppose that $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$, $f : E_1 \to E_1$ and $g : E_2 \to E_2$ are single-valued mappings, and let $p, H : E_1 \to 2^{E_1}$ and $q, F : E_2 \to 2^{E_2}$ be four multi-valued mappings. Let $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ be two multi-valued mappings such that for each $z \in E_1$, $M(.,z) : E_1 \to 2^{E_1}$ is a P_1-η_1-accretive mapping with $f(E_1) \cap \mathrm{D}(M(.,z)) \neq \emptyset$, and for each $t \in E_2$, $N(.,t) : E_2 \to 2^{E_2}$ is a P_2-η_2-accretive mapping with $g(E_2) \cap \mathrm{D}(N(.,t)) \neq \emptyset$. For given two arbitrary real constants $\rho, \gamma > 0$, we consider the problem of finding $(x,y) \in E_1 \times E_2$, $w \in p(x)$, $u \in H(x)$, $v \in F(y)$, $\nu \in q(y)$, $z' \in E_1$ and $z'' \in E_2$ such that

$$\text{(3.1)} \qquad \begin{cases} S(w,v) + \rho^{-1} R^{M(.,x),P_1}_{\rho,\eta_1}(z') = 0, \\ T(u,\nu) + \gamma^{-1} R^{N(.,y),P_2}_{\gamma,\eta_2}(z'') = 0, \end{cases}$$

where $R^{M(.,x),P_1}_{\rho,\eta_1} = I_1 - P_1 \circ J^{M(.,x),P_1}_{\rho,\eta_1} = I_1 - P_1(J^{M(.,x),P_1}_{\rho,\eta_1}(.))$, $R^{N(.,y),P_2}_{\gamma,\eta_2} = I_2 - P_2 \circ J^{N(.,y),P_2}_{\gamma,\eta_2} = I_2 - P_2(J^{N(.,y),P_2}_{\gamma,\eta_2}(.))$, I_i is the identity mapping on E_i, $J^{M(.,x),P_1}_{\rho,\eta_1}$ is the resolvent operator (or P_1-η_1-proximal-point mapping) associated with P_1-η_1-accretive mapping $M(.,x)$, $J^{N(.,y),P_2}_{\gamma,\eta_2}$ is the resolvent operator (or P_2-η_2-proximal-point mapping) associated with P_2-η_2-accretive mapping $N(.,y)$, and $P_1 \circ J^{M(.,x),P_1}_{\rho,\eta_1}$ (resp. $P_2 \circ J^{N(.,y),P_2}_{\gamma,\eta_2}$) denotes P_1 composition $J^{M(.,x),P_1}_{\rho,\eta_1}$ (resp. P_2 composition $J^{N(.,y),P_2}_{\gamma,\eta_2}$). The problem (3.1) is called a *system of generalized multi-valued resolvent equations* (SGMRE).

Let E_i, P_i, η_i $(i = 1,2)$, $M, N, S, T, F, H, f, g, p$ and q be the same as in the SGMRE (3.1). Corresponding to the SGMRE (3.1), we now consider the following system of generalized variational inclusions (SGVI): Find $(x,y) \in E_1 \times E_2$, $w \in p(x), u \in H(x), v \in F(y)$ and $\nu \in q(y)$ such that

$$\text{(3.2)} \qquad \begin{cases} 0 \in S(w,v) + M(f(x),x), \\ 0 \in T(u,\nu) + N(g(y),y). \end{cases}$$

If $p : E_1 \to E_1$ and $q : E_2 \to E_2$ be two single-valued mappings, $M : E_1 \to 2^{E_1}$ is a P_1-η_1-accretive mapping and $N : E_2 \to 2^{E_2}$ is a P_2-η_2-accretive mapping, then the system (3.1) reduces to the problem of finding $(x, y) \in E_1 \times E_2$, $u \in H(x), v \in F(y), z' \in E_1$ and $z'' \in E_2$ such that

$$\left\{\begin{array}{l} S(p(x), v) + \rho^{-1} R^{M,P_1}_{\rho,\eta_1}(z') = 0, \\ T(u, q(y)) + \gamma^{-1} R^{N,P_2}_{\gamma,\eta_2}(z'') = 0, \end{array}\right. \tag{3.3}$$

where $R^{M,P_1}_{\rho,\eta_1} = I_1 - P_1 \circ J^{M,P_1}_{\rho,\eta_1} = I_1 - P_1(J^{M,P_1}_{\rho,\eta_1}(.))$, $R^{N,P_2}_{\gamma,\eta_2} = I_2 - P_2 \circ J^{N,P_2}_{\gamma,\eta_2} = I_2 - P_2(J^{N,P_2}_{\gamma,\eta_2}(.))$, I_i is the identity mapping on E_i, J^{M,P_1}_{ρ,η_1} is the resolvent operator (or P_1-η_1-proximal-point mapping) associated with P_1-η_1-accretive mapping M, $J^{N,P_2}_{\gamma,\eta_2}$ is the resolvent operator (or P_2-η_2-proximal-point mapping) associated with P_2-η_2-accretive mapping N, and $P_1 \circ J^{M,P_1}_{\rho,\eta_1}$ (resp. $P_2 \circ J^{N,P_2}_{\gamma,\eta_2}$) denotes P_1 composition J^{M,P_1}_{ρ,η_1} (resp. P_2 composition $J^{N,P_2}_{\gamma,\eta_2}$). The problem (3.3) is also called a *system of generalized multi-valued resolvent equations*.

It should be remarked that for appropriate and suitable choices of the mappings $P_i, \eta_i (i = 1, 2)$, $M, N, S, T, F, H, f, g, p, q$ and the underlying spaces E_i $(i = 1, 2)$, the SGVI (3.2) includes various systems of variational inequalities/inclusions and many classes of variational inequality/inclusion problems, see, for example, [2–4, 7, 10–12] and the references therein.

As a direct consequence of the definition of resolvent operator (or P-η-proximal-point mapping) and using some simple arguments, we now present the following assertion which has a key role in obtaining the main results of this paper.

Lemma 3.1. *Suppose that $E_i, P_i, \eta_i (i = 1, 2), M, N, S, T, F, H, f, g, p$ and q are the same as in the* SGMRE (3.1). *Moreover, let for each $i \in \{1, 2\}$, P_i be a strictly η_i-accretive mapping, and the multi-valued mappings $M : E_1 \to 2^{E_1}$ and $N : E_2 \to 2^{E_2}$ be P_1-η_1-accretive and P_2-η_2-accretive, respectively. Then $(x, y, u, v, w, \nu) \in E_1 \times E_2 \times H(x) \times F(y) \times p(x) \times q(y)$ is a solution of the* SGVI (3.2) *if and only if (x, y, u, v, w, ν) satisfies the relations*

$$\left\{\begin{array}{l} f(x) = J^{M(.,x),P_1}_{\rho,\eta_1}[P_1 \circ f(x) - \rho S(w, v)], \\ g(y) = J^{N(.,y),P_2}_{\gamma,\eta_2}[P_2 \circ g(y) - \gamma T(u, \nu)], \end{array}\right. \tag{3.4}$$

where $\rho, \gamma > 0$, $J^{M(.,x),P_1}_{\rho,\eta_1}$ and $J^{N(.,y),P_2}_{\gamma,\eta_2}$ are the same as in the SGMRE (3.1), *and $P_1 \circ f$ (resp. $P_2 \circ g$) denotes P_1 composition f (resp. P_2 composition g).*

Thanks to the above last assertion, we now present the following conclusion in which the equivalence between the SGMRE (3.1) and SGVI (3.2) is established and has a prominent role in constructing iterative algorithms and in the study of convergence analysis of the sequences generated by our proposed iterative algorithms.

Proposition 3.1. *Let $E_i, P_i, \eta_i (i = 1, 2), M, N, S, T, F, H, f, g, p$ and q be the same as in Lemma 3.1. Then (x, y, u, v, w, ν) with $(x, y) \in E_1 \times E_2$, $u \in H(x), v \in F(y), w \in p(x)$ and $\nu \in q(y)$ is a solution of the* SGVI (3.2) *if and only if $(x, y, u, v, w, \nu, z', z'')$, where $(z', z'') \in E_1 \times E_2$, is a solution of the* SGMRE (3.1) *satisfying*

$$\left\{\begin{array}{l} f(x) = J^{M(.,x),P_1}_{\rho,\eta_1}(z'), \\ g(y) = J^{N(.,y),P_2}_{\gamma,\eta_2}(z''), \\ z' = P_1 \circ f(x) - \rho S(w, v), \\ z'' = P_2 \circ g(y) - \gamma T(u, \nu), \end{array}\right. \tag{3.5}$$

where $\rho, \gamma > 0$ and $J^{M(.,x),P_1}_{\rho,\eta_1}$ and $J^{N(.,y),P_2}_{\gamma,\eta_2}$ are the same as in the SGMRE (3.1).

Proof. Utilizing Lemma 3.1, $(x,y,u,v,w,\nu) \in E_1 \times E_2 \times H(x) \times F(y) \times p(x) \times q(y)$ is a solution of the SGVI (3.2) if and only if

$$\begin{cases} f(x) = J_{\rho,\eta_1}^{M(.,x),P_1}[P_1 \circ f(x) - \rho S(w,v)], \\ g(y) = J_{\gamma,\eta_2}^{N(.,y),P_2}[P_2 \circ g(y) - \gamma T(u,\nu)], \end{cases} \Leftrightarrow \begin{cases} f(x) = J_{\rho,\eta_1}^{M(.,x),P_1}(z'), \\ g(y) = J_{\gamma,\eta_2}^{N(.,y),P_2}(z''), \\ z' = P_1 \circ f(x) - \rho S(w,v), \\ z'' = P_2 \circ g(y) - \gamma T(u,\nu), \end{cases}$$

$$\Leftrightarrow \begin{cases} z' = P_1(J_{\rho,\eta_1}^{M(.,x),P_1}(z')) - \rho S(w,v), \\ z'' = P_2(J_{\gamma,\eta_2}^{N(.,y),P_2}(z'')) - \gamma T(u,\nu), \end{cases} \Leftrightarrow \begin{cases} \big(I_1 - P_1 \circ J_{\rho,\eta_1}^{M(.,x),P_1}\big)(z') = -\rho S(w,v), \\ \big(I_2 - P_2 \circ J_{\gamma,\eta_2}^{N(.,y),P_2}\big)(z'') = -\gamma T(u,\nu), \end{cases}$$

$$\Leftrightarrow \begin{cases} S(w,v) + \rho^{-1} R_{\rho,\eta_1}^{M(.,x),P_1}(z') = 0, \\ T(u,\nu) + \gamma^{-1} R_{\gamma,\eta_2}^{N(.,y),P_2}(z'') = 0, \end{cases}$$

where $R_{\rho,\eta_1}^{M(.,x),P_1} = I_1 - P_1 \circ J_{\rho,\eta_1}^{M(.,x),P_1}$ and $R_{\gamma,\eta_2}^{N(.,y),P_2} = I_2 - P_2 \circ J_{\gamma,\eta_2}^{N(.,y),P_2}$. Accordingly, $(x,y,u,v,w,\nu,z',z'') \in E_1 \times E_2 \times H(x) \times F(y) \times p(x) \times q(y)$ is a solution of the SGMRE (3.1). Hence, the solution sets of the SGMRE (3.1) and SGVI (3.2) are the same. This gives us the desired result. □

Lemma 3.2. *[14] Let E be a complete metric space, and $T : E \to CB(E)$ be a multi-valued mapping. Then for any $\varepsilon > 0$ and for any given $x, y \in E$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \leq (1+\varepsilon) D(T(x),T(y)),$$

where $D(.,.)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\}, \quad \forall A,B \in CB(E).$$

We employ Proposition 3.1 and Nadler technique [14] and suggest the following iterative algorithm for approximating a solution of the SGMRE (3.1).

Algorithm 3.1. Let $E_i, P_i, \eta_i (i=1,2), M, N, S, T, F, H, f, g, p$ and q be the same as in the SGMRE (3.1) such that the mappings f and g are onto. For any given $(x_0,y_0),(z_0',z_0'') \in E_1 \times E_2$, $u_0 \in H(x_0)$, $v_0 \in F(y_0)$, $w_0 \in p(x_0)$ and $\nu_0 \in q(y_0)$, compute the sequences $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$, $\{w_n\}_{n=0}^\infty$, $\{\nu_n\}_{n=0}^\infty$, $\{z_n'\}_{n=0}^\infty$ and $\{z_n''\}_{n=0}^\infty$ by the following iterative schemes:

$$\begin{cases} f(x_n) = J_{\rho,\eta_1}^{M(.,x_n),P_1}(z_n'), \\ g(y_n) = J_{\gamma,\eta_2}^{N(.,y_n),P_2}(z_n''), \\ u_n \in H(x_n); \quad \|u_{n+1} - u_n\|_1 \leq (1 + \frac{1}{1+n}) D_1(H(x_{n+1}),H(x_n)), \\ v_n \in F(y_n); \quad \|v_{n+1} - v_n\|_2 \leq (1 + \frac{1}{1+n}) D_2(F(y_{n+1}),F(y_n)), \\ w_n \in p(x_n); \quad \|w_{n+1} - w_n\|_1 \leq (1 + \frac{1}{1+n}) D_1(p(x_{n+1}),p(x_n)), \\ \nu_n \in q(y_n); \quad \|\nu_{n+1} - \nu_n\|_2 \leq (1 + \frac{1}{1+n}) D_2(q(y_{n+1}),q(y_n)), \\ z_{n+1}' = P_1 \circ f(x_n) - \rho S(w_n,v_n), \\ z_{n+1}'' = P_2 \circ g(y_n) - \gamma T(u_n,\nu_n), \end{cases} \tag{3.6}$$

where $n = 0,1,2,\ldots$; $\rho, \gamma > 0$ are positive real constants and for $i=1,2$, D_i is the Hausdorff metric on $CB(E_i)$.

If p, q, M and N are the same as in the system (3.3), then Algorithm 3.1 reduces to the following iterative algorithm.

Algorithm 3.2. Suppose that $E_i, P_i, \eta_i (i=1,2), S, T$ be the same as in the SGMRE (3.1) and M, N, p and q are the same as in the system (3.3). For any given $(x_0,y_0),(z_0',z_0'') \in$

$E_1 \times E_2$, $u_0 \in H(x_0)$ and $v_0 \in F(y_0)$, define the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$ in the following way:

$$\begin{cases} f(x_n) = J_{\rho,\eta_1}^{M,P_1}(z'_n), \\ g(y_n) = J_{\gamma,\eta_2}^{N,P_2}(z''_n), \\ u_n \in H(x_n); \qquad \|u_{n+1} - u_n\|_1 \le (1 + \frac{1}{1+n})D_1(H(x_{n+1}), H(x_n)), \\ v_n \in F(y_n); \qquad \|v_{n+1} - v_n\|_2 \le (1 + \frac{1}{1+n})D_2(F(y_{n+1}), F(y_n)), \\ z'_{n+1} = P_1 \circ f(x_n) - \rho S(p(x_n), v_n), \\ z''_{n+1} = P_2 \circ g(y_n) - \gamma T(u_n, q(y_n)), \end{cases}$$

where $n = 0, 1, 2, \ldots$; $\rho, \gamma > 0$ are positive real constants and for $i = 1, 2$, D_i is the Hausdorff metric on $CB(E_i)$.

We are now in a position to present the main result of this section regarding the strong convergence of the sequences generated by our suggested iterative algorithms to a solution of the SGMRE (3.1). For this end, we need to recall some definitions and a useful result.

Definition 3.6. *A mapping $T : E \times E \to E$ is said to be*

(i) *ξ-Lipschitz continuous in the first argument if, there exists a constant $\xi > 0$ such that*

$$\|T(x,z) - T(y,z)\| \le \xi\|x - y\|, \quad \forall x, y, z \in E;$$

(ii) *ς-Lipschitz continuous in the second argument if, there exists a constant $\varsigma > 0$ such that*

$$\|T(z,x) - T(z,y)\| \le \varsigma\|x - y\|, \quad \forall x, y, z \in E.$$

Definition 3.7. *A multi-valued mapping $S : E \to CB(E)$ is said to be D-Lipschitz continuous with constant λ_S (or λ_S-D-Lipschitz continuous) if, there exists a constant $\lambda_S > 0$ such that*

$$D(S(x), S(y)) \le \lambda_S\|x - y\|, \quad \forall x, y \in E;$$

where $D(.,.)$ is the Hausdorff metric on $CB(E)$.

Lemma 3.3. *[15] Let E be a uniformly smooth Banach space and $\mathcal{F}$ be the normalized duality mapping from E into E^*. Then, for all $x, y \in E$, we have*

(i) *$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle$;*

(ii) *$\langle x - y, J(x) - J(y)\rangle \le 2d^2(x,y)\rho_E\big(\frac{4\|x-y\|}{d(x,y)}\big)$, where $d(x,y) = \sqrt{\frac{\|x\|^2+\|y\|^2}{2}}$.*

Theorem 3.1. *Let for each $i \in \{1,2\}$, E_i be a real uniformly smooth Banach space with the dual space E_i^* and module of smoothness $\rho_{E_i}(t) \le C_i t^2$ for some $C_i > 0$. Assume that for each $i \in \{1,2\}$, $\eta_i : E_i \times E_i \to E_i$ is a τ_i-Lipschitz continuous mapping, and $P_i : E_i \to E_i$ is a ϱ_i-strongly η_i-accretive and ς_i-Lipschitz continuous mapping. Let $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ be two multi-valued mappings such that for each $z \in E_1$, $M(.,z) : E_1 \to 2^{E_1}$ is a P_1-η_1-accretive mapping with $f(E_1) \cap \mathrm{D}(M(.,z)) \ne \emptyset$, and for each $t \in E_2$, $N(.,t) : E_2 \to 2^{E_2}$ is a P_2-η_2-accretive mapping with $g(E_2) \cap \mathrm{D}(N(.,t)) \ne \emptyset$. Suppose that $S : E_1 \times E_2 \to E_1$ is a Lipschitz continuous mapping in the first and second arguments with constants λ_{S_1} and λ_{S_2}, respectively, $T : E_1 \times E_2 \to E_2$ is a Lipschitz continuous mapping in the first and second arguments with constants λ_{T_1} and λ_{T_2}, respectively, $f : E_1 \to E_1$ is an α-strongly accretive, δ_1-Lipschitz continuous onto mapping, and $g : E_2 \to E_2$ is a β-strongly accretive, δ_2-Lipschitz continuous and onto mapping. Let $p, H : E_1 \to CB(E_1)$ be D_1-Lipschitz continuous with constants λ_{D_p} and λ_{D_H}, respectively, and $q, F : E_2 \to CB(E_2)$ be D_2-Lipschitz continuous with constants λ_{D_q} and λ_{D_F}, respectively. If there exist constants σ_i $(i = 1,2)$, $\rho, \gamma > 0$ such that*

$$\|J_{\rho,\eta_1}^{M(.,x'),P_1}(z) - J_{\rho,\eta_1}^{M(.,x''),P_1}(z)\|_1 \le \sigma_1\|x' - x''\|_1, \qquad \forall x', x'', z \in E_1, \tag{3.7}$$

$$\|J_{\gamma,\eta_2}^{N(.,y'),P_2}(t) - J_{\gamma,\eta_2}^{N(.,y''),P_2}(t)\|_2 \le \sigma_2\|y' - y''\|_2, \qquad \forall y', y'', t \in E_2, \tag{3.8}$$

and

$$
(3.9) \qquad \begin{cases} 0 < \frac{(B'+\delta_1(1+\varsigma_1)+\sqrt{\hat{\theta}_1}+\sqrt{\hat{\theta}_4})\tau_1}{\varrho_1(1-B'-\sigma_1)} < 1, \\ 0 < \frac{(B''+\delta_2(1+\varsigma_2)+\sqrt{\hat{\theta}_2}+\sqrt{\hat{\theta}_3})\tau_2}{\varrho_2(1-B''-\sigma_2)} < 1, \end{cases}
$$

where

$$
\hat{\theta}_1 = \frac{1+\rho\lambda_{S_1}\lambda_{D_F}}{1-\rho(\lambda_{S_1}\lambda_{D_p}+\lambda_{S_2}\lambda_{D_F})}, \qquad \hat{\theta}_2 = \frac{\rho\lambda_{S_2}\lambda_{D_F}}{1-\rho(\lambda_{S_1}\lambda_{D_p}+\lambda_{S_2}\lambda_{D_F})},
$$

$$
\hat{\theta}_3 = \frac{1+\gamma\lambda_{T_2}\lambda_{D_q}}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_{D_q})}, \qquad \hat{\theta}_4 = \frac{\gamma\lambda_{T_1}\lambda_{D_H}}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_{D_q})},
$$

$B' = \sqrt{1-2\alpha+64C_1\delta_1^2}$ *and* $B'' = \sqrt{1-2\beta+64C_2\delta_2^2}$ *such that* $2\alpha < 1+64C_1\delta_1^2$ *and* $2\beta < 1+64C_2\delta_2^2$*. Then, the iterative sequences* $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{\nu_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ *and* $\{z''_n\}_{n=0}^{\infty}$ *generated by Algorithm 3.1 converge strongly to* x, y, u, v, w, ν, z' *and* z''*, respectively, and* $(x, y, u, v, w, \nu, z', z'')$ *is a solution of the* SGMRE (3.1).

Proof. Using (3.6), we yield

$$
(3.10) \qquad \begin{aligned} \|z'_{n+1} - z'_n\|_1 &= \|P_1 \circ f(x_n) - \rho S(w_n, v_n) - (P_1 \circ f(x_{n-1}) - \rho S(w_{n-1}, v_{n-1}))\|_1 \\ &\leq \|x_n - x_{n-1} - (P_1 \circ f(x_n) - P_1 \circ f(x_{n-1}))\|_1 \\ &\quad + \|x_n - x_{n-1} - \rho(S(w_n, v_n) - S(w_{n-1}, v_{n-1}))\|_1 \\ &\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|_1 \\ &\quad + \|f(x_n) - f(x_{n-1})\|_1 + \|P_1 \circ f(x_n) - P_1 \circ f(x_{n-1})\|_1 \\ &\quad + \|x_n - x_{n-1} - \rho(S(w_n, v_n) - S(w_{n-1}, v_{n-1}))\|_1. \end{aligned}
$$

Since f is α-strongly accretive and δ_1-Lipschitz continuous, and E_1 is a uniformly smooth Banach space with $\rho_{E_1}(t) \leq C_1 t^2$ for all $t \in E_1$ and for some $t \in E_1$, by using Lemma 3.3, it follows that

$$
\begin{aligned} &\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|_1^2 \\ &\quad \leq \|x_n - x_{n-1}\|_1^2 + 2\langle \mathcal{F}_1(x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))), -(f(x_n) - f(x_{n-1}))\rangle_1 \\ &\quad = \|x_n - x_{n-1}\|_1^2 - 2\langle \mathcal{F}_1(x_n - x_{n-1}), f(x_n) - f(x_{n-1})\rangle_1 \\ &\qquad + 2\langle \mathcal{F}_1(x_n - x_{n-1} - (f(x_n) - f(x_{n-1})) - \mathcal{F}_1(x_n - x_{n-1}), -(f(x_n) - f(x_{n-1}))\rangle_1 \\ &\quad \leq \|x_n - x_{n-1}\|_1^2 - 2\alpha\|x_n - x_{n-1}\|_1^2 \\ &\qquad + 4d_n^2(x_n - x_{n-1} - (f(x_n) - f(x_{n-1})), x_n - x_{n-1}) \\ &\qquad \times \rho_{E_1}\Big(\frac{4\|f(x_n) - f(x_{n-1})\|_1}{d_n(x_n - x_{n-1} - (f(x_n) - f(x_{n-1})), x_n - x_{n-1})}\Big) \\ &\quad \leq (1-2\alpha)\|x_n - x_{n-1}\|_1^2 + 64C_1\|f(x_n) - f(x_{n-1})\|_1^2 \\ &\quad = (1-2\alpha+64C_1\delta_1^2)\|x_n - x_{n-1}\|_1^2, \end{aligned}
$$

where $\mathcal{F}_1$ is the normalized duality mapping from E_1 into E_1^*. The last inequality implies that

$$
(3.11) \qquad \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|_1 \leq \sqrt{1-2\alpha+64C_1\delta_1^2}\|x_n - x_{n-1}\|_1.
$$

Considering the fact that f and P_1 are δ_1-Lipschitz and ς_1-Lipschitz continuous, respectively, we yield

$$
(3.12) \qquad \|f(x_n) - f(x_{n-1})\|_1 \leq \delta_1\|x_n - x_{n-1}\|_1
$$

and

$$\|P_1 \circ f(x_n) - P_1 \circ f(x_{n-1})\|_1 \leq \varsigma_1 \delta_1 \|x_n - x_{n-1}\|_1. \tag{3.13}$$

Making use of Lemma 3.3(i), we obtain

$$\begin{aligned}
&\|x_n - x_{n-1} - \rho(S(W_n, v_n) - S(w_{n-1}, v_{n-1}))\|_1^2 \\
&\leq \|x_n - x_{n-1}\|_1^2 - 2\rho\langle S(w_n, v_n) - S(w_{n-1}, v_{n-1}), \\
&\quad \mathcal{F}_1(x_n - x_{n-1} - \rho(S(w_n, v_n) - S(w_{n-1}, v_{n-1})))\rangle_1 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\rho\|S(w_n, v_n) - S(w_{n-1}, v_{n-1})\|_1 \\
&\quad \times \|x_n - x_{n-1} - \rho(S(w_n, v_n) - S(w_{n-1}, v_{n-1}))\|_1.
\end{aligned} \tag{3.14}$$

Taking into account that S is Lipschitz continuous in the first and second arguments with constants λ_{S_1} and λ_{S_2}, respectively, p and F are λ_{D_1}-D_1-Lipschitz continuous and λ_{D_F}-D_2-Lipschitz continuous, respectively, it follows that

$$\begin{aligned}
\|S(w_n, v_n) - S(w_{n-1}, v_{n-1})\|_1 &\leq \lambda_{S_1}\|w_n - w_{n-1}\|_1 + \lambda_{S_2}\|v_n - v_{n-1}\|_2 \\
&\leq \lambda_{S_1}(1 + \frac{1}{n})D_1(p(x_n), p(x_{n-1})) + \lambda_{S_2}(1 + \frac{1}{n})D_2(F(y_n), F(y_{n-1})) \\
&\leq \lambda_{S_1}\lambda_{D_p}(1 + \frac{1}{n})\|x_n - x_{n-1}\|_1 + \lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{n})\|y_n - y_{n-1}\|_2.
\end{aligned} \tag{3.15}$$

Let $\Omega = \|x_n - x_{n-1} - \rho(S(w_n, v_n) - S(w_{n-1}, v_{n-1}))\|_1$, then substituting (3.15) into (3.14), we get

$$\begin{aligned}
\Omega^2 &\leq \|x_n - x_{n-1}\|_1^2 + 2\rho(\lambda_{S_1}\lambda_{D_p}(1 + \frac{1}{n})\|x_n - x_{n-1}\|_1 + \lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{n})\|y_n - y_{n-1}\|_2)\Omega \\
&= \|x_n - x_{n-1}\|_1^2 + 2\rho\lambda_{S_1}\lambda_{D_p}(1 + \frac{1}{n})\|x_n - x_{n-1}\|_1\Omega + 2\rho\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{n})\|y_n - y_{n-1}\|_2\Omega \\
&\leq (1 + \rho\lambda_{S_1}\lambda_{D_p}(1 + \frac{1}{n}))\|x_n - x_{n-1}\|_1^2 + \rho(\lambda_{S_1}\lambda_{D_p} + \lambda_{S_2}\lambda_{D_F})(1 + \frac{1}{n})\Omega^2 \\
&\quad + \rho\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{n})\|y_n - y_{n-1}\|_2^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\Omega &\leq \hat{\theta}_{1,n}\|x_n - x_{n-1}\|_1^2 + \hat{\theta}_{2,n}\|y_n - y_{n-1}\|_2^2 \\
&\leq \hat{\theta}_{1,n}\|x_n - x_{n-1}\|_1^2 + 2\sqrt{\hat{\theta}_{1,n}\hat{\theta}_{2,n}}\|x_n - x_{n-1}\|_1\|y_n - y_{n-1}\|_2 + \hat{\theta}_{2,n}\|y_n - y_{n-1}\|_2^2 \\
&= \big(\sqrt{\hat{\theta}_{1,n}}\|x_n - x_{n-1}\|_1 + \sqrt{\hat{\theta}_{2,n}}\|y_n - y_{n-1}\|_2\big)^2,
\end{aligned} \tag{3.16}$$

where for all $n \in \mathbb{N}$,

$$\hat{\theta}_{1,n} = \frac{1 + \rho\lambda_{S_1}\lambda_{D_p}(1 + \frac{1}{n})}{1 - \rho(\lambda_{S_1}\lambda_{D_p} + \lambda_{S_2}\lambda_{D_F})(1 + \frac{1}{n})}, \quad \hat{\theta}_{2,n} = \frac{\rho\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{n})}{1 - \rho(\lambda_{S_1}\lambda_{D_p} + \lambda_{S_2}\lambda_{D_F})(1 + \frac{1}{n})}.$$

Therefore, using (3.16), we deduce that

$$\Omega \leq \sqrt{\hat{\theta}_{1,n}}\|x_n - x_{n-1}\|_1 + \sqrt{\hat{\theta}_{2,n}}\|y_n - y_{n-1}\|_2. \tag{3.17}$$

Combining (3.10)–(3.13) and (3.17), yields

$$\begin{aligned}
\|z'_{n+1} - z'_n\|_1 &\leq \big(\sqrt{1 - 2\alpha + 64C_1\delta_1^2} + \delta_1(1 + \varsigma_1) + \sqrt{\hat{\theta}_{1,n}}\big)\|x_n - x_{n-1}\|_1 \\
&\quad + \sqrt{\hat{\theta}_{2,n}}\|y_n - y_{n-1}\|_2 + \|P_1 \circ f(x_{n-1}) - P_1 \circ f(x_{n-1})\|_1 \\
&= (B' + \delta_1(1 + \varsigma_1) + \sqrt{\hat{\theta}_{1,n}})\|x_n - x_{n-1}\|_1 + \sqrt{\hat{\theta}_{2,n}}\|y_n - y_{n-1}\|_2,
\end{aligned} \tag{3.18}$$

where $B' = \sqrt{1-2\alpha+64C_1\delta_1^2}$. In the light of the facts that g is β-strongly accretive and δ_2-Lipschitz continuous, P_2 is ς_2-Lipschitz continuous, T is Lipscchitz continuous in the first and second arguments with constants λ_{T_1} and λ_{T_2}, respectively, H and q are λ_{D_H}-D_1-Lipschitz continuous and λ_{D_q}-D_2-Lipschitz continuous, respectively, and E_2 is a uniformly smooth Banach space with $\rho_{E_2}(t) \leq C_2t^2$ for all $t \in E_2$ and for some $C_2 > 0$, by an argument analogous to the previous one, by using Lemma 3.3 and (3.6), one can show that

$$\|z''_{n+1} - z''_n\|_2 \leq \sqrt{\hat{\theta}_{4,n}}\|x_n - x_{n-1}\|_1 + (B'' + \delta_2(1+\varsigma_2) + \sqrt{\hat{\theta}_{3,n}})\|y_n - y_{n-1}\|_2, \tag{3.19}$$

where

$$\hat{\theta}_{3,n} = \frac{1+\gamma\lambda_{T_2}\lambda_{D_q}(1+\frac{1}{n})}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_{D_q})(1+\frac{1}{n})}, \quad \hat{\theta}_{4,n} = \frac{\gamma\lambda_{T_1}\lambda_{D_H}(1+\frac{1}{n})}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_{D_q})(1+\frac{1}{n})}$$

and $B'' = \sqrt{1-2\beta+64C_2\delta_2^2}$. Let us now define a norm $\|.\|_*$ on $E_1 \times E_2$ by

$$\|(x,y)\|_* = \|x\|_1 + \|y\|_2, \quad \forall (x,y) \in E_1 \times E_2.$$

It can be easily observed that $(E_1 \times E_2, \|.\|_*)$ is a Banach space. Then, adding (3.18) and (3.19), for each $n \in \mathbb{N}$, we get

$$\begin{aligned}
\|(z'_{n+1}, z''_{n+1}) - (z'_n, z''_n)\|_* &= \|z'_{n+1} - z'_n\|_1 + \|z''_{n+1} - z''_n\|_2 \\
&\leq (B' + \delta_1(1+\varsigma_1) + \sqrt{\hat{\theta}_{1,n}} + \sqrt{\hat{\theta}_{4,n}})\|x_n - x_{n-1}\|_1 \\
&\quad + (B'' + \delta_2(1+\varsigma_2) + \sqrt{\hat{\theta}_{2,n}} + \sqrt{\hat{\theta}_{3,n}})\|y_n - y_{n-1}\|_2.
\end{aligned} \tag{3.20}$$

Applying (3.6), (3.11) and Theorem 3.2 of Part I, we obtain

$$\begin{aligned}
\|x_n - x_{n-1}\|_1 &= \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1})) \\
&\quad + J^{M(.,x_n),P_1}_{\rho,\eta_1}(z'_n) - J^{M(.,x_{n-1}),P_1}_{\rho,\eta_1}(z'_{n-1})\|_1 \\
&\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|_1 \\
&\quad + \|J^{M(.,x_n),P_1}_{\rho,\eta_1}(z'_n) - J^{M(.,x_{n-1}),P_1}_{\rho,\eta_1}(z'_{n-1})\|_1 \\
&\leq B'\|x_n - x_{n-1}\|_1 + \|J^{M(.,x_n),P_1}_{\rho,\eta_1}(z'_n) - J^{M(.,x_n),P_1}_{\rho,\eta_1}(z'_{n-1})\|_1 \\
&\quad + \|J^{M(.,x_n),P_1}_{\rho,\eta_1}(z'_{n-1}) - J^{M(.,x_{n-1}),P_1}_{\rho,\eta_1}(z'_{n-1})\|_1 \\
&\leq B'\|x_n - x_{n-1}\|_1 + \frac{\tau_1}{\varrho_1}\|z'_n - z'_{n-1}\|_1 + \sigma_1\|x_n - x_{n-1}\|_1
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\|y_n - y_{n-1}\|_2 &= \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1})) \\
&\quad + J^{N(.,y_n),P_2}_{\gamma,\eta_2}(z''_n) - J^{N(.,y_{n-1}),P_2}_{\gamma,\eta_2}(z''_{n-1})\|_2 \\
&\leq \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|_2 \\
&\quad + \|J^{N(.,y_n),P_2}_{\gamma,\eta_2}(z''_n) - J^{N(.,y_{n-1}),P_2}_{\gamma,\eta_2}(z''_{n-1})\|_2 \\
&\leq B''\|y_n - y_{n-1}\|_2 + \|J^{N(.,y_n),P_2}_{\gamma,\eta_2}(z''_n) - J^{N(.,y_n),P_2}_{\gamma,\eta_2}(z''_{n-1})\|_2 \\
&\quad + \|J^{N(.,y_n),P_2}_{\gamma,\eta_2}(z''_{n-1}) - J^{N(.,y_{n-1}),P_2}_{\gamma,\eta_2}(z''_{n-1})\|_2 \\
&\leq B''\|y_n - y_{n-1}\|_2 + \frac{\tau_2}{\varrho_2}\|z''_n - z''_{n-1}\|_2 + \sigma_2\|y_n - y_{n-1}\|_2.
\end{aligned} \tag{3.22}$$

Recalling (3.21) and (3.22) it follows that for each $n \in \mathbb{N}$,

$$\|x_n - x_{n-1}\|_1 \leq \frac{\tau_1}{\varrho_1(1 - B' - \sigma_1)}\|z'_n - z'_{n-1}\|_1 \tag{3.23}$$

and

$$\|y_n - y_{n-1}\|_2 \leq \frac{\tau_2}{\varrho_2(1 - B'' - \sigma_2)}\|z''_n - z''_{n-1}\|_2. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.20), we yield

$$\begin{aligned}
\|(z'_{n+1}, z''_{n+1}) - (z'_n, z''_n)\|_* &\leq (B' + \delta_1(1+\varsigma_1) + \sqrt{\hat{\theta}_{1,n}} + \sqrt{\hat{\theta}_{4,n}}) \times \|x_n - x_{n-1}\|_1 \\
&\quad + (B'' + \delta_2(1+\varsigma_1) + \sqrt{\hat{\theta}_{2,n}} + \sqrt{\hat{\theta}_{3,n}}) \times \|y_n - y_{n-1}\|_2 \\
&\leq \frac{(B' + \delta_1(1+\varsigma_1) + \sqrt{\hat{\theta}_{1,n}} + \sqrt{\hat{\theta}_{4,n}})\tau_1}{\varrho_1(1 - B' - \sigma_1)} \times \|z'_n - z'_{n-1}\|_1 \\
&\quad + \frac{(B'' + \delta_2(1+\varsigma_2) + \sqrt{\hat{\theta}_{2,n}} + \sqrt{\hat{\theta}_{3,n}})\tau_2}{\varrho_2(1 - B'' - \sigma_2)} \times \|z''_n - z''_{n-1}\|_2 \\
&\leq \varphi_n(\|z'_n - z'_{n-1}\|_1 + \|z''_n - z''_{n-1}\|_2) \\
&= \varphi_n\|(z'_n, z''_n) - (z'_{n-1}, z''_{n-1})\|_*,
\end{aligned} \tag{3.25}$$

where for each $n \in \mathbb{N}$,

$$\varphi_n = \max\Big\{\frac{(B' + \delta_1(1+\varsigma_1) + \sqrt{\hat{\theta}_{1,n}} + \sqrt{\hat{\theta}_{4,n}})\tau_1}{\varrho_1(1 - B' - \sigma_1)}, \frac{(B'' + \delta_2(1+\varsigma_2) + \sqrt{\hat{\theta}_{2,n}} + \sqrt{\hat{\theta}_{3,n}})\tau_2}{\varrho_2(1 - B'' - \sigma_2)}\Big\}.$$

In virtue of the fact that for $i = 1, 2, 3, 4$, we have $\hat{\theta}_{i,n} \to \hat{\theta}_i$, as $n \to \infty$, where

$$\hat{\theta}_1 = \frac{1 + \rho\lambda_{S_1}\lambda_{D_p}}{1 - \rho(\lambda_{S_1}\lambda_{D_p} + \lambda_{S_2}\lambda_{D_F})}, \qquad \hat{\theta}_2 = \frac{\rho\lambda_{S_2}\lambda_{D_F}}{1 - \rho(\lambda_{S_1}\lambda_{D_p} + \lambda_{S_2}\lambda_{D_F})},$$

$$\hat{\theta}_3 = \frac{1 + \gamma\lambda_{T_2}\lambda_{D_q}}{1 - \gamma(\lambda_{T_1}\lambda_{D_H} + \lambda_{T_2}\lambda_{D_q})} \quad \text{and} \quad \hat{\theta}_4 = \frac{\gamma\lambda_{T_1}\lambda_{D_H}}{1 - \gamma(\lambda_{T_1}\lambda_{D_H} + \lambda_{T_2}\lambda_{D_q})},$$

it follows that $\varphi_n \to \varphi$, as $n \to \infty$, where

$$\varphi = \max\Big\{\frac{(B' + \delta_1(1+\varsigma_1) + \sqrt{\hat{\theta}_1} + \sqrt{\hat{\theta}_4})\tau_1}{\varrho_1(1 - B' - \sigma_1)}, \frac{(B'' + \delta_2(1+\varsigma_2) + \sqrt{\hat{\theta}_2} + \sqrt{\hat{\theta}_3})\tau_2}{\varrho_2(1 - B'' - \sigma_2)}\Big\}.$$

Evidently, (3.9) implies that $\varphi \in (0, 1)$. Accordingly, there exists $n_0 \in \mathbb{N}$ and $\hat{\varphi} \in (\varphi, 1)$ such that $\varphi^n \leq \hat{\varphi}$, for all $n \geq n_0$. Consequently, for all $n > n_0$, by (3.25), we obtain

$$\begin{aligned}
\|(z'_{n+1}, z''_{n+1}) - (z'_n, z''_n)\|_* &\leq \varphi^n\|(z'_n, z''_n) - (z'_{n-1}, z''_{n-1})\|_* \\
&\leq \hat{\varphi}\|(z'_n, z''_n) - (z'_{n-1}, z''_{n-1})\|_* \\
&\leq \hat{\varphi}^2\|(z'_{n-1}, z''_{n-1}) - (z'_{n-2}, z''_{n-2})\|_* \\
&\leq \dots \\
&\leq \hat{\varphi}^{n-n_0}\|(z'_{n_0+1}, z''_{n_0+1}) - (z'_{n_0}, z''_{n_0})\|_*.
\end{aligned} \tag{3.26}$$

From (3.26), it follows that for any $m \geq n > n_0$,

$$\begin{aligned}
\|(z'_m, z''_m) - (z'_n, z''_n)\|_* &\leq \sum_{k=n}^{m-1} \|(z'_{k+1}, z''_{k+1}) - (z'_k, z''_k)\|_* \\
&\leq \sum_{k=n}^{m-1} \hat{\varphi}^{k-n_0}\|(z'_{n_0+1}, z''_{n_0+1}) - (z'_{n_0}, z''_{n_0})\|_*.
\end{aligned} \tag{3.27}$$

In view of the fact that $\hat{\varphi} \in (0, 1)$, we conclude that the right-hand side of (3.27) approaches zero, as $n \to \infty$, that is, $\|(z'_m, z''_m) - (z'_n, z''_n)\|_* \to 0$, as $n \to \infty$. Thereby, $\{(z'_n, z''_n)\}_{n=0}^{\infty}$ is a Cauchy sequence in $E_1 \times E_2$. In the light of the completeness of $E_1 \times E_2$, there is $(z', z'') \in E_1 \times E_2$ such that $(z'_n, z''_n) \to (z', z'')$, as $n \to \infty$. Making use of (3.23) and (3.24) it follows that $x_n \to x$ and $y_n \to y$, as $n \to \infty$. Then, using (3.6) and owing to the fact that the mappings H, F, p and q are λ_{D_H}-D_1-Lipschitz continuous, λ_{D_F}-D_2-Lipschitz continuous, λ_{D_p}-D_1-Lipschitz continuous and λ_{D_q}-D_2-Lipschitz continuous, respectively, it can be easily seen that $\{u_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{\nu_n\}_{n=0}^{\infty}$ are Cauchy sequences in E_1 and E_2, respectively. Consequently, there are $u, w \in E_1$ and $v, \nu_1 \in E_2$ such that $u_n \to u$, $w_n \to w$, $v_n \to v$ and $\nu_n \to \nu$, as $n \to \infty$. In view of the fact that $u_n \in H(x_n)$ for each $n \geq 0$, we have

$$\begin{aligned} d_1(u, H(x)) &= \inf\{\|u - s\|_1 : s \in H(x)\} \\ &\leq \|u - u_n\|_1 + d_1(u_n, H(x)) \\ &\leq \|u - u_n\|_1 + D_1(H(x_n), H(x)) \\ &\leq \|u - u_n\|_1 + \lambda_{D_H}\|x_n - x\|_1. \end{aligned}$$

Clearly, the right-hand side of the preceding inequality tends to zero, as $n \to \infty$, and so thanks to the fact that $H(x)$ is closed, it follows that $u \in H(x)$. Following the same argument, we can show that $v \in F(y)$, $w \in p(x)$ and $\nu \in q(y)$. Since $x_n \to x$, $y_n \to y$, $u_n \to u$, $w_n \to w$ and $\nu_n \to \nu$, as $n \to \infty$, and in view of the Lipschitz continuity of the mappings f, g, S and T, it follows that $z'_n \to z' = P_1 \circ f(x) - \rho S(w, v)$ and $z''_n \to z'' = P_2 \circ g(y) - \gamma T(u, \nu)$, as $n \to \infty$. In the meanwhile, using Theorem 3.2 of Part I, for each $n \geq 0$, we obtain

$$\begin{aligned} \|J_{\rho,\eta_1}^{M(.,x_n),P_1}(z'_n) - J_{\rho,\eta_1}^{M(.,x),P_1}(z')\|_1 &\leq \|J_{\rho,\eta_1}^{M(.,x_n),P_1}(z'_n) - J_{\rho,\eta_1}^{M(.,x_n),P_1}(z')\|_1 \\ &\quad + \|J_{\rho,\eta_1}^{M(.,x_n),P_1}(z') - J_{\rho,\eta_1}^{M(.,x),P_1}(z')\|_1 \\ &\leq \frac{\tau_1}{\varrho_1}\|z'_n - z'\|_1 + \varsigma_1\|x_n - x\|_1 \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \|J_{\gamma,\eta_2}^{N(.,y_n),P_2}(z''_n) - J_{\gamma,\eta_2}^{N(.,y),P_2}(z'')\|_2 &\leq \|J_{\gamma,\eta_2}^{N(.,y_n),P_2}(z''_n) - J_{\gamma,\eta_2}^{N(.,y_n),P_2}(z'')\|_2 \\ &\quad + \|J_{\gamma,\eta_2}^{N(.,y_n),P_2}(z'') - J_{\gamma,\eta_2}^{N(.,y),P_2}(z'')\|_2 \\ &\leq \frac{\tau_2}{\varrho_2}\|z''_n - z''\|_2 + \varsigma_2\|y_n - y\|_2. \end{aligned} \tag{3.29}$$

Now, taking into account that $z'_n \to z'$ and $z''_n \to z''$, as $n \to \infty$, (3.28) and (3.29) imply that

$$\begin{aligned} &\|J_{\rho,\eta_1}^{M(.,x_n),P_1}(z'_n) - J_{\rho,\eta_1}^{M(.,x),P_1}(z')\|_1 \to 0 \text{ and} \\ &\|J_{\gamma,\eta_2}^{N(.,y_n),P_2}(z''_n) - J_{\gamma,\eta_2}^{N(.,y),P_2}(z'')\|_2 \to 0, \text{ as } n \to \infty, \text{ as so} \end{aligned}$$

$$J_{\rho,\eta_1}^{M(.,x_n),P_1}(z'_n) \to J_{\rho,\eta_1}^{M(.,x),P_1}(z') \quad \text{and} \quad J_{\gamma,\eta_2}^{N(.,y_n),P_2}(z''_n) \to J_{\gamma,\eta_2}^{N(.,y),P_2}(z''),$$

as $n \to \infty$. Now, in the light of the Lipschitz continuity of the mappings f and g and using (3.6), we deduce that $f(x) = J_{\rho,\eta_1}^{M(.,x),P_1}(z')$ and $g(y) = J_{\gamma,\eta_2}^{N(.,y),P_2}(z'')$. Thereby, thanks to the above-mentioned arguments, it follows that $(x, y, u, v, w, \nu) \in E_1 \times E_2 \times H(x) \times F(y) \times p(x) \times q(y)$ is a solution of the SGMRE (3.1). This completes the proof. □

As an immediate consequence of the last result, we obtain the following corollary which generalizes and improves Theorem 3.1 in [4].

Corollary 3.1. *Suppose that $E_i, P_i, \eta_i (i = 1, 2), S, T, F, H, f$ and g are the same as in Theorem 3.1. Assume that $p : E_1 \to E_1$ and $q : E_2 \to E_2$ are λ_p-Lipschitz continuous and λ_q-Lipschitz continuous, respectively. Moreover, let $M : E_1 \to 2^{E_1}$ and $N : E_2 \to 2^{E_2}$ be P_1-η_1-accretive* *and P_2-η_2-accretive mappings with $f(E_1)\cap \mathrm{D}(M)\neq\emptyset$ and $g(E_2)\cap \mathrm{D}(N)\neq\emptyset$. If there exist constants $\rho,\gamma>0$ such that*

$$\left\{\begin{array}{l} 0<\frac{(B'+\sqrt{\theta_1}+\sqrt{\theta_4})\tau_1}{\varrho_1(1-B')}<1,\\ 0<\frac{(B''+\sqrt{\theta_2}+\sqrt{\theta_3})\tau_2}{\varrho_2(1-B'')}<1, \end{array}\right. \tag{3.30}$$

where

$$\theta_1=\frac{1+\rho\lambda_{S_1}\lambda_{D_F}}{1-\rho(\lambda_{S_1}\lambda_p+\lambda_{S_2}\lambda_{D_F})},\quad \theta_2=\frac{\rho\lambda_{S_2}\lambda_{D_F}}{1-\rho(\lambda_{S_1}\lambda_p+\lambda_{S_2}\lambda_{D_F})},$$
$$\theta_3=\frac{1+\gamma\lambda_{T_2}\lambda_q}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_q)},\quad \theta_4=\frac{\gamma\lambda_{T_1}\lambda_{D_H}}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_q)},$$

and B' and B'' are the same as in (3.9). *Then, the iterative sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$ generated by Algorithm 3.2 converge strongly to x,y,u,v,z' and z'', respectively, and (x,y,u,v,z',z'') is a solution to the system* (3.4).

4. Conclusions

The study of nonlinear equations of evolution in the setting of Banach spaces was the main incentive to introduce the notion of accretive mappings which the beginning of their study comes back to the sixties. Remember that during the past decades, the applications in different branches of sciences have been major motivations and driving forces for developing and generalizing such mappings in different contexts. In one of the pioneering studies in this direction, the introduction of the concept of P-η-accretive mapping was first made by Kazmi and Khan [9] in 2007. The above description motivated us to construct a new iterative algorithm for solving a new system of generalized multi-valued resolvent equations (for short, SGMRE) in the framework of Banach spaces. We have studied the convergence analysis of the sequences generated by our proposed iterative algorithm under some appropriate conditions along with new results which improve and generalize many known corresponding results.

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