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Two-step inertial viscosity subgradient extragradient algorithm with self-adaptive step sizes for solving pseudomonotone equilibrium problems

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ABSTRACT. This paper presents a two-step inertial viscosity subgradient extragradient algorithm with self-adaptive step sizes for finding a solution to pseudomonotone equilibrium problems in the setting of a real Hilbert space, where such a solution also includes some additional properties. Without prior knowledge of the Lipschitz constants of the pseudomonotone bifunction, the strong convergence theorem of the suggested algorithm is provided under some mild constraint qualifications for the scalar sequences. To demonstrate the effectiveness of the constructed algorithm, numerical experiments are performed on Nash-Cournot oligopolistic equilibrium models of electricity markets, Nash-Cournot models, and the image restoration problem.

1. INTRODUCTION

The equilibrium problem started to gain interest after the publication of a paper by Blum and Oettli [5], which has been widely applied to study real world applications, see [23, 29, 30], and the references therein. The equilibrium problem is a problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f\colon H \times H \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (1.1) will be denoted by $EP(f, C)$. Mathematically, the equilibrium problem (1.1) encompasses various significant problems as particular cases. Examples include optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, saddle point problems, and fixed point problems, as highlighted in [1, 10, 15, 22, 25, 33], and the references therein.

To address the equilibrium problem (1.1) with f being a monotone bifunction, one often resorts to approximate solutions using the proximal point method, as outlined below.

$$\begin{cases} x_0 \in H, \\ f(x_{k+1}, y) + \frac{1}{\lambda_k}\langle x_{k+1} - x_k, y - x_{k+1}\rangle \geq 0, \forall y \in C, \end{cases} \tag{1.2}$$

where $\{\lambda_k\} \subset (0, +\infty)$. In [7], the authors demonstrated that the sequence $\{x_k\}$, constructed by Algorithm (1.2), converges weakly to a solution of the equilibrium problem (1.1). However, the proximal point method may not be applied for a weaker assumption, such as a pseudomonotone, see [13]. To overcome this drawback, Tran et al. [31] proposed the following so-called extragradient method for solving the equilibrium problem when

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the bifunction f is pseudomonotone and satisfies Lipschitz-type continuity with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min \left\{ \lambda f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C \right\}, \\ x_{k+1} = \arg\min \left\{ \lambda f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C \right\}, \end{cases} \tag{1.3}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. The authors proved that the sequence $\{x_k\}$ generated by Algorithm (1.3) converges weakly to a solution of the equilibrium problem (1.1). Observe that the extragradient method needs to solve two optimization problems on the feasible set C for finding y_k and x_{k+1} at each iteration. This point may lead to difficulties in use if the feasible set C has a complex structure. In order to improve this situation, Hieu [14] proposed the following so-called subgradient extragradient method for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuity with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in H, \\ y_k = \arg\min \left\{ \lambda_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C \right\}, \\ T_k = \{ z \in H : \langle x_k - \lambda_k v_k - y_k, z - y_k \rangle \leq 0 \}, v_k \in \partial_2 f(x_k, y_k), \\ z_k = \arg\min \left\{ \lambda_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in T_k \right\}, \\ x_{k+1} = \alpha_k x_0 + (1 - \alpha_k) z_k, \end{cases} \tag{1.4}$$

where $0 < \lambda_k < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{\alpha_k\} \subset (0,1)$ such that $\sum_{k=0}^{\infty} \alpha_k = +\infty$ and $\lim_{k\to\infty} \alpha_k = 0$, and $\partial_2 f(x_k, y_k)$ is the subdifferential of $f(x_k, \cdot)$ at y_k. They proved that the sequence $\{x_k\}$ generated by Algorithm (1.4) converges strongly to $P_{EP(f,C)}(x_0)$ where $P_{EP(f,C)}(x_0)$ is the metric projection of x_0 onto $EP(f, C)$. It is worth noting that the subgradient extragradient method also involves solving two optimization problems to find y_k and z_k at each iteration, similar to the extragradient method. However, the second optimization problem for finding z_k is not performed over the feasible set C but only on the half-space T_k. Consequently, the subgradient extragradient method holds a competitive advantage over the extragradient method, particularly when the feasible set C is not straightforward. On the other hand, for the application of the aforementioned algorithms, one must choose suitable step sizes, dependent on the Lipschitz constants of the bifunction f. However, this choice may impose restrictions in practical applications as the Lipschitz constants of the bifunction are often unknown or challenging to estimate.

Meanwhile, inertial-type methods have garnered significant interest from researchers in solving equilibrium problems, as demonstrated in [12, 17, 26, 32, 35, 36] and related references. This method is derived from the heavy ball method, an implicit discretization of second-order dynamics in time [2, 3], and is considered as a means to enhance convergence properties. A key feature of inertial-type techniques is that the next iterate is constructed using two prior iterates. In this context, this iterative scheme is referred to as the one-step inertial method.

In 2021, Thong et al. [30] proposed the following algorithm by using the ideas of one-step inertial and subgradient extragradient methods (shortly, One-step ISE) for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuity:

Algorithm: One-step ISE Algorithm

Initialization. Choose parameters $\lambda_1 > 0$, $\mu \in (0,1)$, $\{\theta_k\} \subset [0,\tau)$ for some $\tau > 0$, $\{\gamma_k\} \subset (a,b) \subset (0, 1-\alpha_k)$, and $\{\epsilon_k\} \subset [0,\infty)$, $\{\alpha_k\} \subset (0,1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$, and $\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0$. Pick $x_0, x_1 \in H$ and set $k = 1$.

Step 1. Choose θ_k such that $0 \le \theta_k \le \overline{\theta}_k$, where

$$\overline{\theta}_k = \begin{cases} \min\left\{\tau, \frac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \tau, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_k = x_k + \theta_k(x_k - x_{k-1}).$$

Step 3. Solve the strongly convex program

$$y_k = \arg\min\left\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}.$$

Step 4. Construct a half-space

$$T_k = \{z \in H : \langle w_k - \lambda_k v_k - y_k, z - y_k\rangle \le 0\},$$

where $v_k \in \partial_2 f(w_k, y_k)$.

Step 5. Solve the strongly convex program

$$z_k = \arg\min\left\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in T_k\right\}.$$

Step 6. Update the next iterate x_{k+1} as

$$x_{k+1} = (1 - \alpha_k - \gamma_k)x_k + \gamma_k z_k,$$

Step 7. Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \dfrac{\mu(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]}\right\}, \\ \qquad\qquad \text{if } f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) > 0, \\ \lambda_k, \qquad\qquad \text{otherwise.} \end{cases}$$

Step 8. Put $k := k + 1$ and go to **Step 1**.

Consequently, the authors proved that the sequence $\{x_k\}$ generated by the One-step ISE Algorithm converges strongly to an element $p^* \in EP(f, C)$ where $\|p^*\| = \min\{\|p\| : p \in EP(f, C)\}$.

In 2022, Rehman et al. [25] proposed the following algorithm by using the techniques of one-step modified inertial and subgradient extragradient methods (shortly, One-step MISE) for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuity:

Algorithm: One-step MISE Algorithm

Initialization. Choose parameters $\lambda_1 > 0$, $\mu \in (0,1)$, $\tau \in (0,1)$, $\eta \in (0, 2-\sqrt{2})$, and $\{\alpha_k\} \subset (0,1)$ such that $\sum_{k=1}^{+\infty} \alpha_k = +\infty$, $\lim_{k\to\infty} \alpha_k = 0$, and $\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0$. Pick $x_0, x_1 \in H$ and set $k = 1$.

Step 1. Choose θ_k such that $0 \le \theta_k \le \overline{\theta}_k$, where

$$\overline{\theta}_k = \begin{cases} \min\left\{\tau, \frac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \tau, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_k = x_k + \theta_k(x_k - x_{k-1}) - \alpha_k\left[x_k + \theta_k(x_k - x_{k-1})\right].$$

Step 3. Solve the strongly convex program

$$y_k = \arg\min\left\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}.$$

Step 4. Construct a half-space

$$T_k = \left\{z \in H : \langle w_k - \lambda_k v_k - y_k, z - y_k\rangle \le 0\right\},$$

where $v_k \in \partial_2 f(w_k, y_k)$.

Step 5. Update the next iterate x_{k+1} by solving the strongly convex program

$$x_{k+1} = \arg\min\left\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in T_k\right\}.$$

Step 7. Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \dfrac{(2-\sqrt{2}-\eta)\mu\|w_k - y_k\|^2 + (2-\sqrt{2}-\eta)\mu\|x_{k+1} - y_k\|^2}{2\left[f(w_k, x_{k+1}) - f(w_k, y_k) - f(y_k, x_{k+1})\right]}\right\}, \\ \qquad\qquad \text{if } f(w_k, x_{k+1}) - f(w_k, y_k) - f(y_k, x_{k+1}) > 0, \\ \lambda_k, \qquad\qquad \text{otherwise.} \end{cases}$$

Step 8. Put $k := k+1$ and go to **Step 1**.

Accordingly, the authors proved that the sequence $\{x_k\}$ generated by the One-step MISE Algorithm converges strongly to $p^* \in EP(f, C)$. It is emphasized that both the One-step ISE and One-step MISE Algorithms used a self-adaptive process to deal with the unknown knowledge of the Lipschitz constants of the bifunction f.

In this paper, drawing from the literature mentioned earlier, our main focus will be on the algorithm for solving the equilibrium problem (1.1). Additionally, we consider the problem (1.1) in conjunction with a contraction mapping $h\colon H \to H$ and finding a point p^* in $EP(f, C)$ such that $\|h(p^*) - p^*\| \le \|h(p^*) - p\|$, for all $p \in EP(f, C)$. It is worth pointing out that this type of problem will not only yield a solution point for $EP(f, C)$ but also address concerns related to an optimization problem, as discussed in [21] and reference therein for more information. We introduce a new iterative algorithm for finding solutions to pseudomonotone equilibrium problems. To illustrate the convergence of the introduced algorithm and to compare it with other noteworthy algorithms, we perform numerical examples in the context of practical applications.

This paper is organized as follows: Section 2 provides relevant definitions and properties that will be used in subsequent sections. In Section 3, we present the two-step inertial viscosity subgradient extragradient algorithm with self-adaptive step sizes and prove the convergence theorem. Section 4 discusses the performance of the introduced algorithm by comparing it to well-known algorithms that have appeared previously.

2. Preliminaries

This section will present the definitions and some important basic properties that will be used throughout this paper. The notation $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of the real numbers and the natural numbers, respectively.

Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$, and its corresponding norm $\|\cdot\|$. We will start by reviewing the definitions and some useful facts that will be utilized in this paper. For a Hilbert space H, we know that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

and

$$\begin{aligned} \|(1+\alpha)x - (\alpha-\beta)y - \beta z\|^2 &= (1+\alpha)\|x\|^2 - (\alpha-\beta)\|y\|^2 - \beta\|z\|^2 \\ &\quad +(1+\alpha)(\alpha-\beta)\|x-y\|^2 + \beta(1+\alpha)\|x-z\|^2 \\ &\quad -\beta(\alpha-\beta)\|y-z\|^2, \end{aligned} \tag{2.5}$$

for each $x, y, z \in H$, and for each $\alpha, \beta \in \mathbb{R}$, see [16].

Definition 2.1. *Let C be a nonempty closed convex subset of H. A bifunction $f : H \times H \to \mathbb{R}$ is said to be:*

(i) monotone on C if

$$f(x,y) + f(y,x) \leq 0, \forall x, y \in C;$$

(ii) pseudomonotone on C if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \forall x, y \in C;$$

(iii) Lipschitz-type continuity on H if there exists two positive constants c_1 and c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \forall x, y, z \in H. \tag{2.6}$$

Remark 2.1. *A monotone bifunction is a pseudomonotone bifunction, but the converse is not true in general, for instance, see* [18].

For each $x \in H$, we denote the metric projection of x onto a nonempty closed convex subset C of H by $P_C(x)$, that is

$$\|x - P_C(x)\| \leq \|x - y\|, \forall y \in C.$$

Lemma 2.1. [6, 11] *Let C be a nonempty closed convex subset of H. Then,*

(i) $P_C(x)$ is singleton and well-defined for each $x \in H$;
(ii) $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in C$.

For a function $f : H \to \mathbb{R}$, the subdifferential of f at $x \in H$ is defined by

$$\partial f(x) = \{z \in H : f(y) - f(x) \geq \langle z, y - x\rangle, \forall y \in H\}.$$

The function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$.

Lemma 2.2. [6] *For any $x \in H$, the subdifferentiable $\partial f(x)$ of a continuous convex function f is a weakly closed and bounded convex set.*

Lemma 2.3. [9] *Let C be a nonempty convex subset of H and $f : C \to \mathbb{R}$ be a convex subdifferentiable and lower semicontinuous function on C. Then, x^* is a solution to the following convex problem:*

$$\min\{f(x) : x \in C\}$$

if and only if $0 \in \partial f(x^*) + N_C(x^*)$, *where $N_C(x^*)$ is the normal cone of C at x^*, that is $N_C(x^*) := \{z \in H : \langle z, y - x^*\rangle \le 0, \forall y \in C\}$.*

We end this section by recalling some auxiliary results for obtaining the main results.

Lemma 2.4. [21] *Let $\{a_k\}$ and $\{c_k\}$ be sequences of non-negative real numbers such that*

$$a_{k+1} \le (1-\alpha_k)a_k + \alpha_k b_k + c_k, \forall k \in \mathbb{N} \cup \{0\},$$

where $\{\alpha_k\}$ is a sequence in $(0,1)$ and $\{b_k\}$ is a sequence in $\mathbb{R}$. Assume that $\sum\limits_{k=0}^{\infty} c_k < \infty$. If $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$ and $\limsup\limits_{k\to\infty} b_k \le 0$, then $\lim\limits_{k\to\infty} a_k = 0$.

Lemma 2.5. [20] *Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ such that $a_{k_i} < a_{k_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_n\}$ of positive integers such that $\lim\limits_{n\to\infty} m_n = \infty$ and the following properties hold:*

$$a_{m_n} \le a_{m_n+1} \text{ and } a_n \le a_{m_n+1},$$

for all (sufficiently large) numbers $n \in \mathbb{N}$. Indeed, m_n is the largest number k in the set $\{1, 2, \dots, n\}$ such that

$$a_k < a_{k+1}.$$

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. The following assumptions on the bifunction $f\colon H \times H \to \mathbb{R}$ will be assumed in this work:

- $(A1)$ $f(\cdot, y)$ is sequentially weakly upper semicontinuous on C, for each fixed $y \in C$, that is if $\{x_k\} \subset C$ is a sequence converging weakly to $x \in C$, then $\limsup\limits_{k\to\infty} f(x_k, y) \le f(x, y)$;
- $(A2)$ $f(x, \cdot)$ is convex, subdifferentiable and lower semicontinuous on H, for each fixed $x \in H$;
- $(A3)$ f is psuedomonotone on C;
- $(A4)$ f is Lipschitz-type continuity on H.

Remark 3.2. *(i) If the bifunction f satisfies the assumptions $(A3)$ and $(A4)$, then $f(x, x) = 0$, for each $x \in C$, see* [32].

(ii) If the bifunction f satisfies the assumptions $(A1) - (A3)$, then the solution set $EP(f, C)$ is closed and convex, see [24, 31] *for more detail.*

Now, to find a solution to the equilibrium problem (1.1) along with a ρ-contraction mapping $h : H \to H$, we introduce the following two-step inertial viscosity subgradient extragradient algorithm:

Algorithm: Two-step Inertial Viscosity Subgradient Extragradient Algorithm (Two-step IVSE Algorithm)

Initialization. Choose parameters $\lambda_1 > 0$, $\tau \in [0,1)$, $\varsigma \in [0,1)$, $\mu \in (0,1)$, $\{\beta_k\} \in (0,1)$ with $0 < \inf \beta_k \leq \sup \beta_k < 1$, and $\{\epsilon_k\} \subset [0,\infty)$, $\{\alpha_k\} \subset \left(0, \frac{1}{2-\rho}\right)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$, and $\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0$. Pick $x_{-1}, x_0, x_1 \in H$ and set $k = 1$.

Step 1. Choose θ_k such that $0 \leq \theta_k \leq \overline{\theta}_k$, where

$$\overline{\theta}_k = \begin{cases} \min\left\{\tau, \frac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \tau, & \text{otherwise}, \end{cases}$$

and choose δ_k such that $0 \leq \delta_k \leq \overline{\delta}_k$, where

$$\overline{\delta}_k = \begin{cases} \min\left\{\varsigma, \frac{\epsilon_k}{\|x_k - x_{k-2}\|}, \frac{\epsilon_k}{\|x_{k-1} - x_{k-2}\|}\right\}, & \text{if } x_k \neq x_{k-2} \text{ and } x_{k-1} \neq x_{k-2}, \\ \varsigma, & \text{otherwise}. \end{cases}$$

Step 2. Compute

$$w_k = x_k + \theta_k(x_k - x_{k-1}) + \delta_k(x_{k-1} - x_{k-2}).$$

Step 3. Solve the strongly convex program

$$y_k = \arg\min\left\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}.$$

Step 4. Select $v_k \in \Delta_k$ and construct a half-space

$$T_k = \{z \in H : \langle w_k - \lambda_k v_k - y_k, z - y_k\rangle \leq 0\},$$

where

$$\Delta_k = \{s \in \partial_2 f(w_k, y_k) : \lambda_k s + y_k = w_k - t, \exists t \in N_C(y_k)\}.$$

Step 5. Solve the strongly convex program

$$z_k = \arg\min\left\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in T_k\right\}.$$

Step 6. Update the next iterate x_{k+1} as

$$x_{k+1} = \alpha_k h(w_k) + (1 - \alpha_k)(\beta_k w_k + (1 - \beta_k) z_k).$$

Step 7. Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \dfrac{\mu(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]}\right\}, & \\ \qquad \text{if } f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) > 0, & \\ \lambda_k, \qquad \text{otherwise}. & \end{cases}$$

Step 8. Put $k := k + 1$ and go to **Step 1**.

Remark 3.3. *(i) The terms $\theta_k(x_k - x_{k-1})$ and $\delta_k(x_{k-1} - x_{k-2})$, included in the Two-step IVSE Algorithm, are intended to enhance convergence properties. Consequently, the next iterate in the Two-step IVSE Algorithm is constructed using three prior iterates, forming what is referred to as the two-step inertial method. It is worth noting that if $\delta_k = 0$, for each $k \in \mathbb{N}$, the Two-step IVSE Algorithm reduces to the one-step inertial viscosity subgradient extragradient algorithm (shortly, One-step IVSE). Furthermore, we emphasize* *that the choice of parameters θ_k and δ_k can significantly influence the numerical behavior of the Two-step IVSE Algorithm, potentially leading to superior performance.*

(ii) The step size λ_k in the Two-step IVSE Algorithm is self-adaptive, supporting straightforward computations, as highlighted in [25, 28, 30]. *This feature enables the implementation of the Two-step IVSE Algorithm without the need for prior knowledge of the Lipschitz constants of the bifunction.*

(iii) In particular, in the case that h is a constant operator, say $h(x) = c$ for some $c \in H$ and for all $x \in H$, one sees that the sequence $\{x_k\}$ generated by the Two-step IVSE Algorithm converges strongly to $P_{EP(f,C)}(c)$, which is the case that was considered in [14, 30].

(iv) In step 4, the nonemptiness of the set Δ_k is guaranteed. Indeed, it is demonstrated by the definition of y_k and Lemma 2.3 that

$$0 \in \partial_2 \left\{ \lambda_k f(w_k, y_k) + \frac{1}{2}\|y_k - w_k\|^2 \right\} + N_C(y_k).$$

Then, there exist $s \in \partial_2 f(w_k, y_k)$ and $t \in N_C(y_k)$ such that

$$\lambda_k s + y_k - w_k + t = 0.$$

This confirms that a solution exists within Δ_k, ensuring its nonemptiness.

The following lemma establishes crucial relations in the convergence analysis of the sequence generated by the Two-step IVSE Algorithm.

Lemma 3.6. *Let $f\colon H \times H \to \mathbb{R}$ be a bifunction which satisfies $(A1) - (A4)$. Suppose that the solution set $EP(f, C)$ is nonempty. Let $w_k \in H$. If y_k, z_k, and λ_{k+1} are constructed as in the process of the Two-step IVSE Algorithm, then the following result hold:*

$$\|z_k - p\|^2 \le \|w_k - p\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2, \forall p \in EP(f, C).$$

Proof. The proof of this Lemma follows the technique in [34, Lemma 3.2]. Nevertheless, for the sake of completeness and detail, we have attached its detailed proof in the Appendix section. □

Now, we present the strong convergence theorem for the Two-step IVSE Algorithm.

Theorem 3.1. *Let $f\colon H \times H \to \mathbb{R}$ be a bifunction which satisfies $(A1) - (A4)$, and $h\colon H \to H$ be a ρ-contraction mapping. Suppose that the solution set $EP(f, C)$ is nonempty. Then, the sequence $\{x_k\}$ generated by the Two-step IVSE Algorithm converges strongly to $\tilde{p} \in EP(f, C)$ such that $\tilde{p} = P_{EP(f,C)}h(\tilde{p})$.*

Proof. Let $p \in EP(f, C)$. Firstly, we note that $\{\lambda_k\}$ is a nonincreasing sequence. On the other hand, by the Lipschitz-type continuity of f on H, there exists two positive constants c_1 and c_2 such that

$$\begin{aligned} f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) &\le c_1\|w_k - y_k\|^2 + c_2\|y_k - z_k\|^2 \\ &\le \max\{c_1, c_2\}(\|w_k - y_k\|^2 + \|y_k - z_k\|^2). \end{aligned}$$

So, by the definition of λ_k, we get

$$\lambda_{k+1} \ge \min\left\{\lambda_k, \frac{\mu}{2\max\{c_1, c_2\}}\right\} \ge \ldots \ge \min\left\{\lambda_1, \frac{\mu}{2\max\{c_1, c_2\}}\right\}.$$

This implies that $\{\lambda_k\}$ is bounded from below. Consequently, we know that the limit of $\{\lambda_k\}$ exists. Now, let $\eta \in (0, 1-\mu)$ be fixed. These imply that

$$\lim_{k\to\infty}\left(1-\frac{\mu\lambda_k}{\lambda_{k+1}}\right) = 1-\mu > \eta > 0.$$

Thus, there exists $k_0 \in \mathbb{N}$ such that

$$1-\frac{\mu\lambda_k}{\lambda_{k+1}} \geq \eta > 0,$$

for each $k \geq k_0$. By Lemma 3.6 and the above facts, we have

$$\|z_k - p\|^2 \leq \|w_k - p\|^2 - \eta\|w_k - y_k\|^2 - \eta\|y_k - z_k\|^2, \tag{3.7}$$

for each $k \geq k_0$. This implies that

$$\|z_k - p\| \leq \|w_k - p\|, \tag{3.8}$$

for each $k \geq k_0$.

Now, let $u_k = \beta_k w_k + (1-\beta_k)z_k$. In view of the inequality (3.8), we get

$$\begin{aligned} \|u_k - p\| &\leq \beta_k\|w_k - p\| + (1-\beta_k)\|z_k - p\| \\ &\leq \|w_k - p\|, \end{aligned} \tag{3.9}$$

for each $k \geq k_0$. It follows from the definition of x_{k+1} that

$$\begin{aligned} \|x_{k+1} - p\| &\leq \alpha_k\|h(w_k) - p\| + (1-\alpha_k)\|u_k - p\| \\ &\leq \rho\alpha_k\|w_k - p\| + \alpha_k\|h(p) - p\| + (1-\alpha_k)\|w_k - p\| \\ &= (1-(1-\rho)\alpha_k)\|w_k - p\| + (1-\rho)\alpha_k\frac{\|h(p)-p\|}{1-\rho}, \end{aligned}$$

for each $k \geq k_0$. Using this one together with the definition of w_k, we obtain, for each $k \geq k_0$, that

$$\begin{aligned} \|x_{k+1} - p\| &\leq (1-(1-\rho)\alpha_k)\|x_k - p\| + (1-(1-\rho)\alpha_k)\theta_k\|x_k - x_{k-1}\| \\ &\quad + (1-(1-\rho)\alpha_k)\delta_k\|x_{k-1} - x_{k-2}\| + (1-\rho)\alpha_k\frac{\|h(p)-p\|}{1-\rho} \\ &= (1-(1-\rho)\alpha_k)\|x_k - p\| + (1-\rho)\alpha_k\left(\sigma_k + \psi_k + \frac{\|h(p)-p\|}{1-\rho}\right), \end{aligned} \tag{3.10}$$

where $\sigma_k = \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\|$ and $\psi_k = \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\delta_k}{\alpha_k}\|x_{k-1} - x_{k-2}\|$. Due to the choices of the sequences $\{\theta_k\}$ and $\{\delta_k\}$, we have

$$\sigma_k = \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leq \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\epsilon_k}{\alpha_k},$$

and

$$\psi_k = \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\delta_k}{\alpha_k}\|x_{k-1} - x_{k-2}\| \leq \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\epsilon_k}{\alpha_k},$$

for each $k \in \mathbb{N}$. It follows from the properties of $\lim_{k\to\infty}\frac{\epsilon_k}{\alpha_k} = 0$ and $\lim_{k\to\infty}\alpha_k = 0$ that

$$\lim_{k\to\infty}\sigma_k = 0, \tag{3.11}$$

and

$$\lim_{k\to\infty}\psi_k = 0. \tag{3.12}$$

Thus, there exist constants $M_0, M_1 > 0$ such that

$$\sigma_k = \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leq M_0, \tag{3.13}$$

and

$$\psi_k = \left(\frac{1-(1-\rho)\alpha_k}{1-\rho}\right)\frac{\delta_k}{\alpha_k}\|x_{k-1} - x_{k-2}\| \leq M_1, \tag{3.14}$$

for each $k \in \mathbb{N}$. Using this one with the inequality (3.10), we get that

$$\begin{aligned}
\|x_{k+1} - p\| &\leq (1-(1-\rho)\alpha_k)\|x_k - p\| + (1-\rho)\alpha_k\left(M_0 + M_1 + \frac{\|h(p)-p\|}{1-\rho}\right)\\
&\leq \max\left\{\|x_k - p\|, M_0 + M_1 + \frac{\|h(p)-p\|}{1-\rho}\right\}\\
&\leq \cdots\\
&\leq \max\left\{\|x_{k_0} - p\|, M_0 + M_1 + \frac{\|h(p)-p\|}{1-\rho}\right\},
\end{aligned}$$

for each $k \geq k_0$. This implies that the sequence $\{\|x_k - p\|\}$ is bounded. Subsequently, $\{x_k\}$ is a bounded sequence.

Furthermore, by the definition of w_k and (2.5), we see that

$$\begin{aligned}
\|w_k - p\|^2 &= \|(1+\theta_k)(x_k - p) - (\theta_k - \delta_k)(x_{k-1} - p) - \delta_k(x_{k-2} - p)\|^2\\
&= (1+\theta_k)\|x_k - p\|^2 - (\theta_k - \delta_k)\|x_{k-1} - p\|^2 - \delta_k\|x_{k-2} - p\|^2\\
&\quad +(1+\theta_k)(\theta_k - \delta_k)\|x_k - x_{k-1}\|^2 + \delta_k(1+\theta_k)\|x_k - x_{k-2}\|^2\\
&\quad -\delta_k(\theta_k - \delta_k)\|x_{k-1} - x_{k-2}\|^2\\
&\leq (1+\theta_k)\|x_k - p\|^2 - (\theta_k - \delta_k)\|x_{k-1} - p\|^2 - \delta_k\|x_{k-2} - p\|^2\\
&\quad +(2\theta_k - \delta_k - \theta_k\delta_k)\|x_k - x_{k-1}\|^2 + 2\delta_k\|x_k - x_{k-2}\|^2\\
&\quad +(\delta_k - \theta_k\delta_k)\|x_{k-1} - x_{k-2}\|^2\\
&\leq \|x_k - p\|^2 + \theta_k(\|x_k - p\|^2 - \|x_{k-1} - p\|^2) + \delta_k\big(\|x_{k-1} - p\|^2\\
&\quad -\|x_{k-2} - p\|^2\big) + 2\theta_k\|x_k - x_{k-1}\|^2 + 2\delta_k\|x_k - x_{k-2}\|^2\\
&\quad +\delta_k\|x_{k-1} - x_{k-2}\|^2,
\end{aligned} \tag{3.15}$$

for each $k \in \mathbb{N}$. This, along with the relation (3.7), implies that

$$\begin{aligned}
\|z_k - p\|^2 - \|x_k - p\|^2 &\leq \theta_k(\|x_k - p\|^2 - \|x_{k-1} - p\|^2) + \delta_k\big(\|x_{k-1} - p\|^2\\
&\quad -\|x_{k-2} - p\|^2\big) + 2\theta_k\|x_k - x_{k-1}\|^2 + 2\delta_k\|x_k - x_{k-2}\|^2\\
&\quad +\delta_k\|x_{k-1} - x_{k-2}\|^2 - \eta\|w_k - y_k\|^2 - \eta\|y_k - z_k\|^2,
\end{aligned} \tag{3.16}$$

for each $k \geq k_0$.

On the other hand, by the definition of u_k and the relation (3.7), we have

$$\begin{aligned}
\|u_k - p\|^2 &\leq \beta_k\|w_k - p\|^2 + (1-\beta_k)\|z_k - p\|^2\\
&\leq \beta_k\|w_k - p\|^2 + (1-\beta_k)\left(\|w_k - p\|^2 - \eta\|w_k - y_k\|^2 - \eta\|y_k - z_k\|^2\right)\\
&= \|w_k - p\|^2 - (1-\beta_k)\eta\left(\|w_k - y_k\|^2 + \|y_k - z_k\|^2\right),
\end{aligned}$$

for each $k \geq k_0$. Thus, applying the above inequality and the inequality (3.15) to the definition of x_{k+1}, we obtain that

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \alpha_k \|h(w_k) - p\|^2 + (1-\alpha_k)\|u_k - p\|^2 \\ &\leq \alpha_k \|h(w_k) - p\|^2 + (1-\alpha_k)\big(\|w_k - p\|^2 - (1-\beta_k)\eta\big(\|w_k - y_k\|^2 \\ &\quad +\|y_k - z_k\|^2\big)\big) \\ &\leq \alpha_k \|h(w_k) - p\|^2 + (1-\alpha_k)\|x_k - p\|^2 + (1-\alpha_k)\theta_k\big(\|x_k - p\|^2 \\ &\quad -\|x_{k-1} - p\|^2\big) + (1-\alpha_k)\delta_k(\|x_{k-1} - p\|^2 - \|x_{k-2} - p\|^2) \\ &\quad +2(1-\alpha_k)\theta_k\|x_k - x_{k-1}\|^2 + 2(1-\alpha_k)\delta_k\|x_k - x_{k-2}\|^2 \\ &\quad +(1-\alpha_k)\delta_k\|x_{k-1} - x_{k-2}\|^2 - (1-\alpha_k)(1-\beta_k)\eta\big(\|w_k - y_k\|^2 \\ &\quad +\|y_k - z_k\|^2\big), \end{aligned} \tag{3.17}$$

for each $k \geq k_0$. This implies, for each $k \geq k_0$, that

$$\begin{aligned} &(1-\beta_k)\eta\|w_k - y_k\|^2 + (1-\beta_k)\eta\|y_k - z_k\|^2 \\ \leq\ &\|x_k - p\|^2 - \|x_{k+1} - p\|^2 + (1-\alpha_k)\theta_k(\|x_k - p\|^2 - \|x_{k-1} - p\|^2) \\ &+(1-\alpha_k)\delta_k(\|x_{k-1} - p\|^2 - \|x_{k-2} - p\|^2) + 2(1-\alpha_k)\theta_k\|x_k - x_{k-1}\|^2 \\ &+2(1-\alpha_k)\delta_k\|x_k - x_{k-2}\|^2 + (1-\alpha_k)\delta_k\|x_{k-1} - x_{k-2}\|^2 + \alpha_k M_2, \end{aligned} \tag{3.18}$$

where $M_2 = \sup_{k\geq k_0}\{|\|h(w_k) - p\|^2 - \|x_k - p\|^2| + (1-\beta_k)\eta\|w_k - y_k\|^2 + (1-\beta_k)\eta\|y_k - z_k\|^2\}$.

Now, since $P_{EP(f,C)}h$ is a contraction on H, we know that there exists $\tilde{p} \in EP(f,C)$ such that $\tilde{p} = P_{EP(f,C)}h(\tilde{p})$. Next, we will show that the sequence $\{x_k\}$ converges strongly to $\tilde{p}$. We investigate the following two possible cases.

Case 1. Suppose that $\|x_{k+1} - \tilde{p}\| \leq \|x_k - \tilde{p}\|$, for all $k \geq k_0$. This means that $\{\|x_k - \tilde{p}\|\}_{k\geq k_0}$ is a nonincreasing sequence. Consequently, by using this one together with the boundness property of $\{\|x_k - \tilde{p}\|\}$, we know that the limit of $\|x_k - \tilde{p}\|$ exists. Since $\lim_{k\to\infty} \theta_k\|x_k - x_{k-1}\|^2 = 0$, $\lim_{k\to\infty} \delta_k\|x_k - x_{k-2}\|^2 = 0$, $\lim_{k\to\infty} \delta_k\|x_{k-1} - x_{k-2}\|^2 = 0$, and the properties of the control sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{\theta_k\}$, it follows from the inequality (3.18) that

$$\lim_{k\to\infty} \|w_k - y_k\| = 0, \tag{3.19}$$

and

$$\lim_{k\to\infty} \|y_k - z_k\| = 0. \tag{3.20}$$

These imply that

$$\lim_{k\to\infty} \|w_k - z_k\| = 0. \tag{3.21}$$

In addition, since $\lim_{k\to\infty} \theta_k\|x_k - x_{k-1}\| = 0$ and $\lim_{k\to\infty} \delta_k\|x_{k-1} - x_{k-2}\| = 0$, we get

$$\lim_{k\to\infty} \|x_k - w_k\| = 0. \tag{3.22}$$

This, combined with (3.19), implies that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.23}$$

On the other hand, by the definition of x_{k+1} and the inequality (3.9), we have

$$\begin{aligned}
\|x_{k+1}-\tilde{p}\|^2 &= \|\alpha_k(h(w_k)-\tilde{p})+(1-\alpha_k)(u_k-\tilde{p})\|^2 \\
&\leq (1-\alpha_k)^2\|u_k-\tilde{p}\|^2+2\alpha_k\langle h(w_k)-\tilde{p},x_{k+1}-\tilde{p}\rangle \\
&\leq (1-\alpha_k)^2\|w_k-\tilde{p}\|^2+2\alpha_k\langle h(w_k)-\tilde{p},x_{k+1}-\tilde{p}\rangle \\
&= (1-\alpha_k)^2\|w_k-\tilde{p}\|^2+2\alpha_k\langle h(w_k)-h(\tilde{p}),x_{k+1}-\tilde{p}\rangle \\
&\quad +2\alpha_k\langle h(\tilde{p})-\tilde{p},x_{k+1}-\tilde{p}\rangle \\
&\leq (1-\alpha_k)^2\|w_k-\tilde{p}\|^2+2\alpha_k\rho\|w_k-\tilde{p}\|\|x_{k+1}-\tilde{p}\| \\
&\quad +2\alpha_k\langle h(\tilde{p})-\tilde{p},x_{k+1}-\tilde{p}\rangle \\
&\leq (1-\alpha_k)^2\|w_k-\tilde{p}\|^2+\alpha_k\rho\left(\|w_k-\tilde{p}\|^2+\|x_{k+1}-\tilde{p}\|^2\right) \\
&\quad +2\alpha_k\langle h(\tilde{p})-\tilde{p},x_{k+1}-\tilde{p}\rangle \\
&= \left((1-\alpha_k)^2+\alpha_k\rho\right)\|w_k-\tilde{p}\|^2+\alpha_k\rho\|x_{k+1}-\tilde{p}\|^2 \\
&\quad +2\alpha_k\langle h(\tilde{p})-\tilde{p},x_{k+1}-\tilde{p}\rangle, \qquad (3.24)
\end{aligned}$$

for each $k \geq k_0$. Consider, for each $k \in \mathbb{N}$,

$$\begin{aligned}
\|w_k-\tilde{p}\|^2 &\leq (\|x_k-\tilde{p}\|+\theta_k\|x_k-x_{k-1}\|+\delta_k\|x_{k-1}-x_{k-2}\|)^2 \\
&\leq \|x_k-\tilde{p}\|^2+\theta_k\|x_k-x_{k-1}\|^2+\delta_k\|x_{k-1}-x_{k-2}\|^2 \\
&\quad +2\theta_k\|x_k-\tilde{p}\|\|x_k-x_{k-1}\|+2\delta_k\|x_k-\tilde{p}\|\|x_{k-1}-x_{k-2}\| \\
&\quad +2\delta_k\|x_k-x_{k-1}\|\|x_{k-1}-x_{k-2}\| \\
&\leq \|x_k-\tilde{p}\|^2+3M_3\theta_k\|x_k-x_{k-1}\|+5M_3\delta_k\|x_{k-1}-x_{k-2}\|,
\end{aligned}$$

where $M_3 = \sup_{k\in\mathbb{N}}\{\|x_k-\tilde{p}\|,\|x_k-x_{k-1}\|,\|x_{k-1}-x_{k-2}\|\}$. Combining this with the inequality (3.24), for each $k \geq k_0$, we have that

$$\begin{aligned}
&\|x_{k+1}-\tilde{p}\|^2 \\
\leq\ & \left(\frac{(1-\alpha_k)^2+\alpha_k\rho}{1-\alpha_k\rho}\right)\|x_k-\tilde{p}\|^2+3M_3\left(\frac{(1-\alpha_k)^2+\alpha_k\rho}{1-\alpha_k\rho}\right)\theta_k\|x_k-x_{k-1}\| \\
&+5M_3\left(\frac{(1-\alpha_k)^2+\alpha_k\rho}{1-\alpha_k\rho}\right)\delta_k\|x_{k-1}-x_{k-2}\|+\left(\frac{2\alpha_k}{1-\alpha_k\rho}\right)\langle h(\tilde{p})-\tilde{p},x_{k+1}-\tilde{p}\rangle \\
\leq\ & \left(1-\frac{2(1-\rho)\alpha_k}{1-\alpha_k\rho}\right)\|x_k-\tilde{p}\|^2+3M_3\left(\frac{1-(1-\rho)\alpha_k}{1-\alpha_k\rho}\right)\theta_k\|x_k-x_{k-1}\| \\
&+5M_3\left(\frac{1-(1-\rho)\alpha_k}{1-\alpha_k\rho}\right)\delta_k\|x_{k-1}-x_{k-2}\|+\frac{2(1-\rho)\alpha_k}{1-\alpha_k\rho}\left(\frac{\alpha_k\|x_k-\tilde{p}\|^2}{2(1-\rho)}\right. \\
&\left.+\frac{1}{1-\rho}\langle h(\tilde{p})-\tilde{p},x_{k+1}-\tilde{p}\rangle\right) \\
\leq\ & \left(1-\frac{2(1-\rho)\alpha_k}{1-\alpha_k\rho}\right)\|x_k-\tilde{p}\|^2+\frac{2(1-\rho)\alpha_k}{1-\alpha_k\rho}\left(3M_3\left(\frac{1-(1-\rho)\alpha_k}{2(1-\rho)}\right)\frac{\theta_k}{\alpha_k}\|x_k-x_{k-1}\|\right. \\
&\left.+5M_3\left(\frac{1-(1-\rho)\alpha_k}{2(1-\rho)}\right)\frac{\delta_k}{\alpha_k}\|x_{k-1}-x_{k-2}\|+\frac{\alpha_k M_4}{2(1-\rho)}+\frac{1}{1-\rho}\langle h(\tilde{p})-\tilde{p},x_{k+1}-\tilde{p}\rangle\right),
\end{aligned}$$

where $M_4 = \sup_{k \geq k_0} \{\|x_k - \tilde{p}\|^2\}$. Put $\gamma_k = \frac{2(1-\rho)\alpha_k}{1-\alpha_k \rho}$, for each $k \in \mathbb{N}$. Hence, the above inequality implies that

$$\begin{aligned} \|x_{k+1} - \tilde{p}\|^2 \leq\ & (1-\gamma_k)\|x_k - \tilde{p}\|^2 + \gamma_k \Bigg(\frac{3M_3\sigma_k}{2} + \frac{5M_3\psi_k}{2} + \frac{\alpha_k M_4}{2(1-\rho)} \\ & + \frac{1}{(1-\rho)} \langle h(\tilde{p}) - \tilde{p}, x_{k+1} - \tilde{p} \rangle \Bigg), \end{aligned} \tag{3.25}$$

for each $k \geq k_0$. Moreover, by the property of the sequence $\{\alpha_k\}$, we note that

$$\sum_{k=1}^{\infty} \gamma_k = \infty. \tag{3.26}$$

Now, let $x^* \in \omega_w(x_k)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ which converges weakly to x^*. Utilizing (3.23), we observe that the subsequence $\{y_{k_n}\}$ of $\{y_k\}$ also converges weakly to x^*. Since C is closed and convex set, so it is weakly closed, therefore we can confirm that $x^* \in C$.

Next, in view of the inequalites (6.41), (6.44), and (6.46) in the Appendix section, we obtain

$$\begin{aligned} \lambda_{k_n} f(y_{k_n}, y) \geq\ & \lambda_{k_n} f(y_{k_n}, z_{k_n}) + \langle w_{k_n} - z_{k_n}, y - z_{k_n} \rangle \\ \geq\ & \lambda_{k_n} f(w_{k_n}, z_{k_n}) - \lambda_{k_n} f(w_{k_n}, y_{k_n}) - \frac{\mu \lambda_{k_n}}{2\lambda_{k_n+1}} \|w_{k_n} - y_{k_n}\|^2 \\ & - \frac{\mu \lambda_{k_n}}{2\lambda_{k_n+1}} \|y_{k_n} - z_{k_n}\|^2 + \langle w_{k_n} - z_{k_n}, y - z_{k_n} \rangle \\ \geq\ & \langle y_{k_n} - w_{k_n}, y_{k_n} - z_{k_n} \rangle - \frac{\mu \lambda_{k_n}}{2\lambda_{k_n+1}} \|w_{k_n} - y_{k_n}\|^2 \\ & - \frac{\mu \lambda_{k_n}}{2\lambda_{k_n+1}} \|y_{k_n} - z_{k_n}\|^2 + \langle w_{k_n} - z_{k_n}, y - z_{k_n} \rangle, \end{aligned} \tag{3.27}$$

for each $y \in C$. It follows from the facts (3.19), (3.20), (3.21), and the boundedness of $\{z_k\}$ that the right-hand side of the above inequality tends to zero. Thus, by using the sequentially weakly upper semicontinuity of f and $\lambda_{k_n} > 0$, we have

$$0 \leq \limsup_{n \to \infty} f(y_{k_n}, y) \leq f(x^*, y), \forall y \in C.$$

This means $x^* \in EP(f, C)$ and so $\omega_w(x_k) \subset EP(f, C)$.

Next, since $x^* \in \omega_w(x_k) \subset EP(f, C)$ is arbitrary and the property of $\tilde{p} = P_{EP(f,C)} h(\tilde{p})$, we observe that

$$\begin{aligned} \limsup_{k \to \infty} \langle x_{k+1} - \tilde{p}, h(\tilde{p}) - \tilde{p} \rangle &= \lim_{n \to \infty} \langle x_{k_n+1} - \tilde{p}, h(\tilde{p}) - \tilde{p} \rangle \\ &= \langle x^* - \tilde{p}, h(\tilde{p}) - \tilde{p} \rangle \leq 0. \end{aligned} \tag{3.28}$$

Hence, by (3.11), (3.12), (3.25), (3.26), (3.28), and Lemma 2.4, we have

$$\lim_{k \to \infty} \|x_k - \tilde{p}\| = 0.$$

This completes the proof for the first case.

Case 2. Suppose that there exists a subsequence $\{\|x_{k_i} - \tilde{p}\|\}$ of $\{\|x_k - \tilde{p}\|\}$ such that

$$\|x_{k_i} - \tilde{p}\| < \|x_{k_i+1} - \tilde{p}\|, \ \forall i \in \mathbb{N}.$$

According to Lemma 2.5, there exists a nondecreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} m_n = \infty$, and

$$\|x_{m_n} - \tilde{p}\| \leq \|x_{m_n+1} - \tilde{p}\| \text{ and } \|x_n - \tilde{p}\| \leq \|x_{m_n+1} - \tilde{p}\|, \ \forall n \in \mathbb{N}. \tag{3.29}$$

It follows from the inequality (3.18) that

$$\begin{aligned} & (1-\beta_{m_n})\eta\|w_{m_n} - y_{m_n}\|^2 + (1-\beta_{m_n})\eta\|y_{m_n} - z_{m_n}\|^2 \\ \leq\ & \|x_{m_n} - p\|^2 - \|x_{m_n+1} - p\|^2 + (1-\alpha_{m_n})\theta_{m_n}(\|x_{m_n} - p\|^2 - \|x_{m_n-1} - p\|^2) \\ & +(1-\alpha_{m_n})\delta_{m_n}(\|x_{m_n-1} - p\|^2 - \|x_{m_n-2} - p\|^2) + 2(1-\alpha_{m_n})\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|^2 \\ & +2(1-\alpha_{m_n})\delta_{m_n}\|x_{m_n} - x_{m_n-2}\|^2 + (1-\alpha_{m_n})\delta_{m_n}\|x_{m_n-1} - x_{m_n-2}\|^2 + \alpha_{m_n}M_2 \\ \leq\ & (1-\alpha_{m_n})\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|(\|x_{m_n} - \tilde{p}\| + \|x_{m_n-1} - \tilde{p}\|) \\ & +(1-\alpha_{m_n})\delta_{m_n}\|x_{m_n-1} - x_{m_n-2}\|(\|x_{m_n-1} - \tilde{p}\| + \|x_{m_n-2} - \tilde{p}\|) \\ & +2(1-\alpha_{m_n})\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|^2 + 2(1-\alpha_{m_n})\delta_{m_n}\|x_{m_n} - x_{m_n-2}\|^2 \\ & +(1-\alpha_{m_n})\delta_{m_n}\|x_{m_n-1} - x_{m_n-2}\|^2 + \alpha_{m_n}M_2, \end{aligned}$$

where $M_2 = \sup_{n\in\mathbb{N}}\{|\|h(w_{m_n}) - \tilde{p}\|^2 - \|x_{m_n} - \tilde{p}\|^2| + (1-\beta_{m_n})\eta\|w_{m_n} - y_{m_n}\|^2 + (1-\beta_{m_n})\eta\|y_{m_n} - z_{m_n}\|^2\}$.

Following the line of proof for Case 1, we can demonstrate that

$$\lim_{n\to\infty}\|w_{m_n} - y_{m_n}\| = 0, \ \lim_{n\to\infty}\|y_{m_n} - z_{m_n}\| = 0, \tag{3.30}$$

$$\lim_{n\to\infty}\|w_{m_n} - z_{m_n}\| = 0, \ \lim_{n\to\infty}\|x_{m_n} - y_{m_n}\| = 0, \tag{3.31}$$

$$\limsup_{n\to\infty}\langle x_{m_n+1} - \tilde{p}, h(\tilde{p}) - \tilde{p}\rangle \leq 0, \tag{3.32}$$

and

$$\begin{aligned} \|x_{m_n+1} - \tilde{p}\|^2 \ \leq\ & (1-\gamma_{m_n})\|x_{m_n} - \tilde{p}\|^2 + \gamma_{m_n}\Bigg(\frac{3M_3\sigma_{m_n}}{2} + \frac{5M_3\psi_{m_n}}{2} \\ & + \frac{\alpha_{m_n}M_4}{2(1-\rho)} + \frac{1}{1-\rho}\langle h(\tilde{p}) - \tilde{p}, x_{m_n+1} - \tilde{p}\rangle\Bigg), \end{aligned} \tag{3.33}$$

where $M_3 = \sup_{n\in\mathbb{N}}\{\|x_{m_n} - \tilde{p}\|, \|x_{m_n} - x_{m_n-1}\|, \|x_{m_n-1} - x_{m_n-2}\|\}$ and $M_4 = \sup_{n\in\mathbb{N}}\{\|x_{m_n} - \tilde{p}\|^2\}$. Thus, the relations (3.29) and (3.33) imply that

$$\begin{aligned} \|x_{m_n+1} - \tilde{p}\|^2 \ \leq\ & (1-\gamma_{m_n})\|x_{m_n+1} - \tilde{p}\|^2 + \gamma_{m_n}\Bigg(\frac{3M_3\sigma_{m_n}}{2} + \frac{5M_3\psi_{m_n}}{2} \\ & + \frac{\alpha_{m_n}M_4}{2(1-\rho)} + \frac{1}{1-\rho}\langle h(\tilde{p}) - \tilde{p}, x_{m_n+1} - \tilde{p}\rangle\Bigg). \end{aligned} \tag{3.34}$$

Using this one together with the relation (3.29) again, we obtain

$$\|x_n - \tilde{p}\|^2 \ \leq\ \frac{3M_3\sigma_{m_n}}{2} + \frac{5M_3\psi_{m_n}}{2} + \frac{\alpha_{m_n}M_4}{2(1-\rho)} + \frac{1}{1-\rho}\langle h(\tilde{p}) - \tilde{p}, x_{m_n+1} - \tilde{p}\rangle.$$

Then, by using (3.11), (3.12), and (3.32), we have

$$\limsup_{n\to\infty}\|x_n - \tilde{p}\|^2 \leq 0.$$

Hence, we can conclude that the sequence $\{x_n\}$ converges strongly to $\tilde{p}$. This completes the proof. $\square$

4. Numerical Experiments

This section will provide examples and numerical results to support the presented Theorem 3.1. Specifically, we will consider the Two-step IVSE Algorithm along with the One-step ISE Algorithm, the One-step MISE Algorithm, and [28, Algorithm 3.1] denoted as Tan et al. Algorithm 3.1. All numerical experiments were implemented in Matlab R2024b and conducted on a MacBook Air with Apple M1 and 8.00 GB of RAM.

Example 4.1. *Let $H = \mathbb{R}^n$ be an n-dimensional vector space equipped with the Euclidean norm. The bifunction $\tilde{f}$, defined according to the Nash-Cournot oligopolistic equilibrium models of electricity markets (see* [8, 24]*), is given by:*

$$\tilde{f}(x,y) = \langle Ax + By, y - x\rangle, \quad \forall x, y \in \mathbb{R}^n,$$

where A, $B \in \mathbb{R}^{n\times n}$ are matrices such that B is symmetric positive semidefinite and $B - A$ is negative semidefinite. Note that $\tilde{f}(x,y) + \tilde{f}(y,x) = (x-y)^T(B-A)(x-y), \forall x, y \in \mathbb{R}^n$. Thus, by the property of $B - A$, we see that $\tilde{f}$ is a monotone operator.

Next, we consider the bifunction f which is generated by

$$f(x,y) = \begin{cases} \tilde{f}(x,y), & \text{if } (x,y) \in C \times C, \\ 0, & \text{otherwise,} \end{cases}$$

where $C = \prod_{i=1}^{n}[-5,5]$ is the constrained box, see [27]*. We observe that the bifunction f satisfies Lipschitz-type continuity, see* [31].

Here, the numerical experiment is considered under the case $n = 10$ and the following setting: the matrices A and B are randomly generated from the interval $[-5,5]$ such that they satisfy the above required properties and the ρ-contraction mapping $h : \mathbb{R}^{10} \to \mathbb{R}^{10}$ is a 10×10 diagonal matrix, in which each entry of the main diagonal is ρ. The control parameters of the Two-step IVSE Algorithm, the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1 are set as follows.

- *In the proposed Two-step IVSE Algorithm, we choose $\lambda_1 = 0.5$, $\tau = 0.6$, $\varsigma = 0.3$, $\mu = 0.9$, $\rho = 0.01$, $\epsilon_k = \frac{0.5}{(k+1)^2}$, and $\alpha_k = \frac{1}{k+2}$.*
- *In the One-step ISE Algorithm, we take $\lambda_1 = 0.5$, $\tau = 0.6$, $\mu = 0.9$, $\epsilon_k = \frac{0.5}{(k+1)^2}$, $\alpha_k = \frac{1}{k+2}$, and $\gamma_k = 0.5(1 - \alpha_k)$.*
- *In the One-step MISE Algorithm, we pick $\lambda_1 = 0.5$, $\tau = 0.6$, $\mu = 0.9$, $\eta = 0.05$, $\epsilon_k = \frac{0.5}{(k+1)^2}$, and $\alpha_k = \frac{1}{k+2}$.*
- *In Tan et al. Algorithm 3.1, we set $\lambda_1 = 0.5$, $\theta = 0.6$, $\mu = 0.9$, $\rho = 0.01$, $\delta = 1.05$, $\epsilon_k = \frac{0.5}{(k+1)^2}$, $\alpha_k = \frac{1}{k+2}$, $\beta_k = 0.5$, and $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$ when $S = I_{\mathbb{R}^{10}}$ is identity mapping on $\mathbb{R}^{10}$.*

Besides, the starting points $x_{-1} = x_0 = x_1 \in \mathbb{R}^{10}$ are randomly generated from the interval $[-5,5]$. The Two-step IVSE Algorithm was tested along with the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1 by using the stopping criteria $\frac{\|x_{k+1}-x_k\|}{\|x_k\|+1} < 10^{-6}$.

In the first experiment, we fix the parameter $\beta_k = 0.01 + \frac{1}{k+1}$. We conducted experiments to evaluate the performance of each set of parameters over 10 trials, with independently randomized initial points in each trial. The presented results represent the average performance across those 10 *trials. Observe that when $\delta_k = 0$, the two-step inertial method in our proposed algorithm reduces to the one-step inertial method, as provided in* [28, 30].

TABLE 1. Influence of parameters θ_k and δ_k in Two-step IVSE Algorithm where $\beta_k = 0.01 + \frac{1}{k+1}$ for the equilibrium problems in Example 4.1

Two-step IVSE	$\theta_k = 0$		$\theta_k = 0.25\overline{\theta}_k$		$\theta_k = 0.5\overline{\theta}_k$		$\theta_k = 0.75\overline{\theta}_k$		$\theta_k = \overline{\theta}_k$		*One-step ISE*		*One-step MISE*		*Tan et al. Alg.3.1*	
	Iter	*Time*	*Iter*	*Time*	*Iter*	*Time*	*Iter*	*Time*	*Iter*	*Time*	*Iter*	*Time*	*Iter*	*Time*	*Iter*	*Time*
$\delta_k = 0$	158.8	0.09	147.0	0.06	134.7	0.06	121.4	0.05	106.2	0.05						
$\delta_k = 0.25\overline{\delta}_k$	152.4	0.08	140.3	0.06	127.6	0.06	113.3	0.05	104.9	0.05						
$\delta_k = 0.5\overline{\delta}_k$	146.1	0.07	133.6	0.06	120.2	0.05	106.2	0.05	104.4	0.05	247.6	0.11	220.1	0.09	142.6	0.07
$\delta_k = 0.75\overline{\delta}_k$	139.4	0.07	126.5	0.06	110.9	0.05	108.1	0.05	108.8	0.05						
$\delta_k = \overline{\delta}_k$	132.1	0.06	118.5	0.06	107.1	0.05	110.1	0.05	118.0	0.05						

From Table 1, we presented the number of iterations (Iter) and the CPU time (Time) in seconds. The best choice of the involved parameters for both cases is $\theta_k = \overline{\theta}_k$ and $\delta_k = 0.5\overline{\delta}_k$. This means that the number of iterations and the CPU time for the Two-step IVSE Algorithm in these cases are better than in all other considered cases. Furthermore, when the parameter $\delta_k = 0$, reducing the Two-step IVSE Algorithm to the One-step IVSE Algorithm, we observe that it yields a higher number of iterations and CPU time compared to other cases, for each fixed considered parameter $\theta_k \neq \overline{\theta}_k$. In conclusion, both the number of iterations and the CPU time of the Two-step IVSE Algorithm are almost superior to those of the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1.

In the next experiment, we consider the influence of parameter β_k by fixing the parameters $\theta_k = \overline{\theta}_k$ and $\delta_k = 0.5\overline{\delta}_k$. We conducted experiments to evaluate the performance of each set of parameters over 10 trials, with independently randomized initial points in each trial. The presented results represent the average performance across those 10 trials.

TABLE 2. Influence of parameter β_k in Two-step IVSE Algorithm where $\theta_k = \overline{\theta}_k$ and $\delta_k = 0.5\overline{\delta}_k$ for the equilibrium problems in Example 4.1

	Two-step IVSE		*One-step ISE*		*One-step MISE*		*Tan et al. Alg.3.1*	
β_k	*Iter*	*Time*	*Iter*	*Time*	*Iter*	*Time*	*Iter*	*Time*
$0.01 + \frac{1}{k+1}$	94.3	0.04						
0.5	140.9	0.07	280.5	0.11	260.1	0.11	133.6	0.06
$0.99 - \frac{1}{k+1}$	1119.5	0.40						

Based on Table 2, we can suggest that setting the parameter $\beta_k = 0.01 + \frac{1}{k+1}$ results in better numbers of iterations and CPU time compared to other cases. Furthermore, the numbers of iterations and CPU time of the Two-step IVSE Algorithm in this case, $\beta_k = 0.01 + \frac{1}{k+1}$, outperform those of the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1.

Example 4.2. *Let $H = \mathbb{R}^n$ be an n-dimensional vector space equipped with the Euclidean norm. We consider a classical form of the bifunction which given by the Nash-Cournot models, see* [19],

$$\tilde{f}(x, y) = \langle Px + q^n(y + x), y - x\rangle, \quad \forall x, y \in \mathbb{R}^n,$$

where

$$P = \begin{pmatrix} 0 & q & q & \cdots & q \\ q & 0 & q & \cdots & q \\ q & q & 0 & \cdots & q \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ q & q & \cdot & \cdots & 0 \end{pmatrix}_{n\times n},$$

where q is a positive real number. We know that the bifunctions $\tilde{f}$ is pseudomonotone and it is not monotone on C, see [4].

The numerical experiment is conducted under the case $n = 10$ and the following setting: the constrained box C, the bifunction f, the contraction mapping h, and the control parameters are all specified as in Example 4.1, with the parameter β_k fixed at $0.01 + \frac{1}{k+1}$. We note that f satisfies Lipschitz-type continuity, see [27]*. In addition, the positive real number q is randomly generated from the interval $(0, 1)$ and the starting points $x_{-1} = x_0 = x_1 \in \mathbb{R}^{10}$ are randomly generated from the interval $[-5, 5]$. Here, we conducted experiments to evaluate the performance of each set of parameters over 10 trials, with independently randomized initial points in each trial. The presented results represent the average performance across those 10 trials. The Two-step IVSE Algorithm was tested along with the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1 by using the stopping criteria $\frac{\|x_{k+1}-x_k\|}{\|x_k\|+1} < 10^{-6}$.*

TABLE 3. Influence of parameters θ_k and δ_k in Two-step IVSE Algorithm where $\beta_k = 0.01 + \frac{1}{k+1}$ for the equilibrium problems in Example 4.2

Two-step IVSE	$\theta_k = 0$		$\theta_k = 0.25\overline{\theta}_k$		$\theta_k = 0.5\overline{\theta}_k$		$\theta_k = 0.75\overline{\theta}_k$		$\theta_k = \overline{\theta}_k$		One-step ISE		One-step MISE		Tan et al. Alg.3.1	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
$\delta_k = 0$	623.3	0.22	551.0	0.19	520.9	0.18	487.1	0.17	551.6	0.20						
$\delta_k = 0.25\overline{\delta}_k$	602.2	0.22	542.5	0.19	415.5	0.14	504.6	0.18	456.1	0.16						
$\delta_k = 0.5\overline{\delta}_k$	500.2	0.18	477.5	0.16	432.3	0.16	484.6	0.17	410.1	0.14	683.8	0.23	671.9	0.23	421.2	0.15
$\delta_k = 0.75\overline{\delta}_k$	415.6	0.14	542.4	0.19	512.6	0.18	555.5	0.19	448.7	0.16						
$\delta_k = \overline{\delta}_k$	468.8	0.16	534.3	0.18	518.0	0.18	500.7	0.18	614.3	0.22						

Table 3 reveals that the selected parameters, $\theta_k = \overline{\theta}_k$ and $\delta_k = 0.5\overline{\delta}_k$, result in improved numbers of iterations and CPU time compared to all other considered cases. Additionally, we observe that both the number of iterations and the CPU time of the Two-step IVSE Algorithm outperform those of the One-step ISE and One-step MISE Algorithms. Meanwhile, the iterations and the CPU time of the Two-step IVSE Algorithm, with some choices for the parameters θ_k and δ_k, exhibit better performance than those of Tan et al. Algorithm 3.1.

Example 4.3. *Here, we regard the image restoration problem, when all images have $n := m_1 \times m_2$ pixels and each pixel value belongs into the range $[0, 255]$. Let $H = \mathbb{R}^n$ be an n-dimensional vector space equipped with the Euclidean norm and $C = \prod_{i=1}^{n}[0, 255]$ be a constrained box.*

The image restoration problem can be modeled by the linear equation system as follows:

$$r = Qx + v, \tag{4.35}$$

where $x \in \mathbb{R}^n$ is the original image, $r \in \mathbb{R}^n$ is the degraded image, $v \in \mathbb{R}^n$ is additive noise, and $Q \in \mathbb{R}^{n\times n}$ is the blurring matrix. To solve (4.35)*, we aim to estimate the original image, vector x, by utilizing the following minimization problem to minimize the additive noise:*

$$\min_{x\in C} \frac{1}{2}\|Qx - r\|^2,$$

see [30]*. We take into consideration the bifunction f, which is defined by*

$$f(x, y) = g(y) - g(x), \ \forall x, y \in \mathbb{R}^n,$$

where $g(x) := \frac{1}{2}\|Qx - r\|^2$. It is obvious that

$$f(x, y) + f(y, x) = 0, \ \forall x, y \in \mathbb{R}^n.$$

Then, the bifunction f is monotone. Besides, the bifunction f satisfies Lipschitz-type continuity.

During this numerical experiment, the control parameters are imposed as in Example 4.1 by fixing values of the control parameters $\theta_k = \overline{\theta}_k$, $\delta_k = 0.5\overline{\delta}_k$, $\beta_k = 0.01 + \frac{1}{k+1}$, while the contraction mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ is given by $h(x) = \dfrac{x}{4}$. The starting points $x_{-1} = x_0 = x_1 \in \mathbb{R}^n$ are randomly generated from the interval $(0, 1)$. The Two-step IVSE Algorithm was tested along with the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1 *by using the stopping criteria as the number of iterations 2000. Throughout all comparisons, the original images are two RGB images, peppers and Barbara with the sizes* 384×512 *and* 512×512*, respectively. To obtain the degraded images, the motion blur is applied to the original images with a motion length of 21 pixels and motion orientation* $11°$*. We use the peak signal-to-noise ratio (PSNR), in decibel (dB), to measure the quality of the restored image, which is determined by*

$$PSNR = 20 \log_{10} \frac{255^2}{\|x_k - x\|^2},$$

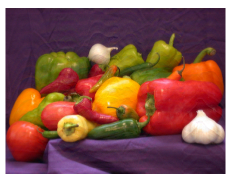
where x *is the original image and* x_k *is the restored image at the* k*-th iteration. The higher PSNR value indicates the higher quality restored image. This means the PSNR value increases as the restored image* x_k *tends to the original image* x*. The restored images at the* 2000*-th iteration are illustrated in Figures 1 and 2, respectively. In the interim, the PSNR values are displayed in Figure 3.*

(A) Original image

(B) Degraded image

(C) Two-step IVSE Algorithm
PSNR = 49.69 dB

(D) One-step ISE Algorithm
PSNR = 46.39 dB



(E) One-step MISE Algorithm
PSNR = 49.14 dB

(F) Tan et al. Algorithm 3.1
PSNR = 41.09 dB

FIGURE 1. Comparison of the restored peppers images in Example 4.3

From Figures 1 and 2, it is evident that the Two-step IVSE Algorithm provides higher PSNR values in comparison to the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1 for both tested images. In addition, the plots in Figure 3 demonstrate that the Two-step IVSE Algorithm yields a more effective solution than the One-step ISE Algorithm, the One-step MISE Algorithm, and Tan et al. Algorithm 3.1.

5. CONCLUSIONS

This paper introduces a two-step inertial viscosity subgradient extragradient algorithm with self-adaptive step sizes, specifically designed for solving pseudomonotone equilibrium problems within the framework of a real Hilbert space. The proposed algorithm not only provides a solution but also incorporates additional properties. Notably, the strong convergence theorem for this algorithm is established without the need for prior knowledge of the Lipschitz constants of the pseudomonotone bifunction, and this convergence

(A) Original image

(B) Degraded image

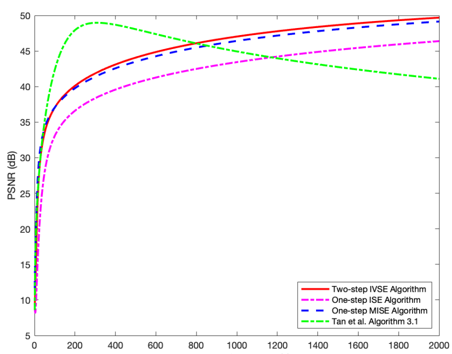
(C) Two-step IVSE Algorithm
PSNR = 40.45 dB

(D) One-step ISE Algorithm
PSNR = 37.22 dB

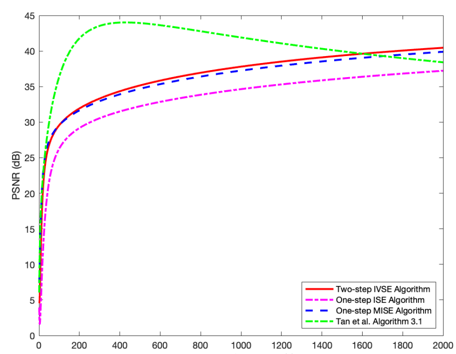
(E) One-step MISE Algorithm
PSNR = 39.89 dB

(F) Tan et al. Algorithm 3.1
PSNR = 38.41 dB

FIGURE 2. Comparison of the restored Barbara images in Example 4.3



(A) The peppers image with size 384×512



(B) The Barbara image with size 512×512

FIGURE 3. The behavior of PSNR values of the RGB images with different sizes in Example 4.3

result holds under mild constraint qualifications for the scalar sequences. To demonstrate the effectiveness of the developed algorithm, numerical experiments are conducted, focusing on Nash-Cournot oligopolistic equilibrium models in electricity markets, Nash-Cournot models, and the image restoration problem. In the context of future research directions, consideration should be given to analyzing the convergence rate of the proposed algorithm.

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Appendix

This section demonstrates the proof of Lemma 3.6.

Proof. Firstly, we show that $C \subset T_k$, for each $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed and $y \in C$. Since $v_k \in \Delta_k$, then there exists $q_k \in N_C(y_k)$ such that

$$\lambda_k v_k + y_k = w_k - q_k. \tag{6.36}$$

Using above equality, in view of $q_k \in N_C(y_k)$, we get

$$\langle w_k - \lambda_k v_k - y_k, y - y_k \rangle = \langle q_k, y - y_k \rangle \leq 0. \tag{6.37}$$

This implies that $y \in T_k$. This shows that $C \subset T_k$, for each $k \in \mathbb{N}$. Consequently, this fact guarantees that the Two-step IVSE Algorithm is well-defined.

Next, we will show the conclusion of the Lemma by utilizing the above facts. Let $p \in EP(f, C)$. By the definition of z_k and Lemma 2.3, we obtain that

$$0 \in \partial_2 \left\{ \lambda_k f(y_k, z_k) + \frac{1}{2}\|z_k - w_k\|^2 \right\} + N_{T_k}(z_k).$$

Then, there exists $v \in \partial_2 f(y_k, z_k)$ and $q \in N_{T_k}(z_k)$ such that

$$\lambda_k v + z_k - w_k + q = 0. \tag{6.38}$$

So, by using the subdifferentiability of f, we have

$$f(y_k, y) - f(y_k, z_k) \geq \langle v, y - z_k \rangle, \forall y \in H. \tag{6.39}$$

In addition, from $q \in N_{T_k}(z_k)$, we get

$$\langle q, z_k - y \rangle \geq 0, \forall y \in T_k.$$

This together with the equality (6.38) yields that

$$\langle w_k - z_k, z_k - y \rangle \geq \lambda_k \langle v, z_k - y \rangle, \forall y \in T_k. \tag{6.40}$$

It follows from the relation (6.39) that

$$\langle w_k - z_k, z_k - y \rangle \geq \lambda_k [f(y_k, z_k) - f(y_k, y)], \forall y \in T_k. \tag{6.41}$$

Note that, since $p \in C \subset T_k$, we have

$$\langle w_k - z_k, z_k - p \rangle \geq \lambda_k [f(y_k, z_k) - f(y_k, p)].$$

Thus, by utilizing the pseudomonotonic of f, we obtain that

$$\langle w_k - z_k, z_k - p\rangle \geq \lambda_k f(y_k, z_k). \tag{6.42}$$

Furthermore, from the subdifferentiability of f and $v_k \in \partial_2 f(w_k, y_k)$, we have

$$f(w_k, y) - f(w_k, y_k) \geq \langle v_k, y - y_k\rangle, \forall y \in H.$$

In particular, from $z_k \in T_k \subset H$, we get

$$f(w_k, z_k) - f(w_k, y_k) \geq \langle v_k, z_k - y_k\rangle. \tag{6.43}$$

Indeed, by the definition of T_k and $z_k \in T_k$, we see that

$$\langle w_k - \lambda_k v_k - y_k, z_k - y_k\rangle \leq 0$$

It follows from the inequality (6.43) that

$$\lambda_k[f(w_k, z_k) - f(w_k, y_k)] \geq \langle y_k - w_k, y_k - z_k\rangle. \tag{6.44}$$

Thus, the relations (6.42) and (6.44) imply that

$$\begin{aligned} \lambda_k[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)] &\geq \langle z_k - w_k, z_k - p\rangle \\ &\quad + \langle y_k - w_k, y_k - z_k\rangle. \end{aligned} \tag{6.45}$$

On the other hand, by the definition of λ_{k+1}, we observe that

$$f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) \leq \frac{\mu(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{2\lambda_{k+1}}. \tag{6.46}$$

Using this one together with the inequality (6.45), we get

$$\langle w_k - z_k, z_k - p\rangle \geq \langle y_k - w_k, y_k - z_k\rangle - \frac{\mu\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{2\lambda_{k+1}}.$$

Due to the above inequality, we note that

$$\begin{aligned} \|w_k - p\|^2 - \|w_k - z_k\|^2 - \|z_k - p\|^2 &= 2\langle w_k - z_k, z_k - p\rangle \\ &\geq 2\langle y_k - w_k, y_k - z_k\rangle - \frac{\mu\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}}. \end{aligned}$$

This implies that

$$\begin{aligned} \|z_k - p\|^2 &\leq \|w_k - p\|^2 - \|w_k - z_k\|^2 - 2\langle y_k - w_k, y_k - z_k\rangle \\ &\quad + \frac{\mu\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}} \\ &= \|w_k - p\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 - 2\langle w_k - y_k, y_k - z_k\rangle \\ &\quad - 2\langle y_k - w_k, y_k - z_k\rangle + \frac{\mu\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}} \\ &= \|w_k - p\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2. \end{aligned}$$

This completes the proof. □

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